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ASYMPTOTIC GROWTH OF POWERS OF IDEALS

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ABSTRACT. Let A be a locally analytically unramified local ring and $J_1, \ldots, J_k, I$ ideals such that $J_i \subseteq \sqrt{I}$ for all i, the ideal I is not nilpotent, and $\bigcap_k I^k = (0)$. Let $C = C(J_1, \ldots, J_k; I) \subseteq \mathbb{R}^{k+1}$ be the cone generated by $\{(m_1, \ldots, m_k, n) \in \mathbb{N}^{k+1} \mid J_1^{m_1} \ldots J_k^{m_k} \subseteq I^n\}$. We prove that the topological closure of C is a rational polyhedral cone. This generalizes results by Samuel, Nagata, and Rees.

Introduction

In this note we continue the study of the asymptotic properties of powers of ideals initiated by Samuel in [8]. Let A be a commutative noetherian ring with identity and I, J ideals in A with $J \subseteq \sqrt{I}$. Also, assume that the ideal I is not nilpotent and $\bigcap_k I^k = (0)$. Then for each positive integer m one can define $v_I(J, m)$ to be the largest integer n such that $J^m \subseteq I^n$. Similarly, $w_J(I, n)$ is defined to be the smallest integer m such that $J^m \subseteq I^n$. Under the above assumptions, Samuel proved that the sequences $\{v_I(J, m)/m\}_m$ and $\{w_J(I, n)/n\}_n$ have limits $l_I(J)$ and $L_J(I)$, respectively, and $l_I(J)L_J(I) = 1$ [8, Theorem 1]. It is also observed that these limits are actually the supremum and infimum of the respective sequences. One of the questions raised in Samuel's paper is whether $l_I(J)$ is always rational. This has been positively answered by Nagata [4] and Rees [5]. The approach used by Rees is described in the next section of this paper.

We consider the following generalization of the problem described above. Let $J_1, \ldots, J_k, I$ be ideals in a locally analytically unramified ring A such that $J_i \subseteq \sqrt{I}$ for all i, I is not nilpotent, and $\bigcap_k I^k = (0)$, and let $C = C(J_1, \ldots, J_k; I) \subseteq \mathbb{R}^{k+1}$ be the cone generated by $\{(m_1, \ldots, m_k, n) \in \mathbb{N}^{k+1} \mid J_1^{m_1} \ldots J_k^{m_k} \subseteq I^n\}$. We prove that the topological closure of C is a rational

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polyhedral cone; i.e., a polyhedral cone bounded by hyperplanes whose equations have rational coefficients. Note that the case $k = 1$ follows from the results proved by Samuel, Nagata, and Rees; the cone C is the intersection of the half-planes given by $n \geq 0$ and $n \leq l_I(J)m_1$. In Section 3 we look at the periodicity of the rate of change of the sequence $\{v_I(J,m)\}_m$, more precisely, the periodicity of the sequence $\{v_I(J,m+1) - v_I(J,m)\}_m$. The last part of the paper describes a method of computing the limits studied by Samuel in the case of monomial ideals.

1. The Rees valuations of an ideal

In this section we give a brief description of the Rees valuations associated to an ideal.

For a noetherian ring A that is not necessarily an integral domain, a discrete valuation on A is defined as follows.

Definition 1.1. Let A be a noetherian ring. We say that $v : A \to \mathbb{Z} \cup \{\infty\}$ is a discrete valuation on A if $\{x \in A \mid v(x) = \infty\}$ is a prime ideal P, v factors through $A \to A/P \to \mathbb{Z} \cup \{\infty\}$, and the induced function on A/P is a rank one discrete valuation on A/P. If I is an ideal in A, then we denote $v(I) := \min\{v(x) \mid x \in I\}$.

If R is a noetherian ring, we denote by $\overline{R}$ the integral closure of R in its total quotient ring $Q(R)$.

Definition 1.2. Let I be an ideal in a noetherian ring A. An element $x \in A$ is said to be integral over I if x satisfies an equation $x^n + a_1x^{n-1} + \cdots + a_n = 0$ with $a_i \in I^i$. The set of all elements in A that are integral over I is an ideal $\overline{I}$, and the ideal I is called integrally closed if $I = \overline{I}$. If all the powers I^n are integrally closed, then I is said to be normal.

Given an ideal I in a noetherian ring A, for each $x \in A$ let $v_I(x) = \sup\{n \in \mathbb{N} \mid x \in I^n\}$. Rees [5] proved that for each $x \in A$ one can define

$$\overline{v}_I(x) = \lim_{k\to\infty} \frac{v_I(x^k)}{k},$$

and for each integer n one has $\overline{v}_I(x) \geq n$ if and only if $x \in \overline{I^n}$. Moreover, there exist discrete valuations $v_1, \ldots, v_h$ on A in the sense defined above, and positive integers $e_1, \ldots, e_h$ such that, for each $x \in A$,

$$\overline{v}_I(x) = \min\Big\{\frac{v_i(x)}{e_i} \mid i = 1, \ldots, h\Big\}. \tag{1.1}$$

We briefly describe a construction of the Rees valuations $v_1, \ldots, v_h$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$ be the minimal prime ideals $\mathfrak{p}$ in A such that $\mathfrak{p} + I \neq A$, and let $\mathcal{R}_i(I)$ be the Rees ring $(A/\mathfrak{p}_i)[It, t^{-1}]$. Denote by $W_{i1}, \ldots, W_{ih_i}$ the rank one discrete valuation rings obtained by localizing the rings $\overline{\mathcal{R}_i(I)}$ at the minimal primes over $t^{-1}\overline{\mathcal{R}_i(I)}$, let w_{ij} ($i = 1, \dots, g$, $1 \le j \le h_i$) be the corresponding discrete valuations, and let $V_{ij} = W_{ij} \cap Q(A/\mathfrak{p}_i)$ ($i = 1, \dots, g$). Then define $v_{ij}(x) := w_{ij}(x + \mathfrak{p}_i)$ and $e_{ij} := w_{ij}(t^{-1})(= v_{ij}(I))$ for all i, and for simplicity, renumber them as $e_1, \dots, e_h$ and $v_1, \dots, v_h$, respectively.

Rees [5] proved that $v_1, \dots, v_h$ are valuations satisfying (1.1). We refer the reader to the original article [5] for more details on this construction.

Remark 1.3. With the notation established above, for every positive integer n we have

$$\overline{I^n} = \bigcap_{i=1}^{h} I^n V_i \cap R.$$

In particular, we have the following.

Remark 1.4. If K, L are ideals in A, $v_1, \dots, v_h$ are the Rees valuations of L, and $v_i(K) \ge v_i(L)$ for all $i = 1, \dots, h$, then $\overline{K} \subseteq \overline{L}$.

The rationality of $l_I(J)$ can now be obtained as consequence of the results of Rees. Indeed, by [8, Theorem 2], if $J = (a_1, \dots a_s)$, then $l_I(J) = \min\{l_I(a_i) \mid i = 1, \dots s\}$, and for each i we have $l_I(a_i) = \overline{v}_I(a_i)$, which is rational.

Finally, recall the following definition.

Definition 1.5. A local noetherian ring $(A, \mathfrak{m})$ is analytically unramified if its $\mathfrak{m}$-adic completion $\hat{A}$ is reduced.

Rees [6] proved that for every ideal I in an analytically unramified ring there exists an integer k such that for all $n \ge 0$, $\overline{I^{n+k}} \subseteq I^n$.

2. The cone structure

Throughout this section A is a locally analytically unramified ring and I and $\underline{J} = J_1, \dots, J_k$ are ideals in A such that $J_i \subseteq \sqrt{I}$ for all i. Let $C = C(J_1, \dots, J_k; I) \subseteq \mathbb{R}^{k+1}$ denote the cone generated by $\{(m_1, \dots, m_k, n) \in \mathbb{N}^{k+1} \mid J_1^{m_1} \dots J_k^{m_k} \subseteq I^n\}$. Also, for $(m_1, \dots, m_k) \in \mathbb{N}^k$, let $v_I(\underline{J}, m_1, \dots, m_k)$ denote the largest nonnegative integer n such that $J_1^{m_1} \dots J_k^{m_k} \subseteq I^n$.

For each Rees valuation v_j of I, denote $\alpha_{ij} = v_j(J_i)/e_j$ for all i, j, where $e_j = v_j(I)$. Then we consider

$$D_j = \left\{ (m_1, \dots, m_k) \in \mathbb{R}^k_{\ge 0} \;\middle|\; \sum_{s=1}^{k} m_s \alpha_{sj} \le \sum_{s=1}^{k} m_s \alpha_{sl} \text{ for all } l \ne j \right\},$$

and we say that a Rees valuation v_j is relevant if $D_j \ne \{0\}$. After a renumbering, assume that $v_1, v_2, \dots, v_r$ ($r \le h$) are the relevant Rees valuations.

Note that each D_j is an intersection of half-spaces (hence a polyhedral cone), $\bigcup_{j=1}^r D_j = \mathbb{R}^k_{\geq 0}$, and two cones D_i, D_j $(i \neq j)$ either intersect along one common face or have only the origin in common. Let

$$E_j = \left\{ (m_1, \ldots, m_k, n) \in \mathbb{R}^{k+1}_+ \;\middle|\; (m_1, \ldots, m_k) \in D_j \text{ and } n < \sum_{s=1}^k m_s \alpha_{sj} \right\}$$

and

$$\overline{E}_j = \left\{ (m_1, \ldots, m_k, n) \in \mathbb{R}^{k+1}_+ \;\middle|\; (m_1, \ldots, m_k) \in D_j \text{ and } n \leq \sum_{s=1}^k m_s \alpha_{sj} \right\}.$$

THEOREM 2.1. *Let A be a locally analytically unramified ring. Then for each $j = 1, \ldots, r$ we have*

$$E_j \cap \mathbb{Q}^{k+1} \subseteq C \cap (D_j \times \mathbb{R}_{\geq 0}) \subseteq \overline{E}_j.$$

Proof. Let $(m_1, \ldots, m_k, n) \in C \cap (D_j \times \mathbb{R}_{\geq 0})$. Then there exists $t \in \mathbb{R}$ such that $tm_1, \ldots, tm_k$ are positive integers and

$$J_1^{tm_1} \ldots J_k^{tm_k} \subseteq I^{tn}.$$

Hence, for each Rees valuation v_j of I we obtain

$$tm_1 v_j(J_1) + \cdots + tm_k v_j(J_k) \geq tn v_j(I),$$

or equivalently,

$$n \leq \sum_{s=1}^k m_s \alpha_{sj}.$$

For the other inclusion, first observe that it is enough to prove that $E_j \cap \mathbb{Z}^{k+1} \subseteq C \cap (D_j \times \mathbb{R}_{\geq 0})$. Indeed, if $E_j \cap \mathbb{Z}^{k+1} \subseteq C \cap (D_j \times \mathbb{R}_{\geq 0})$, then for each $\alpha \in E_j \cap \mathbb{Q}^{k+1}$ there exists a positive integer L such that $\alpha L \in E_j \cap \mathbb{Z}^{k+1} \subseteq C \cap (D_j \times \mathbb{R}_{\geq 0})$. This implies that $\alpha \in (1/L)\big(C \cap (D_j \times \mathbb{R}_{\geq 0})\big) = C \cap (D_j \times \mathbb{R}_{\geq 0})$

Let $(m_1, \ldots, m_k, n) \in E_j \cap \mathbb{Z}^{k+1}$. Set $\alpha = \sum_{s=1}^k m_s \alpha_{sj}$. Since the ring A is analytically unramified, there exists an integer N such that $\overline{I^t} \subseteq I^{t-N}$ for all t. (The convention is that $I^n = A$ for $n \leq 0$.) Let g be the integer part of α. For any Rees valuation v_i of A we then get

$$v_i(I^g) = g e_i \leq \alpha e_i \leq \left(\sum_{s=1}^k m_s \alpha_{si} \right) e_i = v_i(J_1^{m_1} \ldots J_k^{m_k}),$$

and hence, by Remark 1.4,

$$J_1^{m_1} \ldots J_k^{m_k} \subseteq \overline{I^g} \subseteq I^{g-N}.$$

This implies that

$$v_I(\underline{J}, m_1, \ldots, m_k) \geq g - N > \alpha - 1 - N. \tag{2.1}$$

Since $n < \alpha$, we can find $\delta > 0$ such that $n < \alpha - \delta$. Choose l such that $l\delta > N+1$ and $lm_1, \dots, lm_k, ln$ are integers. By (2.1), we obtain $v_I(\underline{J}, lm_1, \dots, lm_k) > l\alpha - N - 1$, and by the choice of l, we also have $nl < l\alpha - N - 1$. Then $nl < v_I(\underline{J}, lm_1, \dots, lm_k)$, which implies that $J_1^{lm_1} \dots J_k^{lm_k} \subseteq I^{ln}$; i.e., $(m_1, \dots, m_k, n) \in C$. □

Corollary 2.2. *The topological closure of C is a rational polyhedral cone.*

Proof. From the previous theorem it follows that the topological closure of $C \cap (D_j \times \mathbb{R}_{\geq 0})$ is $\overline{E}_j$, and hence the topological closure of C is the polyhedral cone bounded by the hyperplanes $n = \sum_{s=1}^{k} m_s \alpha_{sj}$ $(j = 1, \dots, r)$ and the coordinate hyperplanes. □

A detailed example of Corollary 2.2 is given below in Example 2.5.

Corollary 2.3. *Let $a_1, a_2, \dots, a_k$ be real numbers. The limit*

$$\lim_{m_1, \dots, m_k \to \infty} \frac{v_I(\underline{J}, m_1, \dots, m_k)}{a_1 m_1 + \cdots + a_k m_k} \tag{2.2}$$

exists if and only if there exists a rational number l such that $la_s = \alpha_{s1} = \alpha_{s2} = \cdots = \alpha_{sr}$ for all $s = 1, \dots, k$. In this case the limit is equal to l.

Proof. Since the polyhedral cones D_j form a partition of $\mathbb{R}^k_{\geq 0}$, the limit (2.2) exists and is equal to l if and only if for each j we have

$$\lim_{\substack{m_1, \dots, m_k \to \infty \\ (m_1, \dots, m_k) \in D_j}} \frac{v_I(\underline{J}, m_1, \dots, m_k)}{a_1 m_1 + \cdots + a_k m_k} = l. \tag{2.3}$$

On the other hand, (2.3) holds if and only if $la_s = \alpha_{sj}$ for all $s = 1, \dots, k$. Indeed, this limit exists and is equal to l if and only if over D_j the topological closure of C is bounded by the hyperplane $n = la_1 m_1 + \cdots + la_k m_k$, which therefore should coincide with the hyperplane $n = \sum_{s=1}^{k} m_s \alpha_{sj}$.

In conclusion, the limit (2.2) exists and is equal to l if and only if all the hyperplanes $n = \sum_{s=1}^{k} m_s \alpha_{sj}$ $(j = 1, \dots, r)$ coincide with $n = la_1 m_1 + \cdots + la_k m_k$, or equivalently, $la_s = \alpha_{s1} = \alpha_{s2} = \cdots = \alpha_{sr}$ for all $s = 1, \dots, k$. □

Corollary 2.4. *Assume that the ideal I has only one Rees valuation. Then the limit*

$$\lim_{m_1, \dots, m_k \to \infty} \frac{v_I(\underline{J}, m_1, \dots, m_k)}{a_1 m_1 + \cdots + a_k m_k}$$

exists if and only if $l_I(J_1)/a_1 = \cdots = l_I(J_k)/a_k$.

Proof. This is a particular case of the previous Corollary. □

EXAMPLE 2.5. Let $A = \mathbb{R}[[X, Y, Z]]/(XY^2 - Z^9)$ and $I = (x, y, z)A$ be as in [3, Example 3.1]. Then $\mathcal{R}(I) = A[It, t^{-1}]$, $\mathcal{R}(I)/t^{-1}\mathcal{R}(I) \cong \mathbb{R}[xt, yt, zt]/(xt)(yt)^2$, and there are two Rees valuations v_1 and v_2, corresponding to the minimal primes $\mathfrak{p}_1 = (xt, t^{-1})$ and $\mathfrak{p}_2 = (yt, t^{-1})$, over $t^{-1}\mathcal{R}(I)$. As shown in [3, Example 3.1], we have $v_1(x) = 7, v_1(y) = v_1(z) = 1$ and $v_2(x) = v_2(z) = 1, v_2(y) = 4$. Thus $v_1(I) = \min\{v_1(x), v_1(y), v_1(z)\} = 1$. Likewise $v_2(I) = 1$. Set $J_1 = (x, z^2)$ and $J_2 = (y^2, z^3)$. Then $v_1(J_1) = 2, v_2(J_1) = 1$, and $v_1(J_2) = 2, v_2(J_2) = 3$. Therefore, $E_1 = \{(m_1, m_2, n) | n \leq 2m_1 + 2m_2\}$ and $E_1 = \{(m_1, m_2, n) | n \leq m_1 + 3m_2\}$. The boundary planes of E_1 and E_2 in $\mathbb{R}^3$ are $z = 2x + 2y$ and $z = x + 3y$, respectively. Thus, according to the results of Corollary 2.2, the topological closure of the cone generated by $\{(m_1, m_2, n) | J_1^{m_1} J_2^{m_2} \subseteq I^n\}$ is as pictured below.

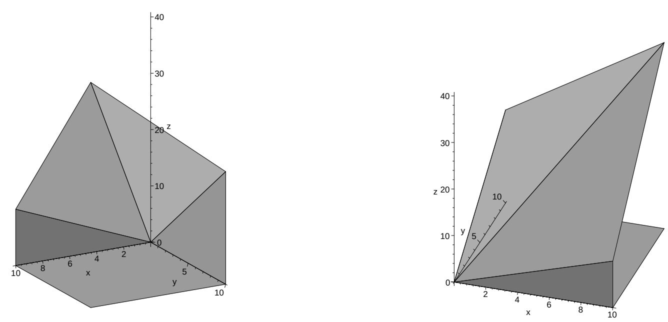

FIGURE 1. View from the front and rotated 90° counter-clockwise around the z-axis.

EXAMPLE 2.6. Let $A = k[[X, Y]]$, with k a field, and $I = (x^3, x^2y, y^2)$. As shown in [7], I has only one associated Rees valuation. Let $J_1 = (x^3y^7), J_2 = (x^4y^6)$, and $J_3 = (x^5y^2)$. Using the methods in Section 4, we can compute $l_I(J_1) = 9/2, l_I(J_2) = 13/3$, and $l_I(J_3) = 8/3$. Then by Corollary 2.4, the limit

$$\lim_{m_1, m_2, m_3 \to \infty} \frac{v_I(J_1, J_2, J_3, m_1, m_2, m_3)}{27m_1 + 26m_2 + 16m_3}$$

exists and equals $1/6$ since

$$\frac{l_I(J_1)}{27} = \frac{l_I(J_2)}{26} = \frac{l_I(J_3)}{16} = \frac{1}{6}.$$

3. Periodic increase

In this section we take a closer look at the sequence $\{v_I(J,m)\}_m$. To simplify the notation we will simply write $v(m)$ instead of $v_I(J,m)$.

We address the question of whether this sequence increases eventually in a periodic way; that is, whether or not there exists a positive integer t such that $v(m+t)-v(m+t-1)=v(m)-v(m-1)$ for $m \gg 0$, or equivalently, $v(m+t)-v(m)=\mathit{constant}$, for $m \gg 0$. Our work is partly motivated by [4, Theorem 8], where Nagata proves that the deviation $v(m)-l_I(J)m$ is bounded. In particular, this implies that there exists a positive constant C such that $0 \leq v(m+t)-v(m)-v(t)<C$ for all m,t.

We begin by defining noetherian filtrations.

Definition 3.1. A family of ideals $\mathcal{F}=\{F_m\}_{m\geq 0}$ in a noetherian ring A is called a filtration if $F_0=A$, $F_{m+1}\subseteq F_m$, and $F_mF_n\subseteq F_{m+n}$ for all $m,n\geq 0$. We say that the filtration $\{F_m\}_{m\geq 0}$ is noetherian if the associated graded ring $\oplus_{m\geq 0}F_m$ is noetherian. Equivalently, the filtration $\mathcal{F}$ is noetherian if and only if there exists t such that $F_{m+t}=F_mF_t$ for all $m\geq t$ ([1, 4.5.12]).

Proposition 3.2. *Let I,J be ideals in a noetherian local ring A such that $J\subseteq\sqrt{I}$, the ideals I,J are not nilpotent, and $\bigcap_k I^k=(0)$. Assume that J is principal and the ring $\mathcal{B}=\oplus_{m,n}J^m\cap I^n$ is noetherian. Then there exists a positive integer t such that $v(m+t)=v(m)+v(t)$ for all $m\geq t$.*

Proof. In the ring $\oplus_{n\geq 0}I^n$ consider the filtration $\{F_m\}$ with $F_m=\oplus_{n\geq 0}J^m \cap I^n$. Since $\mathcal{B}=\oplus_{m\geq 0}F_m$ is noetherian, there exists a positive integer t such that $F_{m+t}=F_mF_t$ for all $m\geq t$. We will prove that this implies $v(m+t)=v(m)+v(t)$ for all $m\geq t$. First note that the inequality $v(m+t)\geq v(m)+v(t)$ always holds. By contradiction, assume that $v(m+t)>v(m)+v(t)$ for some $m\geq t$. This implies that the component of degree $v(m)+v(t)+1$ in F_{m+t} is J^{m+t}, and since $F_{m+t}=F_mF_t$ we then obtain

$$J^{m+t}=J^t(J^m\cap I^{v(m)+1})+J^m(J^t\cap I^{v(t)+1}).$$

Let $J=(z)$. Then we have

$$(z)^{m+t}=z^{m+t}(I^{v(m)+1}:z^m)+z^{m+t}(I^{v(t)+1}:z^t).$$

From the definition of $v(-)$, both $(I^{v(m)+1}:z^m)$ and $(I^{v(t)+1}:z^t)$ are contained in the maximal ideal, and by the Nakayama Lemma, we must have z nilpotent, contradicting our assumptions. ☐

Remark 3.3. It is not always true that the ring $\mathcal{B}$ is noetherian. For such an example see [2, Lemma 5.6].

Note that there are a few other natural conditions that ensure the periodic increase of the sequence $\{v(m)\}_m$. We comment on these below.

Remark 3.4. If the ring $\mathcal{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ is reduced, then we have $v(m) = mv(1)$ for all m. In particular, the sequence $v(m+1) - v(m)$ is constant. Indeed, let $x \in J \setminus I^{v(1)+1}$. The image of x in $I^{v(1)}/I^{v(1)+1} \subseteq \mathcal{G}(I)$ is nonzero, and since $\mathcal{G}(I)$ is reduced, so is the image of x^m in $I^{mv(1)}/I^{mv(1)+1}$. This implies that $J^m \nsubseteq I^{mv(1)+1}$, and hence $v(m) \leq mv(1)$.

The point of view formulated in the above remark can be refined to include the case when J is not necessarily principal, but it comes at the expense of strengthening the hypotheses.

Remark 3.5. Assume that I is normal and $J = (a_1, \dots, a_s)$. Then for every m we have $v_I(J, m) = \min\{v_I((a_j), m) \mid j = 1, \dots, s\}$. Indeed, if $n := \min\{v_I((a_j), m) \mid j = 1, \dots, s\}$, then $a_j^m \in I^n$ for all $j = 1, \dots, s$. This implies that $J^m \subseteq \overline{J^m} = \overline{(a_1^m, \dots, a_s^m)} \subseteq \overline{I^n} = I^n$, so $v_I(J, m) \geq n$. On the other hand, if $v_I(J, m) > n$, we have $J^m \subseteq I^{v_I((a_j),m)+1}$ for some j and hence $a_j^m \in I^{v_I((a_j),m)+1}$, a contradiction. If I is normal and all the rings $\oplus_{m,n}(a_j^m) \cap I^n$ are noetherian ($j = 1, \dots, s$), by Proposition 3.2 we obtain that there exists t_j such that $v_I((a_j), m + t_j) = v_I((a_j), m) + v_I((a_j), t_j)$ for $m \geq t_j$. If we have $t_1 = t_2 = \cdots = t_s = t$ (i.e., the sequences $v_I((a_j), m)$ increase eventually in a periodic way with the same period), then we have $v_I(J, m+t) = v_I(J, m) + v_I(J, t)$ for $m \geq t$. Indeed, by the above observation, $v_I(J, m+t) = v_I((a_j), m+t_j)$ for some j, and hence $v_I(J, m+t) = v_I((a_j), m) + v_I((a_j), t) \leq v_I(J, m) + v_I(J, t)$. The other inequality always holds.

Note that in the situation described in Remark 3.4, when the associated graded ring $\mathcal{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ is reduced (which implies that I is normal), we have $t_1 = t_2 = \cdots = t_s = 1$.

Our final observation introduces a bigraded ring associated to the ideals J and I that can be used in examining the periodicity of the rate of change of the sequence $\{v(m)\}_m$.

Remark 3.6. Let $\mathcal{C}$ be the ring $\oplus_{m\geq 0, n\geq 0} F_{m,n}$, with $F_{m,n} = J^m \cap I^n / J^m \cap I^{n+1}$ and multiplication defined naturally such that $F_{m,n} F_{m',n'} \subseteq F_{m+m',n+n'}$. Let $F_m = \oplus_{n\geq 0} F_{m,n}$. Note that F_m is a filtration on $\mathcal{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ and $F_{m,n} = 0$ for $n < v(m)$, while $F_{m,v(m)} \neq 0$ for all m. As in the above remark, one can check that $v(m+t) = v(m) + v(t)$ is equivalent to $F_{m,v(m)} F_{t,v(t)} \neq 0$.

So, if there exists t such that $F_{t,v(t)}$ contains a nonzerodivisor on $\mathcal{C}$, then $v(m+t) = v(m) + v(t)$ for all m. However, note that $\mathcal{C}$ a domain implies that $F_0 = \mathcal{G}(I)$, the associated graded ring of I, is a domain as well, and then Remark 3.4 applies.

4. Computations

In this section we describe a method of determining $L_J(I) = \inf\{m/n \mid J^m \subseteq I^n\}$ (and $l_I(J) = 1/L_J(I)$) for two monomial ideals I and J in a polynomial ring $k[x_1, \ldots, x_r]$ over a field k. Whenever $J = (a_1, \ldots, a_s)$, one has $L_J(I) = \max\{L_{(a_j)}(I) \mid j = 1, \ldots, s\}$ ([8, Theorem 2]), so we may assume that J is a principal ideal. Let $I = (x_1^{b_{i1}} x_2^{b_{i2}} \ldots x_r^{b_{ir}} \mid i = 1, \ldots, t)$ and $J = (x_1^{c_1} x_2^{c_2} \ldots x_r^{c_r})$.

First observe that $J^m \subseteq I^n$ if and only if there exist nonnegative integers $y_1, \ldots, y_t$ with $y_1 + \cdots + y_t = n$ such that

$$\sum_{i=1}^{t} b_{ij} y_i \leq c_j m \quad \text{for all} \quad j = 1, \ldots, r. \tag{4.1}$$

Set $B_{ij} = (1/c_j) b_{ij}$, $z_i = y_i/(y_1 + \cdots + y_t) = y_i/n$ and $z = (z_1, \ldots, z_t) \in \mathbb{Q}^t$.

So $J^m \subseteq I^n$ if and only if there exist $z_i = y_i/n$ with $y_1 + \cdots + y_t = n$ such that

$$\frac{m}{n} \geq \frac{1}{nc_j} \sum_{i=1}^{t} b_{ij} y_i = \sum_{i=1}^{t} B_{ij} z_i \text{ for all } j = 1, \ldots, r. \tag{4.2}$$

Consider the function $\alpha : \mathbb{R}^t \to \mathbb{R}$, $\alpha(z) = \max_{1 \leq j \leq r}\{\sum_{i=1}^{t} B_{ij} z_i\}$ and the subsets of the rationals $\Lambda_1 = \{m/n \mid J^m \subseteq I^n\}$ and $\Lambda_2 = \{\alpha(z) \mid z_1, \ldots, z_t \in \mathbb{Q}_{\geq 0}, z_1 + \cdots + z_t = 1\}$. We will prove that

$$\inf \Lambda_1 = \inf \Lambda_2 \tag{4.3}$$

The inequality $\geq$ follows from (4.2). For the other inequality, we will show that $\Lambda_2 \subseteq \Lambda_1$. Let $\alpha(z) \in \Lambda_2$ with $z_i = p_i/q$ ($1 \leq i \leq t$, $p_1 + \cdots + p_t = q$, and p_i, q nonnegative integers). The coefficients B_{ij} are rationals, so after clearing the denominators we obtain $\alpha(z) = h/lq$ for some nonnegative integers h, l. By (4.2), since $z_i = lp_i/lq$ for all i, we have $h/lq \in \Lambda_1$, which finishes the proof of (4.3).

Note that

$$\inf \Lambda_2 = \inf \{\alpha(z) \mid z_1, \ldots, z_t \in \mathbb{R}_{\geq 0}, z_1 + \cdots + z_t = 1\},$$

so we need to minimize the function

$$\alpha(z) = \max \left\{ \sum_{i=1}^{t} B_{ij} z_i \,\middle|\, j = 1, \ldots, r \right\}$$

subject to the constraints

$$z_1, \ldots, z_t \geq 0 \quad \text{and} \quad z_1 + \cdots + z_t = 1.$$

Let

$$\Delta_k = \left\{ z \in \mathbb{R}^t_{\geq 0} \,\middle|\, \sum_{i=1}^{t} B_{ik} z_i \geq \sum_{i=1}^{t} B_{ij} z_i \text{ for all } j \neq k \right\}.$$

Clearly $\Delta_1 \cup \cdots \cup \Delta_r = \mathbb{R}^t_{\geq 0}$, so it is enough to minimize the function α on each Δ_k.

In conclusion, for each $k = 1, \ldots, r$, the problem reduces to minimizing the objective function

$$\alpha(z) = \sum_{i=1}^{t} B_{ik} z_i$$

subject to the constraints

$$z_1, \ldots, z_t \geq 0, \quad z_1 + \cdots + z_t = 1$$

and

$$\sum_{i=1}^{t} B_{ik} z_i \geq \sum_{i=1}^{t} B_{ij} z_i \quad \text{for all} \quad j \neq k.$$

This is a classical problem linear programming problem which can be algorithmically solved using the simplex method.

Remark 4.1. In general, the limits $l_I(J)$ and $L_j(I)$ need not be reached by an element of the sequences $\{v_I(J,m)/m\}_m$ and $\{w_J(I,n)/n\}_n$, respectively. However, in the monomial case, as the procedure described above shows, there exists a pair (m, n) with $J^m \subseteq I^n$ and $L_J(I) = n/m$.

Example 4.2. Let $A = k[x, y]$ and $I = (x^3, x^2y, y^2)$, $J = (x^3y^7)$. In this case, $b_{11} = 3, b_{12} = 0, b_{21} = 2, b_{22} = 1, b_{31} = 0, b_{32} = 2,\ c_1 = 3, c_2 = 7$ and $B_{11} = 3/3 = 1, B_{12} = 0/7 = 0, B_{21} = 2/3, B_{22} = 1/7, B_{31} = 0, B_{32} = 2/7$. Then

$$\Delta_1 = \left\{(z_1, z_2, z_3) \in \mathbb{R}^3_{\geq 0} \mid z_1 + (2/3)z_2 \geq (1/7)z_2 + (2/7)z_3\right\}$$

and

$$\Delta_2 = \left\{(z_1, z_2, z_3) \in \mathbb{R}^3_{\geq 0} \mid (1/7)z_2 + (2/7)z_3 \geq z_1 + (2/3)z_2\right\}.$$

By using a computer algebra system that has the simplex method implemented, one can obtain that the minimum on each of the sets Δ_1 and Δ_2 is $2/9$, and hence $L_J(I) = 2/9$.

In fact, the minimum can occur only at the intersection of various regions Δ_k (in our case on $\Delta_1 \cap \Delta_2$), for there are no critical points in the interior of Δ_k.

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