

involve

a journal of mathematics

A generalization of the matrix transpose map and its
relationship to the twist of the polynomial ring by an
automorphism

Andrew McGinnis and Michaela Vancliff



A generalization of the matrix transpose map and its relationship to the twist of the polynomial ring by an automorphism

Andrew McGinnis and Michaela Vancliff

(Communicated by Vadim Ponomarenko)

A generalization of the notion of symmetric matrix was introduced by Cassidy and Vancliff in 2010 and used by them in a construction that produces quadratic regular algebras of finite global dimension that are generalizations of graded Clifford algebras. In this article, we further their ideas by introducing a generalization of the matrix transpose map and use it to generalize the notion of skew-symmetric matrix. With these definitions, an analogue of the result that every $n \times n$ matrix is a sum of a symmetric matrix and a skew-symmetric matrix holds. We also prove an analogue of the result that the transpose map is an antiautomorphism of the algebra of $n \times n$ matrices, and show that the antiautomorphism property of our generalized transpose map is related to the notion of twisting the polynomial ring on n variables by an automorphism.

Introduction

In [Cassidy and Vancliff 2010], a generalization of the notion of symmetric matrix was introduced and used in a construction that produces quadratic regular algebras of finite global dimension that are generalizations of graded Clifford algebras. In the same paper, it was also shown that such a matrix corresponds to a noncommutative analogue of a quadratic form. In this article, we further these ideas by introducing a generalization of the matrix transpose map and use it to generalize the notion of skew-symmetric matrix. In particular, we prove in [Theorem 2.5](#) an analogue of the result that every $n \times n$ matrix is a sum of a symmetric matrix and a skew-symmetric matrix. We also prove, in [Proposition 2.6](#) and [Corollary 2.16](#), an analogue of the result that the transpose map is an antiautomorphism of the algebra of $n \times n$ matrices. This latter property is shown in [Corollary 2.16](#) to be related to the twist of the polynomial ring on n variables by an automorphism.

MSC2010: 15A15, 15B57, 16S50, 16S36.

Keywords: transpose, automorphism, symmetric, skew-symmetric, polynomial ring, twist.

This work was supported in part by the NSF under grants DMS-0900239 and DMS-1302050.

The article is outlined as follows. In [Section 1](#), we define generalizations of symmetric and skew-symmetric matrices together with a few other concepts that will be used in the subsequent section. [Section 2](#) is in two parts: the first defines and explores a generalization of the transpose map, whereas the second ties the behavior of this transpose map to the notion of twisting a polynomial ring by an automorphism.

1. Definitions

In this section, we recall the generalizations of symmetric matrix and quadratic form that were introduced in [\[Cassidy and Vancliff 2010\]](#). We also introduce a generalization of the notion of skew-symmetric matrix.

Throughout, $\mathbb{k}$ denotes a field. We use the notation $M(n, \mathbb{k})$ to denote the vector space of $n \times n$ matrices with entries in $\mathbb{k}$ and $M(m, r, \mathbb{k})$ to denote the vector space of $m \times r$ matrices with entries in $\mathbb{k}$. For any matrix $N \in M(m, r, \mathbb{k})$, we let N_{ij} denote the ij -entry of N .

Definition 1.1. Let $\mu \in M(n, \mathbb{k})$ be such that $\mu_{ij}\mu_{ji} = 1$ for all distinct i, j . A matrix $M \in M(n, \mathbb{k})$ is said to be

- (a) μ -symmetric if $M_{ij} = \mu_{ij}M_{ji}$ for all i, j [\[Cassidy and Vancliff 2010\]](#);
- (b) skew- μ -symmetric if $M_{ij} = -\mu_{ij}M_{ji}$ for all i, j .

If $\mu_{ij} = 1$ for all i, j , then any μ -symmetric matrix is a symmetric matrix, and any skew- μ -symmetric matrix is a skew-symmetric matrix. Consequently, we generalize the notion of transpose in the next section and relate the notions of μ -symmetry and skew- μ -symmetry to that concept.

The notion of μ -symmetry was used in [\[Cassidy and Vancliff 2010\]](#) to produce algebras that may be viewed as quantized graded Clifford algebras. In other words, the main use of μ -symmetry is to “tie together” two or more matrices to a particular matrix μ , and to do so in a symmetrical manner.

Following [\[Vancliff and Veerapen 2013\]](#), we write $M^\mu(n, \mathbb{k})$ for the set of μ -symmetric $n \times n$ matrices with entries in $\mathbb{k}$. Likewise, we write $M^{s\mu}(n, \mathbb{k})$ for the set of skew- μ -symmetric $n \times n$ matrices with entries in $\mathbb{k}$. Clearly, $M^\mu(n, \mathbb{k})$ and $M^{s\mu}(n, \mathbb{k})$ are subspaces of $M(n, \mathbb{k})$.

Mirroring the theory for symmetric matrices and following [\[Cassidy and Vancliff 2010\]](#), a μ -symmetric matrix corresponds to a noncommutative analogue of a quadratic form, provided $\mu_{ii} = 1$ for all i ; this correspondence is summarized as follows.

Definition 1.2 [\[Cassidy and Vancliff 2010\]](#). Let $\mu \in M(n, \mathbb{k})$ be as in [Definition 1.1](#), with the additional assumption that $\mu_{ii} = 1$ for all i . Let (S, μ) denote the quadratic $\mathbb{k}$ -algebra on generators $z_1, \dots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i, j = 1, \dots, n$, and let S_2 denote the span of the homogeneous elements of (S, μ) of degree two. A (noncommutative) quadratic form is defined to be any element of S_2 .

The algebra (S, μ) has no zero divisors and has the same Hilbert series as the polynomial ring on n variables. By [Cassidy and Vanciliff 2010], if $\mu_{ii} = 1$ for all i , then $M^\mu(n, \mathbb{k}) \cong S_2$, as vector spaces, via the map $M \mapsto z^T M z \in S_2$, where $z = (z_1, \dots, z_n)^T$.

In the next section, the algebra (S, μ) will be considered in the special case where $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all $i, j, k = 1, \dots, n$. By [Nafari and Vanciliff 2015, Lemma 2.2], (S, μ) is a twist (see Definition 1.3 below) of the polynomial ring R on n variables by a graded automorphism of R of degree zero if and only if this condition on μ holds.

Definition 1.3 [Artin et al. 1991, §8]. Let $A = \bigoplus_{k \geq 0} A_k$ be a graded $\mathbb{k}$ -algebra and let ϕ be a graded degree-zero automorphism of A . The twist A' of A by ϕ is a graded $\mathbb{k}$ -algebra that is the vector space $\bigoplus_{k \geq 0} A_k$ with a new multiplication $*$ defined as follows: if $a' \in A'_i = A_i$ and $b' \in A'_j = A_j$, then $a' * b' = (a\phi^i(b))'$, where the right-hand side is computed using the original multiplication in A and a, b are the images of a', b' , respectively, in A .

Clearly, the twist of a quadratic algebra is again a quadratic algebra. Moreover, this notion of twist is reflexive and symmetric.

2. Main results

In this section, we define a generalization of the notion of transpose of a matrix and explore properties of this new concept. Our main results are given in Theorem 2.5, Proposition 2.6, Theorem 2.15 and Corollary 2.16.

2A. The transpose map.

Definition 2.1. If $v \in M(r, m, \mathbb{k})$ and $N \in M(m, r, \mathbb{k})$, we define the v -transpose of N , denoted N^{vT} , to be the $r \times m$ matrix with ij -entry given by $v_{ij}N_{ji}$ for all i, j .

Clearly, if $v_{ij} = 1$ for all i, j , then the v -transpose map is the transpose map. Alternatively, we may view the v -transpose as a composition of maps; for this purpose, let $\hat{v} : M(r, m, \mathbb{k}) \rightarrow M(r, m, \mathbb{k})$ be defined by $\hat{v}(K) = (v_{ij}k_{ij})$, where $K = (k_{ij}) \in M(r, m, \mathbb{k})$.

Lemma 2.2. If $v, \hat{v}$ and N are as above, then $N^{vT} = \hat{v}(N^T)$, where N^T denotes the transpose of N . In particular, the v -transpose map is a linear transformation. $\square$

Lemma 2.3. Let μ be as in Definition 1.1. A matrix $M \in M(n, \mathbb{k})$ is μ -symmetric if and only if $M^{\mu T} = M$. Additionally, M is skew- μ -symmetric if and only if $M^{\mu T} = -M$.

Proof. If $M \in M(n, \mathbb{k})$ is μ -symmetric, then $M_{ij} = \mu_{ij}M_{ji}$ for all i, j , so $M = M^{\mu T}$; reversing the argument proves the converse. The proof of skew- μ -symmetric case is similar. $\square$

Proposition 2.4. *Let $\mu \in M(n, \mathbb{k})$ be such that $\mu_{ij}\mu_{ji} = 1$ for all i, j . If $M \in M(n, \mathbb{k})$, then*

- (a) $(M^{\mu T})^{\mu T} = M$,
- (b) $M + M^{\mu T} \in M^\mu(n, \mathbb{k})$,
- (c) $M - M^{\mu T} \in M^{s\mu}(n, \mathbb{k})$.

Proof. (a) We have $[M^{\mu T}]^{\mu T} = (\mu_{ij}M_{ji})^{\mu T} = (\mu_{ij}\mu_{ji}M_{ij}) = (M_{ij}) = M$.

(b)–(c) We have $M \pm M^{\mu T} = (M_{ij} \pm \mu_{ij}M_{ji}) = (\pm\mu_{ij}(M_{ji} \pm \mu_{ji}M_{ij}))$. Thus,

$$[M \pm M^{\mu T}]^{\mu T} = (\pm\mu_{ij}\mu_{ji}(M_{ij} \pm \mu_{ij}M_{ji})) = (\pm(M_{ij} \pm \mu_{ij}M_{ji})) = \pm[M \pm M^{\mu T}],$$

and so the result follows from [Lemma 2.3](#). $\square$

Theorem 2.5. *Suppose $\text{char}(\mathbb{k}) \neq 2$. If $\mu \in M(n, \mathbb{k})$ is such that $\mu_{ij}\mu_{ji} = 1$ for all i, j , then*

$$M(n, \mathbb{k}) = M^\mu(n, \mathbb{k}) \oplus M^{s\mu}(n, \mathbb{k}).$$

Proof. If $M \in M(n, \mathbb{k})$, then $M = \frac{1}{2}(M + M^{\mu T}) + \frac{1}{2}(M - M^{\mu T})$, since $\text{char}(\mathbb{k}) \neq 2$. It follows from [Proposition 2.4](#) that $M(n, \mathbb{k}) = M^\mu(n, \mathbb{k}) + M^{s\mu}(n, \mathbb{k})$. However, the assumption on the characteristic of $\mathbb{k}$ ensures that $M^\mu(n, \mathbb{k}) \cap M^{s\mu}(n, \mathbb{k}) = \{0\}$, which completes the proof. $\square$

A well-known result for symmetric matrices is that if $X \in M(n, \mathbb{k})$ is symmetric, then $P^T X P$ is also symmetric for all $P \in M(n, \mathbb{k})$. This result is a consequence of the fact that $[XY]^T = Y^T X^T$ for all $X, Y \in M(n, \mathbb{k})$; that is, the transpose map is an antiautomorphism of $M(n, \mathbb{k})$. However, the analogues of these results are false in general for μ -symmetry, unless μ satisfies certain conditions as follows.

Proposition 2.6. *If $\mu \in M(n, \mathbb{k})$ is such that $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k , then $[XY]^{\mu T} = Y^{\mu T} X^{\mu T}$ for all $X, Y \in M(n, \mathbb{k})$.*

Proof. Let $X, Y \in M(n, \mathbb{k})$. We have

$$[XY]^{\mu T} = \left(\sum_{k=1}^n X_{ik} Y_{kj} \right)^{\mu T} = \left(\mu_{ij} \sum_{k=1}^n X_{jk} Y_{ki} \right),$$

whereas

$$Y^{\mu T} X^{\mu T} = (\mu_{ik} Y_{ki})(\mu_{kj} X_{jk}) = \left(\sum_{k=1}^n \mu_{ik} \mu_{kj} Y_{ki} X_{jk} \right) = \left(\mu_{ij} \sum_{k=1}^n X_{jk} Y_{ki} \right),$$

where the last equality is a consequence of the condition on μ . $\square$

If $\mu \in M(n, \mathbb{k})$ satisfies the hypotheses of [Propositions 2.4](#) and [2.6](#), then $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k , and $\mu_{ii} = 1$ for all i ; the converse also holds.

Corollary 2.7. *Let $\mu \in M(n, \mathbb{k})$. If $\mu_{ij} = \mu_{ik}\mu_{kj}$ for all i, j, k , and if $\mu_{ii} = 1$ for all i , then $P^{\mu T}XP \in M^\mu(n, \mathbb{k})$ for all $X \in M^\mu(n, \mathbb{k})$ and for all $P \in M(n, \mathbb{k})$.*

Proof. The conditions on μ imply $\mu_{ik}\mu_{ki} = \mu_{ii} = 1$ for all i, k , so that [Lemma 2.3](#) and [Propositions 2.4](#) and [2.6](#) may be applied to compute $[P^{\mu T}XP]^{\mu T}$; namely,

$$[P^{\mu T}XP]^{\mu T} = P^{\mu T}[P^{\mu T}X]^{\mu T} = P^{\mu T}X^{\mu T}[P^{\mu T}]^{\mu T} = P^{\mu T}XP$$

for all $X \in M^\mu(n, \mathbb{k})$ and for all $P \in M(n, \mathbb{k})$. The result follows from [Lemma 2.3](#). $\square$

The hypotheses on μ in the last result coincide with the hypotheses required for the skew polynomial ring (S, μ) , defined in [Definition 1.2](#), to be a twist (in the sense of [Definition 1.3](#)) of the polynomial ring R on n variables by a graded automorphism of R of degree zero. However, the above methods give no insight as to why this should be the case, so further analysis is required to explain this relationship and is the purpose of the next subsection.

2B. The transpose map and twisting the polynomial ring. The goal of this subsection is to show that the result of [Corollary 2.7](#) is directly related to the algebra (S, μ) being a twist of the polynomial ring R as mentioned at the end of [Section 2A](#). Our method will be to show that the result of [Corollary 2.7](#) is directly related to a certain map $\bar{\mu} : M(n, \mathbb{k}) \rightarrow M(n, \mathbb{k})$ (see [Definition 2.14](#)) being an automorphism, in which case $\bar{\mu}$ induces an automorphism of (S, μ) that twists (S, μ) to R .

Throughout this subsection, we assume that $\mu_{ii} = 1$ for all i and that $\mu_{ij}\mu_{ji} = 1$ for all i, j .

Let V denote the span of the homogeneous elements of (S, μ) of degree one. Since (S, μ) is a domain, for each $k = 1, \dots, n$, we may define $\theta_k \in \text{Aut}(S, \mu)$ via $sz_k = z_k\theta_k(s)$ for all $s \in (S, \mu)$. In particular, for every k , we have $\theta_k(z_i) = \mu_{ki}z_i$ for all i , so if we twist (S, μ) by θ_k , we obtain a quadratic algebra in which the image of z_k is central.

Let V^* denote the vector-space dual of V and let $\{z_1^*, \dots, z_n^*\}$ in V^* denote the dual basis to the basis $\{z_1, \dots, z_n\}$ of V . For each k , the linear transformation $\theta_k|_V : V \rightarrow V$ induces a linear map $\theta_k^* : V^* \rightarrow V^*$, where $\theta_k^*(z_i^*) = \mu_{ik}z_i^*$ for all i . Hence θ_k induces a linear map $\bar{\theta}_k : V \otimes_{\mathbb{k}} V^* \rightarrow V \otimes_{\mathbb{k}} V^*$ via

$$\bar{\theta}_k(v \otimes u) = \theta_k(v) \otimes \theta_k^*(u)$$

for all $v \otimes u \in V \otimes_{\mathbb{k}} V^*$.

Remark 2.8. As is well known, $V \otimes_{\mathbb{k}} V^*$ is a $\mathbb{k}$ -algebra under the usual addition and with multiplication given by $(v \otimes u)(v' \otimes u') = (vv')(v \otimes u')$ for all $v, v' \in V$, $u, u' \in V^*$. In fact, $V \otimes_{\mathbb{k}} V^* \cong M(n, \mathbb{k})$, as $\mathbb{k}$ -algebras, via the map that sends $z_i \otimes z_j^*$ to the $n \times n$ matrix with 1 in the ij -entry and zeros elsewhere.

Lemma 2.9. *For every $k = 1, \dots, n$, the linear map $\bar{\theta}_k$ is in $\text{Aut}(V \otimes_{\mathbb{k}} V^*)$.*

Proof. Since $\bar{\theta}_k$ is linear and bijective, it remains to prove that $\bar{\theta}_k$ respects multiplication, and it suffices to consider products of pure tensors. Let $v, v' \in V$ and $u, u' \in V^*$, and write $v' = \sum_{i=1}^n v_i z_i$ and $u = \sum_{j=1}^n u_j z_j^*$, where $v_i, u_j \in \mathbb{k}$ for all i, j . In particular, $uv' = \sum_{i=1}^n u_i v_i$ and

$$\theta_k^*(u)\theta_k(v') = \left(\sum_{j=1}^n u_j \mu_{jk} z_j^* \right) \left(\sum_{i=1}^n v_i \mu_{ki} z_i \right) = \sum_{i=1}^n u_i v_i = uv'.$$

It follows that

$$\begin{aligned} \bar{\theta}_k((v \otimes u)(v' \otimes u')) &= \bar{\theta}_k((uv')(v \otimes u')) \\ &= uv' \theta_k(v) \otimes \theta_k^*(u'), \end{aligned}$$

whereas

$$\begin{aligned} \bar{\theta}_k(v \otimes u) \bar{\theta}_k(v' \otimes u') &= (\theta_k(v) \otimes \theta_k^*(u)) (\theta_k(v') \otimes \theta_k^*(u')) \\ &= \theta_k^*(u) \theta_k(v') (\theta_k(v) \otimes \theta_k^*(u')), \end{aligned}$$

so the result follows. $\square$

In the following, $\mathbb{k}^\times$ denotes the nonzero elements of $\mathbb{k}$.

Lemma 2.10. *For all k, i , we have $\bar{\theta}_k = \bar{\theta}_i$ if and only if $\theta_k \in \mathbb{k}^\times \theta_i$.*

Proof. We have $\theta_k = \lambda \theta_i$ for some $\lambda \in \mathbb{k}^\times$ if and only if $\theta_k^* = \lambda^{-1} \theta_i^*$. The result follows from the definitions of $\bar{\theta}_k$ and $\bar{\theta}_i$. $\square$

Proposition 2.11. *The map θ_k is in $\mathbb{k}^\times \theta_1$ for all k if and only if the algebra (S, μ) is a twist (in the sense of [Definition 1.3](#)) of the polynomial ring on n variables.*

Proof. As mentioned above, for each k , the twist of (S, μ) by θ_k yields an algebra in which the image of z_k is central. Hence, if $\theta_k \in \mathbb{k}^\times \theta_1$ for all k , then twisting by θ_k produces an algebra R in which the image of z_i is central for all i . Since the relations of R are induced by the relations of (S, μ) , it follows that R is the polynomial ring on n variables.

Conversely, suppose (S, μ) is a twist of the polynomial ring R on n variables. It follows that there exists a degree-zero map $\theta \in \text{Aut}(S, \mu)$ such that twisting (S, μ) by θ renders the image of z_k central in R for all k . Writing “ $\cdot$ ” for the multiplication in R , this implies

$$z_k \theta(z_i) = z_k \cdot z_i = z_i \cdot z_k = z_i \theta(z_k)$$

for all i, k . However, since S is a quadratic algebra and since S_2 has a $\mathbb{k}$ -basis $\{z_j z_l : 1 \leq j \leq l \leq n\}$, it follows that $\theta(z_k) \in \mathbb{k}^\times z_k$ for all k . Writing $\theta(z_k) = \lambda_k z_k$, where $\lambda_k \in \mathbb{k}^\times$ for all k , we have $\mu_{ik} = \lambda_k / \lambda_i$ for all i, k and $\lambda_i \theta_i = \theta$ for all i . Thus, $\theta_k \in \mathbb{k}^\times \theta_1$ for all k . $\square$

Corollary 2.12. *We have $\bar{\theta}_k = \bar{\theta}_1$ for all k if and only if (S, μ) is a twist (in the sense of [Definition 1.3](#)) of the polynomial ring on n variables.*

Proof. The result follows by combining [Lemma 2.10](#) with [Proposition 2.11](#). $\square$

Lemma 2.13. *If $\bar{\theta}_k = \bar{\theta}_1$ for all k , then $\bar{\theta}_k((a_{ij})) = (\mu_{ji}a_{ij})$ for all k and for all $(a_{ij}) \in M(n, \mathbb{k})$, where $M(n, \mathbb{k})$ is identified with $V \otimes_{\mathbb{k}} V^*$ as in [Remark 2.8](#).*

Proof. By identifying $M(n, \mathbb{k})$ with $V \otimes_{\mathbb{k}} V^*$, we may write $(a_{ij}) \in M(n, \mathbb{k})$ as

$$(a_{ij}) = \left(z_1 \otimes \sum_{i=1}^n a_{1i} z_i^* \right) + \left(z_2 \otimes \sum_{i=1}^n a_{2i} z_i^* \right) + \cdots + \left(z_n \otimes \sum_{i=1}^n a_{ni} z_i^* \right).$$

If $\bar{\theta}_k = \bar{\theta}_1$ for all k , then

$$\begin{aligned} \bar{\theta}_k((a_{ij})) &= \sum_{j=1}^n \bar{\theta}_k \left(z_j \otimes \sum_{i=1}^n a_{ji} z_i^* \right) = \sum_{j=1}^n \bar{\theta}_j \left(z_j \otimes \sum_{i=1}^n a_{ji} z_i^* \right) \\ &= \sum_{j=1}^n \left(\theta_j(z_j) \otimes \sum_{i=1}^n a_{ji} \theta_j^*(z_i^*) \right) = \sum_{j=1}^n \left(z_j \otimes \sum_{i=1}^n \mu_{ij} a_{ji} z_i^* \right) = (\mu_{ji} a_{ij}). \quad \square \end{aligned}$$

[Lemma 2.13](#) motivates the following definition.

Definition 2.14. Define $\bar{\mu} : M(n, \mathbb{k}) \rightarrow M(n, \mathbb{k})$ by $\bar{\mu}((a_{ij})) = (\mu_{ji}a_{ij})$ for all $(a_{ij}) \in M(n, \mathbb{k})$.

Moreover, $\bar{\mu} = ()^T \circ \hat{\mu} \circ ()^T$, where $\hat{\mu}$ is defined just prior to [Lemma 2.2](#). Clearly, $\bar{\mu}$ is linear; with the assumption on μ at the start of [Section 2B](#), $\bar{\mu}$ is also invertible.

Theorem 2.15. *The map $\bar{\mu}$ is an automorphism of $M(n, \mathbb{k})$ if and only if the algebra (S, μ) is a twist of the polynomial ring on n variables.*

Proof. Identify $M(n, \mathbb{k})$ with $V \otimes V^*$ as in [Remark 2.8](#), so that we may view $\bar{\mu} : V \otimes V^* \rightarrow V \otimes V^*$. In particular, $\bar{\mu}(z_i \otimes z_j^*) = \mu_{ji}(z_i \otimes z_j^*)$ for all i, j . If (S, μ) is a twist of the polynomial ring, then $\bar{\mu} = \bar{\theta}_k$ for all k by [Corollary 2.12](#) and [Lemma 2.13](#). Hence $\bar{\mu}$ is an automorphism by [Lemma 2.9](#).

Conversely, suppose $\bar{\mu}$ is an automorphism. It follows that

$$\bar{\mu}((z_j \otimes z_k^*)(z_k \otimes z_i^*)) = \bar{\mu}(z_j \otimes z_k^*)\bar{\mu}(z_k \otimes z_i^*)$$

for all i, j, k . Hence,

$$\bar{\mu}(z_k^* z_k (z_j \otimes z_i^*)) = \mu_{kj}(z_j \otimes z_k^*) \mu_{ik}(z_k \otimes z_i^*)$$

for all i, j, k , so that we have

$$\mu_{ij}(z_j \otimes z_i^*) = \mu_{ik} \mu_{kj}(z_j \otimes z_i^*)$$

for all i, j, k . It follows that $\mu_{ij} = \mu_{ik} \mu_{kj}$ for all i, j, k , so that (S, μ) is a twist of the polynomial ring by [\[Nafari and Vancliff 2015, Lemma 2.2\]](#). $\square$

Corollary 2.16. *The algebra (S, μ) is a twist of the polynomial ring if and only if $[XY]^{\mu T} = Y^{\mu T} X^{\mu T}$ for all $X, Y \in M(n, \mathbb{k})$.*

Proof. Identify $M(n, \mathbb{k})$ with $V \otimes V^*$ as in [Remark 2.8](#). Considering [Definitions 2.1](#) and [2.14](#), $X^{\mu T} = [\bar{\mu}(X)]^T$ for all $X \in M(n, \mathbb{k})$. By [Theorem 2.15](#), (S, μ) is a twist of the polynomial ring if and only if $\bar{\mu}$ is an automorphism, that is, if and only if $\bar{\mu}(XY) = \bar{\mu}(X)\bar{\mu}(Y)$ for all $X, Y \in M(n, \mathbb{k})$. However, this holds if and only if $[\bar{\mu}(XY)]^T = [\bar{\mu}(X)\bar{\mu}(Y)]^T = [\bar{\mu}(Y)]^T[\bar{\mu}(X)]^T$ for all $X, Y \in M(n, \mathbb{k})$, that is, if and only if $[XY]^{\mu T} = Y^{\mu T}X^{\mu T}$ for all $X, Y \in M(n, \mathbb{k})$. $\square$

In view of this last result, it is clearer why the technical condition on μ is required in [Corollary 2.7](#); the insight is that $\bar{\mu}$ needs to be an automorphism in order to have the μ -transpose map be an antiautomorphism, but that condition on $\bar{\mu}$ allows n automorphisms of (S, μ) to “merge” into one automorphism (denoted θ in the proof of [Proposition 2.11](#)) that twists (S, μ) to the polynomial ring.

References

- [Artin et al. 1991] M. Artin, J. Tate, and M. Van den Bergh, “Modules over regular algebras of dimension 3”, *Invent. Math.* **106**:2 (1991), 335–388. [MR](#) [Zbl](#)
- [Cassidy and Vancliff 2010] T. Cassidy and M. Vancliff, “Generalizations of graded Clifford algebras and of complete intersections”, *J. Lond. Math. Soc.* (2) **81**:1 (2010), 91–112. [MR](#) [Zbl](#)
- [Nafari and Vancliff 2015] M. Nafari and M. Vancliff, “Graded skew Clifford algebras that are twists of graded Clifford algebras”, *Comm. Algebra* **43**:2 (2015), 719–725. [MR](#) [Zbl](#)
- [Vancliff and Veerapen 2013] M. Vancliff and P. P. Veerapen, “Generalizing the notion of rank to noncommutative quadratic forms”, pp. 241–250 in *Noncommutative birational geometry, representations and combinatorics*, edited by A. Berenstein and V. Retakh, *Contemp. Math.* **592**, Amer. Math. Soc., Providence, RI, 2013. [MR](#) [Zbl](#)

Received: 2015-05-27

Revised: 2015-09-05

Accepted: 2015-09-07

mcginnis82292@gmail.com

*Department of Mathematics, University of California at
Riverside, Riverside, CA 92521, United States*

vancliff@uta.edu

*Department of Mathematics, University of Texas at Arlington,
P.O. Box 19408, Arlington, TX 76019, United States*

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Suzanne Lenhart	University of Tennessee, USA
John V. Baxley	Wake Forest University, NC, USA	Chi-Kwong Li	College of William and Mary, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Robert B. Lund	Clemson University, USA
Martin Bohner	Missouri U of Science and Technology, USA	Gaven J. Martin	Massey University, New Zealand
Nigel Boston	University of Wisconsin, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA	Emil Minchev	Ruse, Bulgaria
Pietro Cerone	La Trobe University, Australia	Frank Morgan	Williams College, USA
Scott Chapman	Sam Houston State University, USA	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Joshua N. Cooper	University of South Carolina, USA	Zuhair Nashed	University of Central Florida, USA
Jem N. Corcoran	University of Colorado, USA	Ken Ono	Emory University, USA
Toka Diagana	Howard University, USA	Timothy E. O'Brien	Loyola University Chicago, USA
Michael Dorff	Brigham Young University, USA	Joseph O'Rourke	Smith College, USA
Sever S. Dragomir	Victoria University, Australia	Yuval Peres	Microsoft Research, USA
Behrouz Emamizadeh	The Petroleum Institute, UAE	Y.-F. S. Pétermann	Université de Genève, Switzerland
Joel Foisy	SUNY Potsdam, USA	Robert J. Plemmons	Wake Forest University, USA
Errin W. Fulp	Wake Forest University, USA	Carl B. Pomerance	Dartmouth College, USA
Joseph Gallian	University of Minnesota Duluth, USA	Vadim Ponomarenko	San Diego State University, USA
Stephan R. Garcia	Pomona College, USA	Bjorn Poonen	UC Berkeley, USA
Anant Godbole	East Tennessee State University, USA	James Propp	U Mass Lowell, USA
Ron Gould	Emory University, USA	József H. Przytycki	George Washington University, USA
Andrew Granville	Université Montréal, Canada	Richard Rebarber	University of Nebraska, USA
Jerrold Griggs	University of South Carolina, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Jim Haglund	University of Pennsylvania, USA	James A. Sellers	Penn State University, USA
Johnny Henderson	Baylor University, USA	Andrew J. Sterge	Honorary Editor
Jim Hoste	Pitzer College, USA	Ann Trenk	Wellesley College, USA
Natalia Hritonenko	Prairie View A&M University, USA	Ravi Vakil	Stanford University, USA
Glenn H. Hurlbert	Arizona State University, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
Charles R. Johnson	College of William and Mary, USA	Ram U. Verma	University of Toledo, USA
K. B. Kulasekera	Clemson University, USA	John C. Wierman	Johns Hopkins University, USA
Gerry Ladas	University of Rhode Island, USA	Michael E. Zieve	University of Michigan, USA

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2017 is US \$175/year for the electronic version, and \$235/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY



mathematical sciences publishers

nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

involve

2017 vol. 10 no. 1

Intrinsically triple-linked graphs in $\mathbb{R}P^3$	1
JARED FEDERMAN, JOEL FOISY, KRISTIN MCNAMARA AND EMILY STARK	
A modified wavelet method for identifying transient features in time signals with applications to bean beetle maturation	21
DAVID MCMORRIS, PAUL PEARSON AND BRIAN YURK	
A generalization of the matrix transpose map and its relationship to the twist of the polynomial ring by an automorphism	43
ANDREW MCGINNIS AND MICHAELA VANCLIFF	
Mixing times for the rook's walk via path coupling	51
CAM MCLEMAN, PETER T. OTTO, JOHN RAHMANI AND MATTHEW SUTTER	
The lifting of graphs to 3-uniform hypergraphs and some applications to hypergraph Ramsey theory	65
MARK BUDDEN, JOSH HILLER, JOSHUA LAMBERT AND CHRIS SANFORD	
The multiplicity of solutions for a system of second-order differential equations	77
OLIVIA BENNETT, DANIEL BRUMLEY, BRITNEY HOPKINS, KRISTI KARBER AND THOMAS MILLIGAN	
Factorization of Temperley–Lieb diagrams	89
DANA C. ERNST, MICHAEL G. HASTINGS AND SARAH K. SALMON	
Prime labelings of generalized Petersen graphs	109
STEVEN A. SCHLUCHTER, JUSTIN Z. SCHROEDER, KATHRYN COKUS, RYAN ELLINGSON, HAYLEY HARRIS, ETHAN RARITY AND THOMAS WILSON	
A generalization of Zeckendorf's theorem via circumscribed m-gons	125
ROBERT DORWARD, PARI L. FORD, EVA FOURAKIS, PAMELA E. HARRIS, STEVEN J. MILLER, EYVINDUR PALSSON AND HANNAH PAUGH	
Loewner deformations driven by the Weierstrass function	151
JOAN LIND AND JESSICA ROBINS	
Rank disequilibrium in multiple-criteria evaluation schemes	165
JONATHAN K. HODGE, FAYE SPRAGUE-WILLIAMS AND JAMIE WOELK	