COMPACT AND FINITE-DIMENSIONAL LOCALLY COMPACT VECTOR SPACES[1]

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What can be said about locally compact and, in particular, compact vector spaces over arbitrary topological fields? Using the theory of characters, we shall classify all compact vector spaces in §1. In §2 we shall obtain an analogue for locally compact vector spaces over discrete fields of the following classical theorem: If E is a locally compact vector space over a complete field K whose topology is given by a proper absolute value, then E is finite-dimensional, $(\lambda_1, \cdots, \lambda_n) \to \sum_{k=1}^{n} \lambda_k e_k$ is a topological isomorphism from K^n onto E, where $(e_1, \cdots, e_n)$ is any basis of E, and in particular E is metrizable [5, pp. 27, 29]. More precisely, we shall prove that if E is a finite-dimensional, totally disconnected or connected, locally compact, metrizable vector space over a discrete field, then E is the topological direct sum of a discrete subspace and a finite number of subspaces each of which admits a continuous scalar multiplication over an indiscrete locally compact field that commutes with the given scalar multiplication. An analogous theorem for algebras is also obtained. Our argument also yields a new elementary proof of the known characterization of totally disconnected locally compact fields.

Since a topological vector space over a topological field remains a topological vector space if the scalar field is retopologized with the discrete topology, a study of locally compact vector spaces over discrete fields is essentially the same as a study of arbitrary locally compact vector spaces that does not take into consideration the topology of the scalar field.

If K is a discrete field, a K-vector space E equipped with a topology is a topological K-vector space if and only if E is a topological group under addition and, for each $\lambda \in K$, $x \to \lambda x$ is continuous at zero on E. Indeed, it is immediate that if K is discrete, then $(\lambda, x) \to \lambda x$ is continuous at $(0, 0)$ and $\lambda \to \lambda x$ is continuous at zero. Thus a topological vector space over a discrete field may be viewed simply as an abelian topological group equipped with a field of topological automorphisms that contains the identity automorphism.

1. Compact vector spaces

If E is a commutative locally compact group, we shall denote by $E^\wedge$ the character group of E (we shall regard the circle group as $\mathbf{R}/\mathbf{Z}$ and thus use additive notation). The author is indebted to Richard A. Scoville for suggesting Theorem 1.

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THEOREM 1. *Let E be a locally compact (left) module over a locally compact ring K. For each $u \in E^\wedge$ and each $\lambda \in K$, let $u.\lambda$ be the continuous character of E defined by*

$$(u.\lambda)(x) = u(\lambda x)$$

for all $x \in E$. With scalar multiplication so defined, $E^\wedge$ is a locally compact right K-module. Moreover, if K has an identity element and if E is unitary, so is $E^\wedge$.

Proof. The verification that $E^\wedge$ is a right K-module is immediate. For each compact subset C of E and each neighborhood U of zero in $\mathbf{R}/\mathbf{Z}$, let $W(C, U) = \{u \in E^\wedge : u(C) \subseteq U\}$. To show that $E^\wedge$ is a topological right K-module, let C be a compact subset of E, U a neighborhood of zero in $\mathbf{R}/\mathbf{Z}$. If S is any compact neighborhood of zero in K, then SC is a compact subset of E, and $u.\lambda \in W(C, U)$ for all $\lambda \in S$, $u \in W(SC, U)$. Thus $(\lambda, u) \rightarrow u.\lambda$ is continuous at $(0, 0)$. Given $u \in E^\wedge$, $u^{-1}(U)$ is a neighborhood of zero in E, and hence there is a compact neighborhood S of zero in K such that $SC \subseteq u^{-1}(U)$ since C is compact; clearly $u.\lambda \in W(C, U)$ if $\lambda \in S$, so $\lambda \rightarrow u.\lambda$ is continuous at zero. Given $\lambda \in K$, λC is a compact subset of E, and $u.\lambda \in W(C, U)$ for each $u \in W(\lambda C, U)$; thus $u \rightarrow u.\lambda$ is continuous at zero.

THEOREM 2. *Let E, F, and G be locally compact modules over a locally compact ring. K. If $f : E \rightarrow F$ is a continuous linear transformation, then its adjoint $f^\wedge : F^\wedge \rightarrow E^\wedge$ (defined by $f^\wedge(u) = u \circ f$) is also a continuous linear transformation. If, in addition, $g : F \rightarrow G$ is a continuous linear transformation, then $(g \circ f)^\wedge = f^\wedge \circ g^\wedge$. In particular, if f is a topological isomorphism from E onto F, then $f^\wedge$ is a topological isomorphism from $F^\wedge$ onto $E^\wedge$.*

The proof is immediate.

THEOREM 3. *If E is a locally compact module over a locally compact ring K, then the canonical evaluation mapping is a topological isomorphism from the locally compact K-module E onto the locally compact K-module $E^{\wedge\wedge}$.*

The proof is immediate, given that the evaluation mapping is a topological isomorphism from the locally compact group E onto the locally compact group $E^{\wedge\wedge}$ [7, (24.8), p. 378].

THEOREM 4. *If F is a submodule of a locally compact module E over a locally compact ring K, then the annihilator of F in $E^\wedge$ is a closed submodule of $E^\wedge$.*

The proof is immediate.

With obvious changes, similar statements hold for locally compact right modules over locally compact rings.

The following theorem generalizes the well-known fact that the only compact division rings are the finite fields, since any topological division ring may be regarded as a topological vector space over itself.

THEOREM 5. *If E is a nonzero compact vector space over a topological division ring K, then the topology of K is the discrete topology.*

Proof. Let b be a nonzero vector in E, and let U be a neighborhood of zero that does not contain b. We shall show that $SE \subseteq U$ for some neighborhood S of zero in K. Let V be a neighborhood of zero such that $V + V \subseteq U$. Let S' and W be neighborhoods of zero in K and E respectively such that $W \subseteq V$ and $S'W \subseteq V$. By compactness there exist $a_1, \cdots, a_n \in E$ such that $E = \bigcup_{i=1}^{n} (a_i + W)$. Let S be a neighborhood of zero in K such that $S \subseteq S'$ and $Sa_i \subseteq W$ for all $i \in [1, n]$. Clearly $SE \subseteq W + SW \subseteq V + V \subseteq U$. If λ were a nonzero scalar in S, then $E = \lambda E \subseteq U$, in contradiction to the fact that $b \notin U$. Therefore $S = (0)$, and consequently K is discrete.

Let K be a division ring. Then K is a right vector space over itself. We shall denote by $K^\wedge$ the character left K-vector space of the discrete right K-vector space K. Thus $K^\wedge$ is a compact vector space over the discrete division ring K. For each cardinal number $\mathfrak{m}$, we shall denote by $K^{\wedge\mathfrak{m}}$ the compact K-vector space that is the cartesian product of $\mathfrak{m}$ copies of $K^\wedge$.

THEOREM 6. *Let K be a discrete division ring. If E is a compact K-vector space, then there is a unique cardinal number $\mathfrak{m}$ such that E is topologically isomorphic to the compact K-vector space $K^{\wedge\mathfrak{m}}$.*

Proof. The right K-vector space $E^\wedge$ is discrete, and by a familiar theorem of linear algebra there is a unique cardinal number $\mathfrak{m}$ such that $E^\wedge$ is isomorphic to the K-vector space $K^{(\mathfrak{m})}$, the direct sum of $\mathfrak{m}$ copies of K. By Theorems 2, 3, and [7, (23.22), p. 364], E is topologically isomorphic to $K^{\wedge\mathfrak{m}}$. If the compact K-vector spaces $K^{\wedge\mathfrak{m}}$ and $K^{\wedge\mathfrak{n}}$ are topologically isomorphic, then so are $K^{(\mathfrak{m})}$ and $K^{(\mathfrak{n})}$ by Theorem 3 and [7, (23.21), p. 364], whence $\mathfrak{m} = \mathfrak{n}$ by the invariance of the cardinality of a basis.

THEOREM 7. *If K is a discrete division ring, every one-dimensional subspace of $K^\wedge$ is dense.*

Proof. The assertion follows from Theorem 4 and [7, (24.10), p. 380], since $K^{\wedge\wedge}$ is isomorphic to the one-dimensional K-vector space K by Theorem 3.

THEOREM 8. *If K is a discrete division ring, then there is a nonzero compact metrizable K-vector space if and only if K is countable.*

Proof. Necessity: By [7, (24.15), p. 382], if $K^{\wedge\mathfrak{m}}$ is metrizable, then $K^{(\mathfrak{m})}$ is countable, whence K is countable, provided $\mathfrak{m} > 0$. Sufficiency: If K is countable, clearly $K^\wedge$ has a denumerable fundamental system of neighborhoods of zero.

THEOREM 9. *If E is a nonzero compact vector space over a discrete division ring K, then E is connected if and only if the characteristic of K is zero. If E is a locally compact vector space over a topological division ring of prime characteristic, then E is totally disconnected.*

Proof. The first assertion follows from [7, (24.19), p. 383]. If E is a locally compact vector space over a division ring of prime characteristic p, then each character of E takes E into the pth roots of unity, a discrete set, and consequently the connected component of E is (0).

2. Finite-dimensional locally compact metrizable vector spaces

The suspicion that locally compact vector spaces over indiscrete (i.e., not discrete) locally compact fields play a major role in the general theory of finite-dimensional locally compact vector spaces is suggested by the following theorem.

THEOREM 10. *If E is a one-dimensional indiscrete locally compact metrizable vector space over a discrete field K, then there is an indiscrete locally compact topology $\mathfrak{J}$ on K making K a topological field and E a topological vector space over K; $\mathfrak{J}$.*

Proof. Let b be a nonzero vector of E, and let $\mathfrak{J}$ be the topology on K making $\lambda \rightarrow \lambda b$ a homeomorphism from K onto E. Then $\mathfrak{J}$ is an indiscrete, locally compact, metrizable topology. Since $(x, y) \rightarrow x + y$ and, for each $\alpha \in K$, $x \rightarrow \alpha x$ are continuous on $E \times E$ and E respectively, it is immediate that $(\lambda, \mu) \rightarrow \lambda + \mu$ and, for each $\alpha \in K$, $\lambda \rightarrow \alpha\lambda$ are continuous on $K \times K$ and K respectively. Also as the topological group K; $\mathfrak{J}$ is locally compact, it is complete. It follows easily from a lemma of Montgomery [10, p. 880; 4, Exercise 22a, p. 83] that $(\lambda, \mu) \rightarrow \lambda\mu$ is continuous on $K \times K$. Consequently by a theorem of Otobe [11, Theorem 3], generalized by Kaplansky [9, Theorems 7–9], K; $\mathfrak{J}$ is a topological field. Clearly E is a topological vector space over the topological field K; $\mathfrak{J}$.

Our principal result concerning totally disconnected, finite-dimensional, metrizable, locally compact vector spaces and algebras is the following theorem, whose proof does not invoke known structure theorems for locally compact fields (as is well known, by Theorem 9, if the characteristic of the scalar field is a prime, the hypothesis of total disconnectedness is redundant).

THEOREM 11. *Let E be a totally disconnected, finite-dimensional, locally compact, metrizable vector space (algebra) over a discrete field K, and let*

$$L = \{x \in E : \text{either } x = 0 \text{ or } Kx \text{ is indiscrete}\}.$$

Then L is an open subspace (open ideal) of E, and L is the topological direct sum of subspaces (ideals of E) $E_1, \cdots, E_n$, where for each $i \in [1, n]$, the locally compact group (ring) E_i admits the structure of finite-dimensional topological vector space (algebra) over an indiscrete locally compact field F_i under a scalar multiplication satisfying $\alpha.(\mu x) = \mu(\alpha.x)$ (and also $\alpha.(xy) = (\alpha.x)y$, $\alpha.(yx) = y(\alpha.x)$) for all $\alpha \in F_i$, $\mu \in K$, $x \in E_i$ (and $y \in E$). Moreover, for each $i \in [1, n]$, every closed subspace of the K-vector space E_i is a subspace of the

F_i-vector space E_i, and $F_i \cap K$ is a dense subfield of F_i. If N is any algebraic supplement of L, then N is discrete, and E is the topological direct sum of $E_1, \cdots, E_n, N$. Finally

(a) *if the characteristic of K is a prime, then for each $i \in [1, n]$ there is a prime polynomial h over P, the prime subfield of K, such that F_i is topologically isomorphic to the completion of $P(X)$, the field of rational functions over P, equipped with the h-adic valuation;*

(b) *if the characteristic of K is zero, then for each $i \in [1, n]$ there is a prime number p such that F_i is topologically isomorphic to the field of p-adic numbers.*

Proof. We shall prove the theorem under the assumption that the characteristic of K is a prime, and then indicate the slight changes needed for the case where the characteristic of K is zero.

The compact open subgroups of E form a fundamental system of neighborhoods of zero [3, Corollary 1, p. 58]. If K is countable, then so is E, and therefore E is discrete since a locally compact space is a Baire space [4, Theorem 1, p. 76]. Consequently, we may assume that K is uncountable.

We shall first show that there exist a compact open subgroup V and a scalar λ transcendental over P such that $\lambda V = V$. Let $(W_n)_{n \geq 1}$ be a decreasing fundamental system of neighborhoods of zero consisting of compact open subgroups. As P is finite, the algebraic closure of P in K is countable. As K is uncountable, for some $n \geq 1$ the set S_1 of all scalars λ that satisfy $\lambda^{-1}W_1 \supseteq W_n$ is uncountable, since $\lambda^{-1}W_1$ is a neighborhood of zero for each nonzero scalar λ. Similarly, for some $r \geq n$, the set S_2 of all $\lambda \in S_1$ such that $\lambda W_n \supseteq W_r$ is uncountable. Therefore $W_r \subseteq \lambda W_n \subseteq W_1$ for all $\lambda \in S_2$. As the abelian group W_1/W_r is compact and discrete, it is finite, and therefore there is a subgroup V containing W_r and contained in W_1 such that the set S_3 of all $\lambda \in S_2$ such that $\lambda W_n = V$ is uncountable. Let $\lambda_0 \in S_3$; then $S_3 \lambda_0^{-1}$ is uncountable and therefore contains a scalar λ transcendental over P. Let $\lambda = \lambda_1\lambda_0^{-1}$, where $\lambda_1 \in S_3$. Then $\lambda V = \lambda_1 \lambda_0^{-1} V = \lambda_1 W_n = V$, which is a compact open subgroup since it contains W_r and is contained in W_1.

Next we shall show that there is a fundamental decreasing system $(V_k)_{k \geq 0}$ of neighborhoods of zero consisting of compact open subgroups such that $V_0 = V$ and $\lambda V_k = V_k$ for all $k \geq 0$. Let M be the intersection of all the sets ζV where ζ is a nonzero scalar. Clearly M is a compact subspace of E. As K is uncountable and E metrizable, $M = (0)$ by Theorem 8. Consequently, for every neighborhood U of zero there exist nonzero scalars $\zeta_1, \cdots, \zeta_n$ such that $U \supseteq \zeta_1 V \cap \cdots \cap \zeta_n V$ [2, Theorem 1, p. 97]. Thus as E is metrizable, there is a sequence $(\zeta_k)_{k \geq 1}$ of nonzero scalars such that if $V_k = V \cap \zeta_1 V \cap \cdots \cap \zeta_k V$ for each $k \geq 1$, then $(V_k)_{k \geq 0}$ is a fundamental system of neighborhoods of zero. As $\lambda\zeta V = \zeta\lambda V = \zeta V$ for every scalar ζ, clearly $\lambda V_k = V_k$ for all $k \geq 0$.

Let E_0 be the subspace of E generated by V (shortly we shall see that

$E_0 = L$). Then E_0 is an open subspace of E. We shall show that for each vector $a \in E_0$ there is a sequence $(\nu_k)_{k\geq 1}$ of nonzero elements of $P[\lambda]$, the subring of K generated by P and λ, such that $\nu_k a \to 0$. As E_0 is generated by V, there exist scalars $\rho_1, \cdots, \rho_n$ and elements $a_1, \cdots, a_n$ of V such that $a = \rho_1 a_1 + \cdots + \rho_n a_n$. In the topological K-vector space E_0^n, $\lambda V_k^n = V_k^n$ for all $k \geq 0$, and consequently $\lambda^m V_k^n = V_k^n$ for all $k \geq 0$ and all $m \geq 0$. Thus as V^n is compact, there is a strictly increasing sequence $(m_k)_{k\geq 1}$ of positive integers such that the sequence $(\lambda^{m_k}(a_1, \cdots, a_n))_{k\geq 1}$ converges to some point $(c_1, \cdots, c_n)$. Hence also

$$\lambda^{m_{k+1}}(a_1, \cdots, a_n) \to (c_1, \cdots, c_n),$$

so if $\nu_k = \lambda^{m_{k+1}} - \lambda^{m_k}$ for all $k \geq 1$, then $\nu_k(a_1, \cdots, a_n) \to (0, \cdots, 0)$. Thus $\nu_k a_i \to 0$ for all $i \in [1, n]$, whence $\nu_k a = \nu_k(\rho_1 a_1 + \cdots + \rho_n a_n) \to 0$. As λ is transcendental over P, $\nu_k \neq 0$ for all $k \geq 1$.

In particular, Ka is indiscrete for each nonzero $a \in E_0$, so $E_0 \subseteq L$. If $a \notin E_0$, then $Ka \cap E_0 = (0)$, so Ka is discrete as E_0 is open, and hence $a \notin L$. Therefore $L = E_0$.

Moreover, for each $a \in E_0 = L$ there is a nonzero element ν of $P[\lambda]$ such that $\nu a \in V$. Consequently L is the union of the sets $\nu^{-1}V$ where $\nu \in P[\lambda]$, $\nu \neq 0$. Thus L is σ-compact and metrizable, and hence is separable.

Let H be the set of all prime polynomials in $P[\lambda]$ (as λ is transcendental over P, we may regard $P[\lambda]$ as the ring of polynomials over P). We equip the integral domain $P[\lambda]$ with the topology $\mathfrak{J}$ that is the supremum of all the h-adic topologies on $P[\lambda]$, where $h \in H$. So equipped, $P[\lambda]$ is a metrizable topological ring, and the nonzero ideals of $P[\lambda]$ form a fundamental system of neighborhoods of zero.

We shall show that L is a topological module over $P[\lambda]$; $\mathfrak{J}$. For each $\mu \in P[\lambda]$, $x \to \mu x$ is continuous at zero since $\mu \in K$. For each $k \geq 0$, $\lambda^m V_k = V_k$ for all $m \geq 0$ and $\beta V_k = V_k$ for all nonzero $\beta \in P$; hence $\mu V_k \subseteq V_k$ for all $\mu \in P[\lambda]$. Thus $P[\lambda]V_k = V_k$, and hence $(\mu, x) \to \mu x$ is continuous at $(0, 0)$. Finally, given $a \in L$ and $k \geq 0$, there is a nonzero element ν of $P[\lambda]$ such that $\nu a \in V_k$ as we saw above; $P[\lambda]\nu$ is open in $P[\lambda]$ for $\mathfrak{J}$, and $P[\lambda]\nu a \subseteq P[\lambda]V_k = V_k$; hence $\mu \to \mu a$ is continuous at zero.

For each $h \in H$, let $\mathfrak{o}_h$ be the completion of $P[\lambda]$ for the h-adic valuation. Thus $\mathfrak{o}_h$ is the valuation ring of the completion of the field $P(\lambda)$ for the h-adic valuation. Since the residue field $P[\lambda]/(h)$ is finite, $\mathfrak{o}_h$ is compact. The completion of $P[\lambda]$; $\mathfrak{J}$ may be identified with the compact ring $A = \prod_{h\in H} \mathfrak{o}_h$; indeed, $P[\lambda]^H$ is clearly dense in A; given distinct $h_1, \cdots, h_m \in H$, positive integers $k_1, \cdots, k_m$, and polynomials $\nu_1, \cdots, \nu_m$ in $P[\lambda]$, by the Chinese Remainder Theorem there exists $\mu \in P[\lambda]$ such that $\mu \equiv \nu_i \pmod{h_i^{k_i}}$ for all $i \in [1, m]$; hence $\mu \to (\mu_k)$, where $\mu_k = \mu$ for all $k \in H$, is a topological isomorphism from $P[\lambda]$; $\mathfrak{J}$ onto a dense subring of A. As L is locally compact and hence complete, L admits the structure of unitary topological A-module, and scalar multiplication satisfies $(\mu_k).x = \mu x$ for all $\mu \in P[\lambda]$ (where, as before, $\mu_k = \mu$ for all $k \in H$) [3, pp. 81–82].

If $\alpha \in A$, $\mu \in K$, and $x \in L$, then $\alpha.(\mu x) = \mu(\alpha.x)$. Indeed, there is a sequence (α_n) in $P[\lambda]$ such that $\alpha_n \to \alpha$. Hence $\alpha_n(\mu x) \to \alpha.(\mu x)$ and $\alpha_n x \to \alpha.x$, whence also $\mu(\alpha_n x) \to \mu(\alpha.x)$. As $\mu(\alpha_n x) = \alpha_n(\mu x)$ for all $n \geq 0$, therefore, $\alpha.(\mu x) = \mu(\alpha.x)$.

Let $h \in H$. The canonical injection from $\mathfrak{o}_h$ into A is a topological isomorphism from $\mathfrak{o}_h$ onto an ideal of A; we identify $\mathfrak{o}_h$ with this ideal by means of this topological isomorphism; thus L is a (not necessarily unitary) topological module over $\mathfrak{o}_h$. Let 1_h be the idempotent of A that is the identity element of $\mathfrak{o}_h$, and let $E_h = 1_h.L$. By the preceding paragraph, E_h is a subspace of the K-vector space L, and clearly E_h is closed. Also, E_h is a unitary module over $\mathfrak{o}_h$. Moreover, $\mu.x = \mu x$ for all $\mu \in P[\lambda]$, $x \in E_h$. To see this, let $\mu_k = \mu$ for all $k \in H$, $\nu_k = 0$ for all $k \in H - \{h\}$, $\nu_h = \mu$. By definition, $\mu.x = (\nu_k).x$ and $(\mu_k).x = \mu x$. As $x \in E_h$, $x = 1_h.x$ and therefore $\mu x = (\mu_k).x = [(\mu_k)1_h].x = (\nu_k).x = \mu.x$.

We shall next show that E_h is a divisible torsion-free module over $\mathfrak{o}_h$. Let x be a nonzero element of E_h, and let $\mathfrak{a}_x$ be the annihilator of x in $\mathfrak{o}_h$. If the ideal $\mathfrak{a}_x$ were not the zero ideal, then $\mathfrak{a}_x$ would be open since every nonzero ideal of $\mathfrak{o}_h$ is open, and consequently $\mathfrak{a}_x$ would contain a nonzero element μ of $P[\lambda]$ since $P[\lambda]$ is dense in $\mathfrak{o}_h$ and not discrete for the h-adic topology; thus $\mu x = \mu.x = 0$, which is impossible. Consequently, $\mathfrak{a}_x = (0)$. Let α be a nonzero element of $\mathfrak{o}_h$. We have just seen that $\alpha^\wedge : x \to \alpha.x$, $x \in E_h$, is injective, and by an earlier paragraph, $\alpha^\wedge$ is a linear operator on the K-vector space E_h. Hence as E is finite-dimensional, $\alpha^\wedge$ is surjective. Therefore E_h is a divisible, torsion-free $\mathfrak{o}_h$-module.

Let F_h be the completion of the field $P(\lambda)$ for the h-adic valuation. Thus $\mathfrak{o}_h$ is the valuation ring of F_h, and in particular F_h is the quotient field of $\mathfrak{o}_h$. As E_h is a divisible, torsion-free module over $\mathfrak{o}_h$, scalar multiplication may be extended to $F_h \times E_h$ so that E_h is a vector space over F_h. We shall show that E_h is a topological vector space over F_h. As $\mathfrak{o}_h$ is open in F_h, $(\alpha, x) \to \alpha.x$ on $F_h \times E_h$ is continuous at $(0, 0)$ and, for each $x \in E_h$, $\alpha \to \alpha.x$ on F_h is continuous at zero. Earlier, we saw that L and hence also E_h are locally compact, metrizable, and separable. For each nonzero $\alpha \in \mathfrak{o}_h$, $\alpha^\wedge$ is a continuous automorphism of E_h by the preceding paragraph, and hence $\alpha^\wedge$ is a topological automorphism of E_h by [4, Exercise 18, p. 82]. Thus as each element of F_h either belongs to $\mathfrak{o}_h$ or is the inverse of an element of $\mathfrak{o}_h$, $x \to \beta.x$ is continuous on E_h for all $\beta \in F_h$. Therefore E_h is a topological vector space over the indiscrete locally compact field F_h. By [5, Theorem 3, p. 29], E_h is finite-dimensional over F_h. It is easy to verify that $\beta.(\mu x) = \mu(\beta.x)$ for all $\beta \in F_h$, $\mu \in K$, $x \in E$.

Let D be a closed subspace of the K-vector space E_h, and let $x \in D$, $\alpha \in F_h$. There is a sequence (α_n) in $P(\lambda)$ such that $\alpha_n \to \alpha$, whence $\alpha_n x \to \alpha.x$.

As $\alpha_n x \in D$ for all $n \geq 1$, $\alpha.x \in D$. Thus D is a subspace of the F_h-vector space E_h.

The subspace of L generated by $(E_h)_{h \in H}$ is clearly the direct sum of $(E_h)_{h \in H}$ since $1_h 1_k = 0$ if $h \neq k$. Consequently as E is finite-dimensional, $E_h = (0)$ for all but finitely many $h \in H$. Let $h_1, \cdots, h_n$ be all the elements h of H such that $E_h \neq (0)$, and for each $i \in [1, n]$ let $E_i = E_{h_i}$ and $1_i = 1_{h_i}$, and let J be the complement of $\{h_1, \cdots, h_n\}$ in H. For each $h \in J$, $\mathfrak{o}_h.L = \mathfrak{o}_h.(1_h.L) = (0)$, so $(\bigoplus_{h \in J} \mathfrak{o}_h).L = (0)$; as the annihilator of L in A is closed, it therefore contains $\prod_{h \in J} \mathfrak{o}_h$, canonically identified with an ideal of A. Hence for each $x \in L$,

$$x = 1.x = (\textstyle\sum_{i=1}^{n} 1_i).x \in \sum_{i=1}^{n} E_i .$$

Thus L is the direct sum of $E_1, \cdots, E_n$. As the projection of L on E_i along $\sum_{j \neq i} E_j$ is clearly the continuous mapping $x \to 1_i.x$, L is the topological direct sum of $E_1, \cdots, E_n$ [5, Proposition 10, p. 15].

Let N be an algebraic supplement of L in E. As L is open and as $L \cap N = (0)$, N is discrete and the projection of E on N along L is continuous. Therefore E is the topological direct sum of L and N and hence also of $E_1, \cdots, E_n, N$.

We turn next to the additional statements for the case where E is a topological algebra over K. If Kx is indiscrete and if $xz \neq 0$, then $\mu x \to \mu xz$ is a continuous bijection from Kx onto Kxz, so Kxz is indiscrete; similarly, either $zx = 0$ or Kzx is indiscrete. Consequently, L is an ideal of E.

Since $P[\lambda]$ is dense in A, a continuity argument establishes that $\alpha.xy = (\alpha.x)y$ and $\alpha.yx = y(\alpha.x)$ for all $\alpha \in A$, $x \in L$, and $y \in E$. Consequently for each $h \in H$, E_h is an ideal of E, for if $x \in E_h$ and $y \in E$, then

$$1_h.xy = (1_h.x)y = xy, \qquad 1_h.yx = y(1_h.x) = yx,$$

whence $xy, yx \in E_h$.

Let $h \in H$. By the preceding, $\alpha.(xy) = (\alpha.x)y$ for all $\alpha \in \mathfrak{o}_h$, $x, y \in E_h$. In particular,

$$\alpha.((\alpha^{-1}.x)y) = [\alpha.(\alpha^{-1}.x)]y = xy,$$

so $(\alpha^{-1}.x)y = \alpha^{-1}.xy$ for all nonzero $\alpha \in \mathfrak{o}_h$, $x, y \in E_h$. Thus E_h is a topological algebra over F_h. A continuity argument now establishes that $\alpha.xy = (\alpha.x)y$ and $\alpha.yx = y(\alpha.x)$ for all $\alpha \in F_h$, $x \in E_h$, $y \in E$, since $P(\lambda)$ is dense in F_h. This completes the proof for the prime characteristic case.

Assume now that the characteristic of K is zero. We may also assume, of course, that K is an extension of the field of rational numbers. Let V be a compact open subgroup of E. As before, there exists a sequence $(\zeta_k)_{k \geq 1}$ of nonzero scalars such that if $V_k = V \cap \zeta_1 V \cap \cdots \cap \zeta_k V$, then $(V_k)_{k \geq 0}$ is a fundamental system of neighborhoods of zero. Let E_0 be the subspace of E generated by V. For each vector $a \in E_0$ there is a sequence $(\nu_k)_{k \geq 1}$ of strictly positive integers such that $\nu_k a \to 0$. The proof is entirely similar to the proof of the corresponding statement for the prime characteristic case; we need only replace the sequence $(\lambda^{m_k})_{k\geq 1}$ by a strictly increasing sequence $(m_k)_{k\geq 1}$ of positive integers. As before, $E_0 = L$, L is separable, and L is a topological module over the ring of integers $\mathbf{Z}$ (a subring of K), equipped with the topology $\mathfrak{J}$ for which a fundamental system of neighborhoods of zero consists of all nonzero ideals (thus $\mathfrak{J}$ is the supremum of the family of p-adic topologies on $\mathbf{Z}$ where $p \in H$, the set of all prime numbers). The remainder of the proof is exactly as before.

The existence of compact metrizable vector spaces over countable fields (Theorem 8) shows that the hypothesis of finite-dimensionality cannot be simply omitted. However, it would be interesting to know if the hypothesis of finite-dimensionality could be replaced by the hypothesis that K be uncountable.

As stated, Theorem 11 does not yield Theorem 10 for the totally disconnected case. It is easy, however, to derive the totally disconnected case of Theorem 10 from Theorem 11 and, at the same time, obtain the familiar characterization of totally disconnected locally compact fields. Before doing so, we recall that every finite-dimensional extension K of a locally compact field K_0 whose topology is given by a proper absolute value (actually, every locally compact field admits an absolute value compatible with its topology [8, Theorem 8]) has a unique topology that makes it a topological field and induces on K_0 its given topology. Of the many ways of proving this, here is one based on elementary theorems of Banach algebras: K admits a unique topology making it a locally compact vector space over K_0 [5, Theorem 2, p. 27], and this topology may be defined by a norm; as any multilinear transformation defined on finite-dimensional Hausdorff topological vector spaces over K_0 is continuous (cf. [5, Corollary 2, p. 28]), multiplication on $K \times K$ is continuous. Consequently, the norm of K may be chosen to make K a Banach algebra over K_0, and a standard theorem implies that $x \to x^{-1}$ is continuous on the set of nonzero elements of K.

THEOREM 12. *Let E be a one-dimensional, disconnected, indiscrete, locally compact, metrizable vector space over a discrete field K. There is an algebraic isomorphism Φ from F, where F is the p-adic number field for some prime p if the characteristic of K is zero, and where F is the field $\mathbf{Z}_p((X))$ of formal power series over the prime field $\mathbf{Z}_p$ of p elements, equipped with the X-adic topology, if the characteristic of K is a prime p, onto a subfield K_0 of K such that $[K:K_0] < \infty$. Moreover, E is a topological vector space over K, equipped with the unique locally compact topology compatible with its field structure that induces on K_0 the topology making Φ a topological isomorphism.*

Proof. We first note that if h is a prime polynomial over $\mathbf{Z}_p$ and if F_h is the completion of $\mathbf{Z}_p(X)$ for the h-adic valuation, then there is a topological isomorphism from $\mathbf{Z}_p((X))$ onto a subfield L of F_h such that $[F_h:L] < \infty$.

Indeed, the substitution isomorphism S defined by h from $\mathbf{Z}_p(X)$ onto $\mathbf{Z}_p(h)$ is clearly a topological isomorphism from $\mathbf{Z}_p(X)$, equipped with the X-adic valuation, onto the subfield $\mathbf{Z}_p(h)$ of $\mathbf{Z}_p(X)$, equipped with the h-adic valuation; hence S may be extended to a topological isomorphism from $\mathbf{Z}_p((X))$ onto the closure L of $\mathbf{Z}_p(h)$ in F_h. Thus by [5, Theorem 3, p. 29], F_h is topologically isomorphic to a finite-dimensional extension of $Z_p((X))$.

As E is one-dimensional, E is totally disconnected since the connected component of zero in a topological vector space is a subspace. By the preceding paragraph and Theorem 11, E admits the structure of finite-dimensional topological vector space over F, described in the statement of the theorem, under a scalar multiplication satisfying $\alpha.(\mu x) = \mu(\alpha.x)$ for all $\alpha \in F$, $\mu \in K$, $\mu \in K$, $x \in E$. Let e be a nonzero vector of E, and for each $\alpha \in F$ let λ_α be the unique scalar of K such that $\alpha.e = \lambda_\alpha e$. It is immediate that $\Phi : \alpha \rightarrow \lambda_\alpha$ is an isomorphism from F onto a subfield K_0 of K and that $\alpha.x = \lambda_\alpha x$ for all $\alpha \in F$, $x \in E$. Clearly $[K:K_0] = [E:F] < \infty$. Thus E is a finite-dimensional topological vector space over K_0, equipped with the topology making Φ a topological isomorphism. We equip K with the unique locally compact topology compatible with its field structure that induces on K_0 the topology just assigned. As noted earlier, any multilinear transformation defined on finite-dimensional Hausdorff topological vector spaces over K_0 is continuous, and therefore E is a topological vector space over K.

The only theorems used in the proofs of Theorems 11 and 12 that might not be considered entirely standard are Theorem 8 and an open mapping theorem concerning σ-compact, metrizable, locally compact groups. The proof of the latter theorem is an application of the Baire Category Theorem, however, and in the one-dimensional case the appeal to Theorem 8 may be eliminated. By an elementary argument, every locally compact field is metrizable [12, Lemma 1, p. 171]. Therefore we obtain as a corollary of Theorem 12 a new, elementary proof of the classical structure theorems for totally disconnected locally compact fields [6, 12, Theorem 22, p. 170]: an indiscrete locally compact field of prime characteristic p is topologically isomorphic to a finite extension of $\mathbf{Z}_p((X))$, equipped with the X-adic topology; an indiscrete, totally disconnected, locally compact field of characteristic zero is topologically isomorphic to a finite extension of the p-adic number field for some prime p.

An analogue of Theorem 11 for the connected case exists, but the proof given here, though easy, is not elementary.

THEOREM 13. *If E is a nonzero, connected, locally compact, metrizable vector space (algebra) over a discrete field K that is uncountable, then the characteristic of K is zero, and E admits the structure of finite-dimensional topological vector space (algebra) over the topological field* $\mathbf{R}$ *of real numbers under a scalar multiplication $\alpha.(\mu x) = \mu(\alpha.x)$ for all $\alpha \in \mathbf{R}$, $\mu \in K$, $x \in E$.*

Proof. By the Pontryagin-van Kampen theorem [7, (9.14), p. 95], E, regarded as an additive topological group, is the topological direct sum of a topological group F that is topologically isomorphic to $\mathbf{R}^n$ and a compact group C. If the closed topological group $(\mathbf{Z}c)^-$ generated by an element c of E is compact, then $c \in C$. Indeed, if u is the (continuous) projection of E on F along C, then $u((\mathbf{Z}c)^-)$ is a compact subgroup of F and hence is the zero subgroup, since no nonzero subgroup of $\mathbf{R}^n$ is compact, so $(\mathbf{Z}c)^- \subseteq C$. Consequently, if $c \in C$, then $\lambda c \in C$ for all $\lambda \in K$, since $(\mathbf{Z}(\lambda c))^- = (\lambda \mathbf{Z}c)^- = \lambda(\mathbf{Z}c)^-$, a compact group. Thus C is a compact vector space over K. By Theorem 8, $C = (0)$, so $E = F$. As K has characteristic zero (Theorem 9), we may assume that $K \cap \mathbf{R}$ is dense in $\mathbf{R}$; a continuity argument then establishes the desired identities.

Thus Pontryagin's structure theorem for connected locally compact fields may be deduced from the Pontryagin-van Kampen theorem on connected abelian locally compact groups, Theorem 8, and Frobenius's theorem on finite-dimensional division algebras over $\mathbf{R}$.

The author does not know if the connected component of zero of a finite-dimensional, locally compact, metrizable vector space necessarily admits a topological supplement.

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