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Erdős–Szekeres theorem for cyclic permutations

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We provide a cyclic permutation analogue of the Erdős–Szekeres theorem. In particular, we show that every cyclic permutation of length $(k-1)(\ell-1)+2$ has either an increasing cyclic subpermutation of length $k+1$ or a decreasing cyclic subpermutation of length $\ell+1$, and we show that the result is tight. We also characterize all maximum-length cyclic permutations that do not have an increasing cyclic subpermutation of length $k+1$ or a decreasing cyclic subpermutation of length $\ell+1$.

1. Introduction

The study of the longest monotone subsequence of a finite sequence of numbers has inspired a body of research in mathematics, bioinformatics, and computer science. Erdős and Szekeres [1935] showed in their namesake theorem that any permutation of $\{1, 2, \ldots, k\ell+1\}$ has an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell+1$. As a sequence $[a_1, \ldots, a_n]$ can be represented by a set of n points of the form (i, a_i) in the plane, the Erdős–Szekeres theorem can be interpreted geometrically in the following way: for any set of $k\ell+1$ points in the plane, no two of which are on the same horizontal or vertical line, there exists a polygonal path of either k positive-slope edges or ℓ negative-slope edges. It follows immediately from the Erdős–Szekeres theorem that the expected length of a longest increasing subsequence in a random permutation of length n is at least $\frac{1}{2}\sqrt{n}$. Moreover, the computation of longest increasing subsequences is also used in MUMmer systems for aligning whole genomes [Delcher et al. 1999]. A natural extension of the well-known Erdős–Szekeres theorem is to consider its analogue to cyclic subpermutations.

Definition 1. A cyclic subpermutation τ of a cyclic permutation σ is the restriction of σ on τ, i.e., remove all elements not in τ from σ.

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For example, $(1, 3, 5)$ is a cyclic subpermutation of the cyclic permutation $(1, 2, 3, 4, 5)$.

Definition 2. A cyclic permutation is increasing if it can be written in the form $(j_1, j_2, \dots, j_n)$ with $j_1 < j_2 < \dots < j_n$. Similarly, a cyclic permutation is decreasing if it can be written in the form $(j_1, j_2, \dots, j_n)$ with $j_1 > j_2 > \dots > j_n$.

For example, $(6, 1, 4, 2, 7, 3, 5)$ is a cyclic permutation whose longest increasing cyclic subpermutation is $(1, 2, 3, 5, 6)$ and whose longest decreasing cyclic subpermutations are $(7, 5, 4, 2)$ and $(7, 6, 4, 2)$.

Cyclic permutations can be viewed as circular lists, which arise naturally in the field of phylogenetics since the genomes of bacteria are considered to be circular. Geometrically, an increasing/decreasing cyclic subsequence of a circular list corresponds to a polygonal path of positive/negative-slope edges when the points are drawn on the side of a cylinder. Albert et al. [2007] gave a Monte Carlo algorithm to compute the longest increasing circular subsequence with worst case run-time $O(n^{3/2} \log n)$ and also showed that the expected length $\mu(n)$ of the longest increasing circular subsequence satisfies $\lim_{n\to\infty} \mu(n)/(2\sqrt{n}) = 1$. We extend the Erdős–Szekeres theorem to cyclic permutations and examine the structures of the extremal constructions achieving the lower bound for our theorem.

Definition 3. Given positive integers k and ℓ, let $\alpha(k, \ell)$ be the smallest positive integer n such that for any cyclic permutation of length n, there exists either an increasing cyclic subpermutation of length $k+1$, or a decreasing cyclic subpermutation of length $\ell + 1$.

We show in Section 2 that:

Theorem 4. *For* $k, \ell \geq 1$,

$$\alpha(k, \ell) = (k-1)(\ell-1) + 2.$$

Definition 5. Given positive integers k and ℓ, let $\mathbb{C}_{k,\ell}$ be the set of cyclic permutations of length $(k-1)(\ell-1)+1$ that contain no increasing cyclic subpermutations of length $k+1$, or decreasing cyclic subpermutations of length $\ell+1$. Let $\mathbb{S}_{k,\ell}$ be the set of linear permutations of length $k\ell$ that contain no increasing linear subpermutations of length $k+1$, or decreasing linear subpermutations of length $\ell+1$, and let $\mathbb{Y}_{\ell,k}$ be the set of standard Young tableaux on an $\ell \times k$ rectangular diagram, i.e., the set of $\ell \times k$ matrices where the set of entries is $\{1, 2, \dots, k\ell\}$ and each row and column forms an increasing sequence.

It was observed by Knuth [1998, Exercise 5.1.4.9], see also [Stanley 1999, Example 7.23.19(b)], that the permutations in $\mathbb{S}_{k,\ell}$ are in bijection with $\mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$ via the Robinson–Schensted correspondence. The hook-length formula [Frame et al. 1954] expresses the number of standard Young tableaux and allows us to directly compute $|\mathbb{S}_{k,\ell}|$, which increases rapidly as k, ℓ increase (see sequence A060854 in the On-Line Encyclopedia of Integer Sequences). In particular, without loss of generality, assuming $k \le l$ (since $|\mathbb{S}_{k,\ell}| = |\mathbb{S}_{\ell,k}|$), we have

$$|\mathbb{S}_{k,\ell}| = \left(\frac{(\ell k)!}{1^1 2^2 \cdots k^k (k+1)^k \cdots \ell^k (\ell+1)^{k-1} \cdots (k+\ell-1)} \right)^2.$$

Although the Robinson–Schensted correspondence establishes the bijection between $\mathbb{S}_{k,l}$ and $\mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$, it is an algorithmic procedure which can be difficult to analyze. Romik [2006] gave a simple description of the mapping from pairs of square Young tableaux to elements of $\mathbb{S}_{k,k}$. Before we state the theorem, let us introduce a few definitions.

Definition 6. The *grid-function* of $\vec{a} = [a_1, \ldots, a_{k\ell}] \in \mathbb{S}_{k,\ell}$ is $\gamma_{\vec{a}} : [k\ell] \to [\ell] \times [k]$, defined by $\gamma_{\vec{a}}(t) = (i, j)$, where i is the length of the longest decreasing subsequence of $\vec{a}$ ending at a_t and j is the length of the longest increasing subsequence of $\vec{a}$ ending at a_t.

Definition 7. The *grid-ranking* $R_{\vec{a}} = (r_{ij})$ and *grid-valuation* $V_{\vec{a}} = (v_{ij})$ are $\ell \times k$ matrices defined by $r_{ij} = \gamma_{\vec{a}}^{-1}(i, j)$, and $v_{ij} = a_{\gamma^{-1}(\ell+1-i, j)}$.

Note that the Erdős–Szekeres theorem implies that for a linear permutation $\vec{a} \in \mathbb{S}_{k,\ell}$, the longest increasing subsequence has length k and the longest decreasing subsequence has length ℓ (as both $k(\ell-1)+1$ and $(k-1)\ell+1$ are at most $k\ell$), which means that $\gamma_{\vec{a}}$ indeed defines a function.

Working towards our characterization of $\mathbb{C}_{k,\ell}$, Section 3 reproves the following result of [Romik 2006], partially for the sake of self-containment and partially for its use in the proof of Theorem 9.

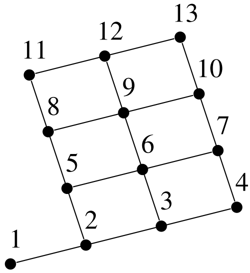
Theorem 8. *For positive integers k, ℓ, the set $\mathbb{S}_{k,\ell}$ is isomorphic to $\mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$. In particular, $\phi : \mathbb{S}_{k,\ell} \to \mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$ defined by $\phi(\vec{a}) = (R_{\vec{a}}, V_{\vec{a}})$ is a bijection.*

In contrast to the exponential size of $\mathbb{S}_{k,l}$, the set $\mathbb{C}_{k,l}$ has at most two elements and we can characterize them precisely. In particular, in Section 4, we show the following theorem:

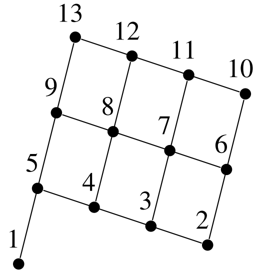
Theorem 9. *For $k, \ell \ge 1$, let $\mathbb{C}_{k,\ell}$ denote the set of cyclic permutations of length $(k-1)(\ell-1)+1$ that contain no increasing cyclic subpermutations of length $k+1$, or decreasing cyclic subpermutations of length $\ell+1$. Then we have*:

(1) *If* $\min(k, \ell) \le 2$ *then* $|\mathbb{C}_{k,\ell}| = 1$ *and the single element of* $\mathbb{C}_{k,\ell}$ *is the decreasing cyclic permutation when* $k \le 2$ *and the increasing cyclic permutation when* $k \ge 3$.

(2) *If* $\min(k, \ell) \ge 3$ *then* $|\mathbb{C}_{k,\ell}| = 2$, *and* $(1, a_1, \ldots, a_{(k-1)(\ell-1)}) \in \mathbb{C}_{k,\ell}$ *precisely when the sequence satisfies one of the following*:



structure (i)



structure (ii)

Figure 1. Extremal examples for $k = 4$ and $\ell = 5$.

(i) *For each* $(i, j) \in [\ell - 1] \times [k - 1]$,

$$a_{(j-1)(\ell-1)+i} = (\ell - 1 - i)(k - 1) + j + 1.$$

(ii) *For each* $(i, j) \in [\ell - 1] \times [k - 1]$,

$$a_{(i-1)(k-1)+j} = (j - 1)(\ell - 1) + (\ell - i) + 1.$$

Note that when $\min(k, \ell) = 2$, the structures described in parts (2i) and (2ii) are the same and coincide with the single structure described in part (1). Figure 1 illustrates the structures in parts (2i) and (2ii) for $k = 4$ and $\ell = 5$. The two extremal examples are $(1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)$ and $(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)$ respectively.

2. Proof of Theorem 4

In this section, we determine $\alpha(k, \ell)$ exactly.

Lemma 10. *For* $k, \ell \geq 1$,

$$\alpha(k, \ell) \leq (k - 1)(\ell - 1) + 2.$$

Proof. The statement is obviously true when $\min(k, \ell) = 1$, so assume $\min(k, \ell) \geq 2$. Without loss of generality $\pi = (1, a_1, a_2, \ldots, a_{(k-1)(\ell-1)+1})$. Consider the sequence $[a_1, a_2, \ldots, a_{(k-1)(\ell-1)+1}]$. By the Erdős–Szekeres theorem, it has either an increasing subsequence of length k or a decreasing subsequence of length ℓ. If there is an increasing subsequence $[a_{i_1}, a_{i_2}, \ldots, a_{i_k}]$, then $(1, a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ forms an increasing cyclic subpermutation of π of length $k + 1$. Otherwise, if there is a decreasing subsequence $[a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}]$, then $(a_{i_1}, a_{i_2}, \ldots, a_{i_\ell}, 1)$ forms a decreasing cyclic subpermutation of π of length $\ell + 1$. $\square$

Lemma 11. *For* $k, \ell \geq 1$,

$$\alpha(k, \ell) > (k - 1)(\ell - 1) + 1.$$

In particular, if $\min(k, \ell) \geq 2$ *and* $\pi = (1, a_1, \ldots, a_{(k-1)(\ell-1)})$, *where the sequence* a_i *is given by one of the formulas in Theorem 9 part (2i) or (2ii), then* π *does not have an increasing cyclic subpermutation of length* $k+1$ *or a decreasing cyclic subpermutation of length* $\ell+1$.

Proof. The lemma is trivial when $\min(k, \ell) = 1$. Assume $\min(k, \ell) \geq 2$ and $\pi = (1, a_1 \ldots, a_{(k-1)(\ell-1)})$, where $[a_1, \ldots, a_{(k-1)(\ell-1)}]$ is given by Theorem 9 part (2i); i.e., for each $(i, j) \in [\ell-1] \times [k-1]$, we have $a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1)+j+1$. (The example given in Figure 1 for $k=4$ and $\ell=5$ is $\pi = (1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)$.) The other case can be handled analogously.

We claim π does not have an increasing cyclic subpermutation of length $k+1$ nor does it have a cyclic subpermutation of length $\ell+1$. Starting from a_1, we can partition the sequence $A = [a_1, \ldots, a_{(k-1)(\ell-1)}]$ into $k-1$ decreasing subsequences $D_1, \ldots, D_{k-1}$, each consisting of $\ell-1$ consecutive elements of the original sequence. In particular, $D_i = [a_t, a_{t+1}, \ldots, a_{t+\ell-2}]$, where $t = (i-1)(\ell-1)+1$. In Figure 1, this partition corresponds to $[11, 8, 5, 2]$, $[12, 9, 6, 3]$, $[13, 10, 7, 4]$. Let L be the longest increasing cyclic subpermutation of π. Suppose $L = (a_{i_1}, a_{i_2}, \ldots, a_{i_t})$, where $a_{i_1} < a_{i_2} < \cdots < a_{i_t}$. L and D_i have at most two common elements for each i, as the elements in D_i are decreasing in A. If $a_{i_1} = 1$, then L can contain at most one element from each of the D_i. Since there are at most $k-1$ D_i's, it follows that L has length at most k. If $a_{i_1} \neq 1$, then $a_{i_1} \in D_j$ for some $j \in [k-1]$. In this case, $1 \notin L$. Furthermore, L can have at most two elements from D_j, and at most one element from D_i for each $i \in [k-1] \backslash \{j\}$. Thus L has length at most k.

We can also partition A into $\ell-1$ increasing subsequences $C_1, \ldots, C_{\ell-1}$ of length $k-1$. In particular, let $C_i = [c_i, c_i+1, \ldots, c_i+k-2]$, where $c_i = 2+(i-1)(k-1)$. In the example above, C_1, C_2, C_3, C_4 would correspond to $[2, 3, 4]$, $[5, 6, 7]$, $[8, 9, 10]$, $[11, 12, 13]$. Similar to the analysis above, let L be the longest decreasing cyclic subpermutation of π. Suppose $L = (a_{i_1}, a_{i_2}, \ldots, a_{i_t})$, where $a_{i_1} > a_{i_2} > \cdots > a_{i_t}$. As before, L can have at most two common elements with each C_i. If $a_{i_t} = 1$, then L can contain at most one element from each of the C_i. Since there are at most $\ell-1$ C_i's, it follows that L has length at most ℓ. If $a_{i_t} \neq 1$, observe that if for some j L and C_j have two common elements, then every other C_i $(i \neq j)$ can contain at most one element from L since numbers in C_t are strictly larger than all numbers in C_s for $s < t$. Thus L has length at most ℓ. □

Theorem 4 follows from Lemma 10 and 11.

3. The structure of the extremal examples in the linear Erdős–Szekeres problem

We will first consider the linear problem; i.e., subpermutations will be linear subpermutations. We will emphasize this by using the vector notation $\vec{a} = [a_1, \ldots, a_n]$

when talking about linear permutations. Recall the definitions of $\gamma_{\vec{a}}$, $R_{\vec{a}}$, $V_{\vec{a}}$ in Definitions 6 and 7. It is easy to see that $\gamma_{\vec{a}}$ is an injective (and therefore bijective) function, since for $t_1 < t_2$ we have $a_{t_1} \neq a_{t_2}$ and either every increasing sequence ending at a_{t_1} can be extended to an increasing sequence ending at a_{t_2}, or every decreasing sequence ending at a_{t_1} can be extended to a decreasing sequence ending at a_{t_2}. The following are immediate from the definitions and prior statements in the lemma:

Lemma 12. *Let $\vec{a} \in \mathbb{S}_{k,\ell}$. The following are true*:

(1) *Let $t_1, t_2 \in [k\ell]$ such that $t_1 < t_2$, and define i_1, i_2, j_1, j_2 by $\gamma_{\vec{a}}(t_q) = (i_q, j_q)$ for $q \in [2]$. If $a_{t_1} < a_{t_2}$ then $j_1 < j_2$ and if $a_{t_1} > a_{t_2}$ then $i_1 < i_2$.*

(2) *Let $i_2 \le i_1$, $j_2 \le j_1$ and $\gamma_{\vec{a}}(t_q) = (i_q, j_q)$, where $q \in [2]$. Then $t_2 \le t_1$.*

(3) $R_{\vec{a}} \in \mathbb{Y}_{\ell,k}$.

(4) *For any $i \in [\ell]$, $j \in [k]$, the sequence $[a_{\gamma_{\vec{a}}^{-1}(i,1)}, \dots, a_{\gamma_{\vec{a}}^{-1}(i,k)}]$ is an increasing subsequence of $\vec{a}$ and the sequence $[a_{\gamma_{\vec{a}}^{-1}(1,j)}, \dots, a_{\gamma_{\vec{a}}^{-1}(\ell,j)}]$ is a decreasing subsequence of $\vec{a}$.*

(5) $V_{\vec{a}} \in \mathbb{Y}_{\ell,k}$.

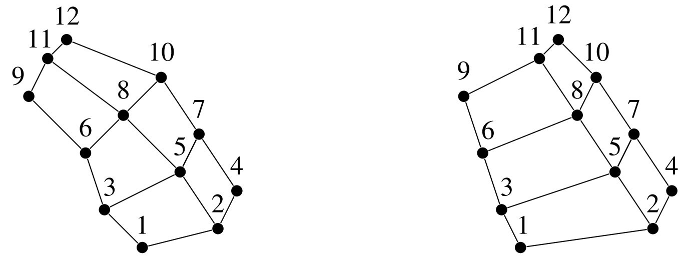
(6) *$\phi : \mathbb{S}_{k,\ell} \to \mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$ defined by $\phi(\vec{a}) = (R_{\vec{a}}, V_{\vec{a}})$ is an injective function*

Proof. Part (1) follows from the fact that if $a_{t_1} < a_{t_2}$ ($a_{t_1} > a_{t_2}$) then any increasing (decreasing) subsequence of $\vec{a}$ ending at a_{t_1} can be extended to a longer increasing (decreasing) subsequence ending at a_{t_2}. This in turn implies (2), which gives (3). Part (2) implies that for any $i \in [\ell]$, $j \in [k]$ the sequences $[\gamma^{-1}(i, 1), \gamma^{-1}(i, 2), \dots, \gamma^{-1}(i, k)]$ and $[\gamma^{-1}(1, j), \gamma^{-1}(2, j), \dots, \gamma^{-1}(\ell, j)]$ are increasing, and this together with (1) gives (4). Part (5) follows from (4). Parts (3) and (5) give that ϕ is a well-defined function, and it follows from the definitions that ϕ must be injective, so (6) is true. $\square$

The proof of Theorem 8 is finished by showing:

Lemma 13. *Let $R = (r_{ij})$, $V = (v_{ij}) \in \mathbb{Y}_{\ell,k}$ and define the sequence $\vec{a} = [a_1, \dots, a_{k\ell}]$ by $a_t = v_{ij}$ if and only if $t = r_{\ell+1-i,j}$. Then $\vec{a} \in \mathbb{S}_{k,\ell}$, $R = R_{\vec{a}}$ and $V = V_{\vec{a}}$. Consequently, the function ϕ defined in Lemma 12 is a bijection.*

Proof. From the fact that the entries of V (and also the entries of R) are unique, it follows that $\vec{a}$ is a well-defined permutation of $[k\ell]$. To show $\vec{a} \in \mathbb{S}_{k,\ell}$, it is enough to show that $\vec{a}$ does not have an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell + 1$. Assume to the contrary that $[a_{t_1}, \dots, a_{t_{k+1}}]$ is an increasing subsequence of length $k+1$ of $\vec{a}$. For each $q \in [k+1]$ define (i_q, j_q) by $a_{t_q} = v_{i_q j_q}$. By the pigeonhole principle there is a $q_1 < q_2$ such that $j_{q_1} = j_{q_2}$. Since $V \in \mathbb{Y}_{\ell,k}$, $t_{q_1} < t_{q_2}$ and $a_{t_1} < a_{t_2}$, this implies $i_{q_1} < i_{q_2}$, so $\ell + 1 - i_{q_1} > \ell + 1 - i_{q_2}$, which together with $R \in \mathbb{Y}_{k,\ell}$ gives $t_{q_1} > t_{q_2}$, a contradiction. The statement that



(9, 11, 12, 6, 3, 8, 1, 10, 5, 7, 2, 4) (9, 6, 3, 1, 11, 12, 8, 10, 5, 7, 2, 4)

Figure 2. Two examples of extremal sequences for the linear Erdős–Szekeres theorem for $k = 4$ and $\ell = 5$ with distorted grid representation. They have the same valuation but different ranking.

$\vec{a}$ does not have a decreasing subsequence of length $\ell + 1$ follows similarly, so $\vec{a} \in \mathbb{S}_{k,\ell}$. Fix an $i \in [\ell]$ and define the sequence $\vec{t} = [t_1, \ldots, t_k]$ by $t_q = r_{i,q}$. Since $R \in \mathbb{Y}_{\ell,k}$, we know $\vec{t}$ is an increasing sequence. Moreover, since $a_{t_q} = v_{\ell+1-i,q}$ and $V \in \mathbb{Y}_{\ell,k}$, we know $[a_{t_1}, \ldots, a_{t_k}]$ is an increasing subsequence of $\vec{a}$. Similarly for any $j \in [k]$ define $\vec{w} = [w_1, \ldots, w_\ell]$ by $w_q = r_{q,j}$; then $\vec{w}$ is increasing and $[a_{w_1}, \ldots, a_{w_\ell}]$ is a decreasing subsequence of $\vec{a}$. This implies that for each $i \in [\ell]$ and $j \in [k]$, we have $\gamma_{\vec{a}}(r_{i,j}) = (i', j')$, where $i' \geq i$ and $j' \geq j$. Since both $\gamma_{\vec{a}}$ and γ are bijections from $[k\ell]$ to $[\ell] \times [k]$, we get that $\gamma_{\vec{a}}(r_{i,j}) = (i, j)$ and so $r_{ij} = \gamma_{\vec{a}}^{-1}(i, j)$. Thus we obtain $R = R_{\vec{a}}$. Since for $V_{\vec{a}} = (v^\star_{ij})$ we have by definition $v^\star_{ij} = a_{\gamma_{\vec{a}}^{-1}}(\ell + 1 - i, j) = a_{r_{\ell+1-i,j}} = v_{ij}$, we obtain $V = V_{\vec{a}}$. So $\phi(\vec{a}) = (R, V)$; therefore ϕ is surjective, which together with Lemma 12 part (6) gives that ϕ is a bijection. □

We remark that similar ideas appear in [Aube et al. 2007] to find the longest increasing subsequence of a sequence. Fix $k, \ell \geq 1$ and set $n = k\ell$. Note that the results above imply that if we represent the sequence $\vec{a} = [a_1, \ldots, a_n]$ as the set of n points (t, a_t) and connect two points (t_1, a_{t_1}) and (t_2, a_{t_2}) precisely when $\gamma_{\vec{a}}(t_1)$ and $\gamma_{\vec{a}}(t_1)$ agree in one of the coordinates and differ by 1 on the other, then we get a (potentially somewhat distorted) $\ell \times k$ grid where the slope of the line from t_1 to t_2 is positive exactly when $\gamma_{\vec{a}}(t_2)$ agrees with $\gamma_{\vec{a}}(t_1)$ on the first coordinate, and negative otherwise. The grid may be distorted in the sense that it is formed by quadrangles that are not necessarily rectangles and are not necessarily isomorphic, and the grid "balances on one of its corners"; in fact it balances on the grid-point indexed by $(\ell + 1, 1)$ with sequence value 1. Indeed, any sequence $[a_1, \ldots, a_n]$ that is a permutation of $[n]$ is in $\mathbb{S}_{k,\ell}$ precisely when such a grid can be fit on its n-point representation in the plane (where the corner on which the distorted grid balances is the grid-point $(\ell + 1, 1)$ and has height 1). See Figure 2 for an illustration.

4. The structure of the extremal examples in the circular Erdős–Szekeres problem

We devote this section to the proof of Theorem 9. The statement is obvious when $\min(k, \ell) = 1$, so we assume that $\min(k, \ell) \geq 2$. For this case we have shown in Lemma 11 that the structures described in Theorem 9 are all in $\mathbb{C}_{k,\ell}$; the proof of Theorem 9 is finished by showing that these structures are the only elements of $\mathbb{C}_{k,\ell}$. Moreover, since any cyclic permutation of length at least 3 that is not the increasing (decreasing) permutation contains a decreasing (increasing) subpermutation of length at least 3, the statement follows when $\min(k, \ell) = 2$. So it is enough to focus on the case when $\min(k, \ell) \geq 3$.

We will define $\mathbb{C}^{\star}_{k,\ell}$ as the set of those sequences in $\mathbb{S}_{k-1,\ell-1}$ that, taken as cyclic permutations have no increasing cyclic subpermutation of length $k+1$, and no decreasing cyclic subpermutations of length $\ell+1$. For the ease of reference, given a sequence $\vec{\rho} \in \mathbb{C}^{\star}_{k,\ell}$ we will use ρ to denote the cyclic permutation corresponding to $\vec{\rho}$.

As an increasing (decreasing) cyclic subpermutation of a cyclic permutation either starts (ends) with 1 or does not contain 1, the following is obvious:

Lemma 14. *Let* $k, \ell \in \mathbb{Z}$ *with* $\min(k, \ell) \geq 2$. *Then* $(1, a_1, \ldots, a_{(k-1)(\ell-1)}) \in \mathbb{C}_{k,\ell}$ *if and only if* $[a_1 - 1, a_2 - 1, \ldots, a_{(k-1)(\ell-1)} - 1] \in \mathbb{C}^{\star}_{k,\ell}$.

By the above lemma, to characterize the extremal examples in the cyclic Erdős–Szekeres theorem it is enough to determine $\mathbb{C}^{\star}_{k,\ell}$. The proof of Theorem 9 is concluded by showing:

Lemma 15. *Let* $k, \ell \in \mathbb{Z}$ *with* $\min(k, \ell) \geq 3$ *and* $\vec{\rho} = [a_1, \ldots, a_{(k-1)(\ell-1)}] \in \mathbb{C}^{\star}_{k,\ell}$. *Then we have one of the following*:

(1) *For* $i \in [\ell-1]$ *and* $j \in [k-1]$, *we have* $a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1)+j$.

(2) *For* $i \in [\ell-1]$ *and* $j \in [k-1]$, *we have* $a_{(i-1)(k-1)+j} = (j-1)(\ell-1)+(\ell-i)$.

Proof. Let $\vec{\rho} = [a_1, \ldots, a_{(k-1)(\ell-1)}] \in \mathbb{C}^{\star}_{k,\ell} \subseteq \mathbb{S}_{k-1,\ell-1}$. For shortness, we will use γ for $\gamma_{\vec{\rho}}$. For each $i \in [\ell-1]$, define the sequence $C_i = [c_{i,1}, \ldots, c_{i,k-1}]$ by $c_{i,j} = a_{\gamma^{-1}(i,j)}$ and for each $j \in [k-1]$, let $D_j = [c_{1,j}, c_{2,j}, \ldots, c_{\ell-1,j}]$. Clearly, $C_1, \ldots, C_{\ell-1}$ and $D_1, \ldots, D_{k-1}$ partition the elements of $\vec{\rho}$. By Lemma 12 part (4) the C_i are increasing and the D_j are decreasing subsequences of $\vec{\rho}$. As $\vec{\rho} \in \mathbb{C}^{\star}_{k,\ell}$, the cyclic permutation ρ does not have an increasing cyclic subpermutation of length $k+1$ or decreasing cyclic subpermutation of length $\ell+1$. We have two possibilities.

<u>Case 1</u>: $\gamma^{-1}(\ell-1, 1) < \gamma^{-1}(1, k-1)$.

As for each $j \in [k-1]$, D_j is an decreasing subsequence of $\vec{\rho}$ we get

$$a_{\gamma^{-1}(1,j)} > a_{\gamma^{-1}(2,j)} > \cdots > a_{\gamma^{-1}(\ell-1,j)}.$$

Using this for $j \in \{1, k-1\}$, we have that for each $i \in [\ell-2]$

$$(c_{1,k-1}, c_{2,k-1}, \dots, c_{\ell-i,k-1}, c_{\ell-i-1,1}, c_{\ell-i,1}, \dots, c_{k-1,1})$$

is a cyclic subpermutation of length $\ell+1$ of the cyclic permutation ρ. Since this cannot be a decreasing subpermutation, we must have $c_{\ell-i,k-1} < c_{\ell-i-1,1}$. Let $i^\star \in [\ell-i-1]$ and $j \in [k-1]$. As D_1 is decreasing and $C_{i^\star}$ is increasing, we have $c_{\ell-i,k-1} < c_{\ell-i-1,1} \le c_{i^\star,1} \le c_{i^\star,j}$ and consequently $c_{\ell-i,k-1} \le (k-1)i$. Using that $C_{\ell-i}$ is increasing, induction on i gives that $c_{\ell-i,j} = a_{\gamma^{-1}(\ell-i,j)} = (i-1)(k-1)+j$.

Since C_1 and $C_{\ell-1}$ are both increasing subsequences of $\vec{\rho}$ and $C_{\ell-1}$ contains the smallest $k-1$ elements of $[(k-1)(\ell-1)]$, we must have for each $j \in [k-2]$ that $\gamma^{-1}(1, j+1) > \gamma^{-1}(\ell-1, j)$; otherwise

$$(c_{\ell-1,j}, c_{\ell-1,j+1}, \dots, c_{\ell-1,k-1}, c_{1,1}, c_{1,2}, \dots, c_{1,j+1})$$

would form an increasing cyclic subpermutation of length $k+1$ of ρ. Using the fact that D_j is a subpermutation and induction on j, for each $j \in [k-1]$ we get $\gamma^{-1}(\ell-i, j) = (j-1)(\ell-1)+\ell-i$.

Combining these we must have for $i \in [\ell-1]$ and $j \in [k-1]$ that $a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1)+j$, giving case (1) of this lemma.

Case 2: $\gamma^{-1}(l-1, 1) > \gamma^{-1}(1, k-1)$.

As before, we get that for each $j \in [k-2]$ the sequence

$$(c_{l-1,1}, c_{l-1,2}, \dots, c_{l-1,k-j}, c_{1,k-j-1}, c_{1,k-j}, \dots, c_{1,k-1})$$

is a cyclic subpermutation of length $k+1$ of ρ, and as it cannot be increasing, we have $c_{l-1,k-j} > c_{1,k-j-1}$. Using the same logic as in Case 1 we obtain for each $j \in [k-1]$ and $i \in [\ell-1]$ that $a_{\gamma^{-1}(i,j)} = (j-1)(\ell-1)+(\ell-i)$.

Again, for each $i \in [\ell-1]$ we have $\gamma^{-1}(i+1, 1) > \gamma^{-1}(i, k-1)$; otherwise

$$(c_{i,k-1}, c_{i+1,k-1}, \dots, c_{\ell-1,k-1}, c_{1,1}, c_{2,1}, \dots, c_{i+1,1})$$

forms a decreasing cyclic subpermutation of length $\ell+1$ of ρ. We obtain that $\gamma^{-1}(i, j) = (i-1)(k-1)+j$. Combining these we must have for $i \in [\ell-1]$ and $j \in [k-1]$ that $a_{(i-1)(\ell-1)+j} = (j-1)(\ell-1)+(\ell-i)$, giving case (2) of this lemma. □

For $k, \ell \ge 2$, set $n = (k-1)(\ell-1)$ and consider the sequence $\vec{\rho} = [1, a_1, \dots, a_n]$; i.e., use the sequence representation of the cyclic permutation of ρ that starts with 1. It is worth noting that $\rho \in \mathbb{C}_{k,\ell}$ precisely when taking the $n+1$ points representing $\vec{\rho}$ in the plane and putting in the grid lines corresponding to $[a_1-1, \dots, a_n-1]$ described in the end of the previous section to the n points of the form (i, a_i), they form a nondistorted grid, i.e., a grid with rectangles (and not just quadrangles) that are of the same size (in fact, the ratio of the side length of each rectangle is $(k-1)/(\ell-1)$), and the point $(1, 1)$ lies on either the first or the last line with positive slope, as in Figure 1.

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