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ALEXANDER-SPANIER COHOMOLOGY OF FOLIATED MANIFOLDS

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Abstract. For a smooth foliated manifold $(M, \mathcal{F})$, the basic and the foliated cohomologies are defined by using the de Rham complex of M. These cohomologies are related with the cohomology of the manifold by the de Rham spectral sequence of $\mathcal{F}$.

A foliated manifold is an example of a space with two topologies, one coarser than the other. For these spaces one can define a continuous cohomology that, for a foliated manifold, corresponds to the continuous foliated (or leafwise) cohomology.

In this paper we introduce a construction for spaces with two topologies based upon the Alexander-Spanier continuous cochains. It allows us to define a spectral sequence, similar to the de Rham spectral sequence for a foliation. In particular, continuous basic and foliated cohomologies are defined and related with the cohomology of the space.

For a smooth foliated manifold, we also consider Alexander-Spanier differentiable cochains. We compare the continuous and differentiable cohomologies, and the latter with the de Rham cohomology. We prove that all three spectral sequences are isomorphic from E_2 onwards if $\mathcal{F}$ is a Riemannian foliation. As a consequence, we conclude that this spectral sequence is a topological invariant of the Riemannian foliation.

We also compute some examples. In particular, we give an isomorphism between the E_2 term for a G-Lie foliation and the *reduced cohomology* of G (in the sense of S.-T. Hu) with coefficients in the reduced foliated cohomology of $\mathcal{F}$.

1. Introduction

Let $(M, \mathcal{F})$ be a C^∞ foliated manifold. Associated to $\mathcal{F}$, there is a filtration of the de Rham complex of M, $A^*(M)$: a smooth form of degree i is said to be of filtration $\geq p$ if it vanishes whenever $i-p+1$ of the vectors are tangent to the foliation. The associated spectral sequence $E_{2,dR}(\mathcal{F})$ is called the *de Rham spectral sequence*, and converges to the de Rham cohomology of M (see (10) below). This spectral sequence is an important C^∞ invariant of the foliation.

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It relates the basic cohomology and the foliated cohomology of $\mathcal{F}$ with the cohomology of M. The goal of this paper is to prove that, for Riemannian foliations, the de Rham spectral sequence is a topological invariant from E_2 onwards. For basic cohomology this is a result by El Kacimi and Nicolau [9]. J.A. Álvarez López and the author [1] have given a different proof by using approximations of continuous foliated maps by smooth maps.

Here we construct two new spectral sequences associated to $\mathcal{F}$. One starts with two real Alexander-Spanier cochain complexes of M, the continuous one, ${}_{\text{cont}}AS^*(M)$, and the differentiable one, ${}_{\text{diff}}AS^*(M)$, with a filtration similar to that considered above (see (6)). In this way, one gets two spectral sequences, $E_{2,\text{cont}}(\mathcal{F})$ and $E_{2,\text{diff}}(\mathcal{F})$, respectively. As every smooth Alexander-Spanier cochain is also continuous, the inclusion induces an homomorphism of spectral sequences

$$J_r : E_{r,\text{diff}}(\mathcal{F}) \longrightarrow E_{r,\text{cont}}(\mathcal{F})\,.$$

There is also a homomorphism

$$\Lambda_r : E_{r,\text{diff}}(\mathcal{F}) \longrightarrow E_{r,dR}(\mathcal{F})$$

between the differentiable Alexander-Spanier spectral sequence and the de Rham spectral sequence (see (14)). Such a homomorphism will be called *quasi-isomorphism* if the homomorphism induced on the level of spectral sequences is an isomorphism of the terms E_r for r bigger or equal than 2. It seems reasonable to conjecture that Λ is a quasi-isomorphism for an arbitrary C^∞ foliation. The Main Theorem of this paper states that J and Λ are quasi-isomorphisms if $\mathcal{F}$ is a Riemannian foliation on a closed manifold (Theorem 8).

The proof has two steps. In the first and fundamental step, the theorem is proved for the particular case of Lie foliations. If $\mathcal{F}$ is the foliation by points on a Lie group, the assertion that J is a quasi-isomorphism reduces to the classical theorem that the continuous and differentiable cohomologies of a Lie group coincide (cf. [19]). The proof for a Lie foliation is a generalization of the classical proof. To prove that Λ is a quasi-isomorphism, the second terms of the spectral sequences are computed (Proposition 3 and Proposition 4) and they are presented as a Lie group cohomology or a Lie algebra cohomology,

$$E^{p,q}_{r,\text{diff}}(\mathcal{F}) \,=\, H^p_\square(G,\overline{H}^q_{\mathcal{F}})\,, \quad E^{p,q}_{r,dR}(\mathcal{F}) \,=\, H^p(\mathfrak{g},\overline{H}^q_{\mathcal{F}})\,,$$

where $\overline{H}^q_{\mathcal{F}}$ stands for the *reduced foliated cohomology*, the quotient of the space $H^q_{\mathcal{F}}$, equipped with the C^∞-topology, by the closure of 0, and $H^p_\square(G,\ )$ refers to the group cohomology defined by Hu [12]. Finally, a theorem by S. Świerczkowski [28] is used to conclude that they are isomorphic.

The second step takes into account Molino's structure theorem to reduce the general case of a Riemannian foliation to the particular case above. It reduces essentially to technicalities about spectral sequences.

Throughout this paper we use the terminology of sheaves. In fact, spectral sequences are constructed from resolutions of the constant sheaf $\mathbb{R}_M$, which are made up of sheaves of *basic forms* or *basic Alexander-Spanier cochains*. The homomorphisms between spectral sequences are also defined by homomorphisms between these resolutions.

The paper is organized as follows. In Section 2, we introduce the Alexander-Spanier spectral sequence in the very general framework of a space with two topologies, one finer than the other. We relate the spectral sequence with the *continuous cohomology* defined by Bott and Haefliger [4] in this setting. Section 3 deals with foliated manifolds and their associated spectral sequences. Some properties of foliated homotopy are proved and the homomorphism between the Alexander-Spanier and de Rham cohomology is defined. Section 4 is devoted to the proof of the Main Theorem for a Lie foliation. In Section 5 we use Molino's structure theorem to reduce the general case of Riemannian foliations to the particular case of Lie foliations.

Codimension one foliations without holonomy provide examples of foliations whose continuous spectral sequence is not isomorphic to the de Rham spectral sequence.

The results of this work were announced in [17].

2. Alexander-Spanier spectral sequence

Let X be a topological space, and let X' be a space with the same set as X, but with a finer topology. Let U be an open set in X. A map

$$\varphi\colon\, U^{p+1} \longrightarrow \mathbb{R}$$

is said to be a *basic Alexander-Spanier p-cochain* in U if it is locally constant when one considers in U^{p+1} the topology induced by X'.

The reason for the term "basic" is that if one considers the decomposition of X by the connected components of X' and the equivalence relation that this partition defines on X, then, under suitable hypotheses, these cochains correspond to the cochains on the quotient space.

For each U, the vector space of basic Alexander-Spanier cochains in U, with the obvious restriction maps, defines a presheaf, which generates the *sheaf of basic Alexander-Spanier cochains* $AS^*_{(X'|X)}$. With the usual differential

$$\delta\,\varphi(x_0,\ldots,x_p) \,=\, \sum_{i=0}^{p} (-1)^i\,\varphi(x_0,\ldots,\hat{x}_i,\ldots,x_p) \tag{1}$$

we have a resolution

$$AS^0_{(X'|X)} \xrightarrow{\ \delta\ } AS^1_{(X'|X)} \xrightarrow{\ \delta\ } AS^2_{(X'|X)} \xrightarrow{\ \delta\ } \cdots \tag{2}$$

of the constant sheaf $\mathbb{R}_X$. In fact, the sequence (2) is pointwise homotopically trivial: let $\epsilon\colon \mathbb{R}_X \to AS^0_{(X'|X)}$ be the obvious map, let $\eta_x\colon AS^0_{(X'|X)_x} \to \mathbb{R}_X$

be the map that assigns to a cochain φ the constant function with value $\varphi(x)$, and let $D_x \colon AS^p_{(X'|X)_x} \to AS^{p-1}_{(X'|X)_x}$ be given by

$$D_x(\varphi)(x_0,\ldots,x_{p-1}) \,=\, \varphi(x,x_0,\ldots,x_{p-1})\,.$$

We have

$$\text{(3)} \qquad \begin{cases} \delta D_x \,+\, D_x\delta \,=\, 1\,, & \text{in positive degrees,} \\ D_x\delta \,=\, 1-\epsilon\eta_x\,, & \text{in degree zero,} \\ \eta_x\epsilon \,=\, 1\,, & \text{on } \mathbb{R}_X\,. \end{cases}$$

As a consequence, there is a spectral sequence

$$\text{(4)} \qquad E_2^{p,q}(X' \mid X) \,=\, H^pH^q(X, AS^\star_{(X'|X)}) \,\Rightarrow\, H^{p+q}(X,\mathbb{R})\,.$$

$E_2^{p,0}(X' \mid X)$ is the cohomology of the sections of the sheaves $AS^*_{(X'|X)}$ and will be called the *basic cohomology* of $(X' \mid X)$.

In the definition of a cochain, we can consider continuous (or smooth, in the appropriate case) rather than arbitrary functions. In this case we use the terms *continuous* or *differentiable* Alexander-Spanier cohomology. The above constructions for the resolution and spectral sequence hold also in the continuous and differentiable cases. In this work, we are mainly concerned with the continuous and the differentiable cohomologies. If it is necessary to avoid confusion, we shall write

$${}_dAS_{(X'|X)}\,, \qquad {}_{\text{cont}}AS_{(X'|X)}\,, \qquad {}_{\text{diff}}AS_{(X'|X)}\,,$$

for the discrete (arbitrary functions), continuous or differentiable Alexander-Spanier sheaves.

Remark 1. Bott and Haefliger [4] define *continuous cohomology* of spaces with two topologies. Let Δ^q be the Euclidean q-simplex. One considers on $\mathrm{Map}\,(\triangle^q,\, X')$ the pull back of the compact open topology on $\mathrm{Map}\,(\triangle^q,\, X)$ by the map induced by the identity $X' \to X$. A *continuous cochain* is a continuous map from $\mathrm{Map}\,(\triangle^q,\, X')$ to $\mathbb{R}$. Mostow [20] proves that the continuous cohomology is the cohomology of X with values in the sheaf of continuous functions on X, locally constant in X'; i.e., $H^q(X, {}_{\text{cont}}AS^0_{(X'|X)})$ is the continuous cohomology of Bott and Haefliger.

Example 1. Let G be a topological group and let BG be its Milnor classifying space. Denote by G_δ the group G with the discrete topology. Then BG_δ is the same set as BG, but with a finer topology. As BG is the semi-simplicial space associated to the nerve NG of G, to compute the spectral sequence

$$\text{(5)} \qquad E_r(BG_\delta \mid BG) \,\Rightarrow\, H(BG)$$

one can use a theorem by Segal ([26, Proposition 5.1]), which asserts that

$$E_1(BG_\delta \mid BG) \,=\, H^q(BG, AS^p_{(BG_\delta|BG)}) \,\cong\, H^q_\delta(AS^p(N^*G))\,,$$

to conclude that this spectral sequence is associated to the double complex

$$AS^p(N^qG)$$

with p as filtrant degree. The differential is $D = \delta_{1,0} + \delta_{0,1}$, where

$$\delta_{1,0}\colon AS^p(N^qG) \to AS^{p+1}(N^qG)$$

is the differential of Alexander-Spanier cochains and

$$\delta_{0,1}\colon AS^p(N^qG) \to AS^p(N^{q+1}G)$$

is induced by the simplicial structure of NG. For a Lie group, this spectral sequence is very close to that considered by Bott and Hochschild (cf. [3]), constructed from the Čech-de Rham complex of G,

$$A^p(N^qG)\,.$$

Bott and Hochschild proved that the E_1 term of this spectral sequence is isomorphic to

$$H_c^{q-p}(G, S^p\mathfrak{g}^*)\,,$$

where $\mathfrak{g}$ is the Lie algebra of G considered as a G-module under the adjoint action, $S^q\mathfrak{g}^*$ denotes the q-th symmetric power, and the subscript c denotes the smooth (or equivalent continuous) cohomology of G with values in $S^q\mathfrak{g}^*$, as defined by van Est [30]. (For another construction of this spectral sequence see [13].) The Bott spectral sequence is a direct summand of (5), and its terms are isomorphic from the term E_2 onwards. In the particular case $q = 0$, as $A^0(NG) = AS^0(NG)$, we have

$$H^*(BG, {}_{\mathrm{cont}}AS^0_{(BG_\delta|BG)}) \;\cong\; H_c(G)\,.$$

(For this isomorphism, see also [20, Corollary 7.6].)

Example 2 (cf. [23]). If $f\colon X \to Y$ is a continuous and closed map such that each $f^{-1}(y)$ is compact and relatively Hausdorff in X, and

$$X' = \coprod_{y\in Y} f^{-1}(y)\,,$$

then $E_r^{p,q}(X' \mid X)$ is the Leray spectral sequence of f. In fact, since X' is defined by a continuous map f, there is an isomorphism

$$AS^*_{(X'|X)} \;\cong\; f^*AS_Y\,.$$

One can compute the Leray spectral sequence from the fine resolution

$$\mathcal{H}^q(f,\mathbb{R}_X)\otimes AS^0_Y \;\longrightarrow\; \mathcal{H}^q(f,\mathbb{R}_X)\otimes AS^1_Y \;\longrightarrow\; \cdots$$

of $\mathcal{H}^q(f,\mathbb{R}_X)$, the Leray sheaf of f. So for the second term of the Leray spectral sequence we have

$$H^p(Y,\mathcal{H}^q(f,\mathbb{R}_X)) \;\cong\; H^p(\Gamma(Y,\mathcal{H}^q(f,\mathbb{R}_X)\otimes AS^*_Y))\,.$$

Now under the above hypothesis on f (see [5, Proposition 4.6]), we have

$$\mathcal{H}^q(f,\mathbb{R}_X)\otimes AS^*_Y \;\cong\; \mathcal{H}^q(f,\mathcal{H}^q(f,AS^*_{(X'|X)}))\,,$$

and, finally,

$$\begin{aligned} H^p(\Gamma(Y,\mathcal{H}^q(f,\mathbb{R}_X)\otimes AS^*_Y)) \;&\cong\; H^p(\Gamma(Y,\mathcal{H}^q(f,AS^*_{(X'|X)}) \\ &\cong\; H^pH^q(M,AS^*_{(X'|X)})\,. \end{aligned}$$

It is possible to give an alternative description of the spectral sequence. To do that, we define a filtration $\{F^pAS^i(X)\}$ of the Alexander-Spanier cochains of X, $AS^*(X)$. We say that $\varphi\in F^pAS^i(X)$ if there exists an open cover $\mathcal{U}$ of X such that

$$\varphi(x_0,\dots,x_{p-1},x_p,\dots,x_i) \;=\; \varphi(y_0,\dots,y_{p-1},x_p,\dots,x_i) \tag{6}$$

if $x_j,y_j,\ 0\le j\le p-1$, belong to the same connected component of U in X', $U\in\mathcal{U}$. Obviously

$$F^0AS^i(X)=AS^i(X)\,,\quad F^pAS^i(X)\supset F^{p+1}AS^i(X)$$

and

$$\delta\left(F^pAS^i(X)\right)\subset F^pAS^{i+1}(X)\,.$$

A p-cochain φ is basic if $\varphi\in F^pAS^p(X)$ and $\delta\varphi\in F^{p+1}AS^{p+1}(X)$. Let $F^pAS^i_X$ be the sheaves of germs of cochains of filtrant degree p. The link between the spectral sequence defined by the filtration and that initially defined is given by the following resolution of AS^p_X:

$$F^pAS^p_X \xrightarrow{\;\overline{\delta}\;} \frac{F^pAS^{p+1}_X}{F^{p+1}AS^{p+1}_X} \xrightarrow{\;\overline{\delta}\;} \frac{F^pAS^{p+2}_X}{F^{p+1}AS^{p+2}_X} \longrightarrow \cdots\,,$$

where $\overline{\delta}$ is induced by δ. Let us check the exactness of this sequence for $q>0$. We define a map

$$E_x\colon\ \left(F^pAS^{p+q}_X\right)_x \longrightarrow \left(F^pAS^{p+q-1}_X\right)_x\,,$$

$E_x(\varphi)=\varphi_x$, by $\varphi_x(x_0,\dots,x_{p+q-1})=\varphi(x_0,\dots,x_{p+q-1},x)$. One gets

$$\varphi\;+\;(-1)^{p+q+1}\delta(\varphi_x)\;=\;(-1)^{p+q+1}(\delta\varphi)_x\,.$$

Let us briefly discuss the naturality of the Alexander-Spanier spectral sequence. Let $(Y'\mid Y)$ be another space with two topologies. A map

$$f\colon (X'\mid X)\;\longrightarrow\;(Y'\mid Y)$$

is a continuous map $f\colon X\to Y$ that is also continuous as a map from X' to Y'. Such a map f defines a differential morphism

$$f^*\colon\ AS^i_{(Y'|Y)}\;\longrightarrow\;AS^i_{(X'|X)}$$

by $f^*(\varphi) = \varphi \circ f^{i+1}$, and a homomorphism of spectral sequences

$$f_r^* \colon E_r^{p,q}(Y' \mid Y) \longrightarrow E_r^{p,q}(X' \mid X)\,. \tag{7}$$

3. Foliated manifolds

A foliation $\mathcal{F}$ on a topological manifold M is a decomposition of the manifold into connected topological submanifolds, with the leaves L_x, $x \in M$, all of the same dimension, the dimension of $\mathcal{F}$, and the additional condition that, locally, the decomposition is modeled on the decomposition of $\mathbb{R}^n$ into the cosets $x + \mathbb{R}^l$ of the standardly embedded subspace $\mathbb{R}^l$. An open set with this condition is said to be a *distinguished open set*. The number $k = n - l$ is the codimension of $\mathcal{F}$. We can consider smooth foliations, $C^{r,s}$-foliations, singular foliations, where leaves of several dimensions are permitted, or just a *lamination*, a metric space decomposed into leaves (see [6] for precise definitions).

Let $(M, \mathcal{F})$ be a foliated manifold, $M = \bigcup_{x\in M} L_x$. Denote by $M^{\mathcal{F}}$ the set M with the leaf topology, for which a basis is formed by the connected components of intersections of open sets of M with leaves. The Alexander-Spanier sheaf and the spectral sequence of the foliated manifold will be that associated to $(M^{\mathcal{F}} \mid M)$. We use the notations

$$_{\text{cont}}AS^*_{\mathcal{F}}\,, \qquad _{\text{diff}}AS^*_{\mathcal{F}}$$

for the continuous and differentiable Alexander-Spanier sheaves, respectively, and

$$E^{p,q}_{r,\text{cont}}(\mathcal{F}) \qquad \text{and} \qquad E^{p,q}_{r,\text{diff}}(\mathcal{F})$$

for the corresponding spectral sequences.

The de Rham spectral sequence. For smooth foliations (or C^r-foliations, with $r \geq 1$) one can construct the *de Rham spectral sequence* of $\mathcal{F}$. Let $(A^*(M), d)$ be the de Rham complex of M and denote by A^*_M the de Rham sheaf of M. A smooth form η is said to be *basic* if it satisfies

$$i_Y\,\eta = 0 \quad\text{and}\quad i_Y\,d\eta = 0 \tag{8}$$

for all $Y \in \Gamma\mathcal{F}$, the algebra of vector fields tangent to the foliation, where i_Y is the interior product by Y. The algebra $A^*_{\mathcal{F}}(M)$ of basic forms is a differential subcomplex of the de Rham complex $A^*(M)$.

The sheaves $A^*_{\mathcal{F}}$ of germs of basic forms define a resolution of $\mathbb{R}_M$, the constant sheaf on M,

$$A^0_{\mathcal{F}} \xrightarrow{\ d\ } A^1_{\mathcal{F}} \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } A^l_{\mathcal{F}} \longrightarrow 0\,, \tag{9}$$

where d is the exterior derivative and l is the dimension of $\mathcal{F}$. Associated to this resolution, we have the de Rham spectral sequence of $\mathcal{F}$,

$$E^{p,q}_{2,dR}(\mathcal{F}) \;=\; H^p(H^q(M, A^\star_{\mathcal{F}})) \;\Rightarrow\; H^{p+q}(M, \mathbb{R})\,. \tag{10}$$

$E^{p,0}_{2,dR}(\mathcal{F})$ is the basic de Rham cohomology of $\mathcal{F}$. We denote by

$$H^q_{\mathcal{F}} \,=\, H^q(M, A^0_{\mathcal{F}}) \tag{11}$$

the differentiable foliated cohomology of the foliation. Note that

$$_{\mathrm{diff}}AS^0_{\mathcal{F}} \,=\, A^0_{\mathcal{F}}\,,$$

the sheaf of smooth functions on M that are locally constant along the leaves.

To describe the de Rham spectral sequence of $\mathcal{F}$ it is also possible to start with a filtration of the de Rham complex $A^*(M)$. A smooth form of degree i is said to be of filtration $\geq p$ if it vanishes whenever $i-p+1$ of the vectors are tangent to the foliation. We shall denote the ideal of all forms of filtration degree $\geq p$ by $F^pA(M)$. We will use further the sheaves

$$A^{p,q} := \, F^pA^{p+q}/F^{p+1}A^{p+q} \tag{12}$$

with the operator

$$d_{\mathcal{F}} : \, A^{p,q} \,\longrightarrow\, A^{p,q+1}$$

induced by the exterior derivative. Checking that it is a resolution of $A_{\mathcal{F}}$ reduces to a form of the Poincaré Lemma (cf. [29]). As A^0_M-modules, these sheaves are fine.

Homotopy. Let $(M', \mathcal{F}')$ be another foliated manifold. By a foliated map

$$f\colon \,(M, \mathcal{F}) \,\longrightarrow\, (M', \mathcal{F}')$$

we understand a smooth map which takes leaves into leaves, or, more precisely, is such that the induced map $df\colon TM \to TM'$ satisfies $df(T\mathcal{F}) \subset T\mathcal{F}'$. Now the map $f^*\colon A^i(M') \to A^i(M)$ is a filtration preserving homomorphism, and we have induced homomorphisms

$$f^{p,q}_r\colon \,E^{p,q}_r(\mathcal{F}') \,\longrightarrow\, E^{p,q}_r(\mathcal{F})$$

between the de Rham (or Alexander-Spanier) spectral sequences of the foliations.

We need two different types of homotopy in this framework. One can consider in $M \times I$ two foliations: the first one is $\mathcal{F} \times I$, with leaves $L \times I$, where L is a leaf of $\mathcal{F}$, the second one is $\mathcal{F} \times I_\delta$, where I_δ is the closed unit interval with the discrete topology, with leaves $L \times \{t\}$. An s-homotopy, for $s = 1$ or 2, will be a smooth function $H\colon M \times I \to M'$ such that the image of each leaf is contained in a leaf, given the first or the second foliation in $M \times I$, respectively.

Lemma 1. *Let $H\colon M \times I \to M'$ be an s-homotopy between foliated maps h_0, h_1 from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$. Then the induced homomorphisms in the spectral sequences are identical, i.e.,*

$$h^*_0 = h^*_1\colon \,E_r(\mathcal{F}') \,\longrightarrow\, E_r(\mathcal{F})\,,$$

for $r \geq s$.

Proof. It is enough to consider the maps

$$i_t \colon M \longrightarrow M \times I\,, \qquad i = 0, 1\,,$$

given by $i_t(x) = (x, t)$. The identity map of $M \times I$ gives an s-homotopy between i_1 and i_2, for $s = 1$ or 2, as we consider the foliation $\mathcal{F} \times I$ or $\mathcal{F} \times I_\delta$, respectively.

For the de Rham spectral sequence, the result is well known (cf. [24]): one can construct a cochain homotopy as usual, starting with a smooth homotopy, and check the filtration requirements.

For $s = 1$, in any case, the lemma follows from the general homotopy invariance of sheaf cohomology. In fact, for the de Rham sheaves, it is

$$A^*_{\mathcal{F}\times I} \cong \pi^* A^*_{\mathcal{F}}\,,$$

where $\pi \colon M \times I \to M$ is the projection. Analogously, for the Alexander-Spanier sheaves we have

$$AS_{(\mathcal{F}'\times I|\mathcal{F}\times I)} \cong \pi^* AS_{(\mathcal{F}'|\mathcal{F})}\,.$$

A general proof of the homotopy axiom for the Alexander-Spanier cohomology was given by Spanier [27]. For $s = 2$ we must check the compatibility of the homotopy $S \colon AS^{i+1}(M \times I) \to AS^i(M)$ with the filtration, that is,

$$S(F^p AS(M \times I)) \subset F^{p-1}(AS(M))\,,$$

where the filtration of $AS(M \times I)$ is defined from the foliation $\mathcal{F} \times I_\delta$. This is a consequence of the fact that, locally, the homotopy is induced by maps like

$$(x_0, \ldots, x_q) \mapsto \sum_{i=0}^{q} (-1)^i \left((x_0, t_0), (x_1, t_0), \ldots, (x_i, t_0), (x_i, t_1), \ldots, (x_q, t_1)\right),$$

with $t_0, t_1 \in I$, t_0 close to t_1. □

A spectral sequences morphism. There exists an onto morphism of differential sheaves

$$\Lambda \colon {}_{\mathrm{diff}}AS^*_{\mathcal{F}} \longrightarrow A^*_{\mathcal{F}}\,.$$

If we take for an open set U of M a p-cochain φ given by the product of $p+1$ smooth functions $f_i \colon U \to \mathbb{R}$, $0 \le i \le p$,

$$\varphi(x_0, x_1, \ldots, x_p) = f_0(x_0) f_1(x_1) \ldots f_p(x_p)\,,$$

then

$$\Lambda(\varphi) = f_0\, df_1 \wedge \ldots \wedge df_p\,.$$

In the general case, for $x \in U$ and $Z_1, \ldots, Z_p \in T_x M$,

$$\text{(13)}\quad \Lambda(\varphi)_x(Z_1, \ldots, Z_p) = \frac{1}{p!} \sum_{\tau \in \mathcal{S}_p} \operatorname{sgn}(\tau) \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_p} \varphi(x, \exp_x \varepsilon_1 Z_{\tau(1)}, \ldots, \exp_x \varepsilon_p Z_{\tau(p)}) \Big|_{\varepsilon_i = 0},$$

where $\varepsilon_i \in \mathbb{R}$, $1 \leq i \leq p$. As a morphism of resolutions of $\mathbb{R}_M$, Λ defines a spectral sequence homomorphism

$$\Lambda_r^{p,q} : E_{r,\mathrm{diff}}^{p,q}(\mathcal{F}) \longrightarrow E_{r,dR}^{p,q}(\mathcal{F}) \tag{14}$$

that converges to an isomorphism.

4. Cohomology of Lie foliations

Let $\mathcal{F}$ be a Lie foliation on M with dense leaves. A suitable description of this structure is the following one: there exists a homomorphism

$$\Pi_1 \colon \pi_1(M) \longrightarrow G ,$$

where G is a simply connected Lie group, a covering map $\pi \colon \tilde{M} \to M$ associated to the homomorphism, with group of deck transformations Γ, and a locally trivial fibration $\Pi \colon \tilde{M} \to G$, equivariant with respect to the action of Γ over $\tilde{M}$ and over G by the left product, if we identify Γ with the image of Π_1 in G. The fibers of Π are the leaves of the lifting foliation $\tilde{\mathcal{F}}$, and Γ is dense in G [10].

A vector field X on M is said to be *foliated* if, for every vector field $Y \in \Gamma\mathcal{F}$, the Lie bracket $[X, Y]$ is also tangent to $\mathcal{F}$. Denote by $\mathfrak{X}(M, \mathcal{F})$ the algebra of foliated vector fields of $\mathcal{F}$. A Lie foliation is *transitive*, that is, at each point of M the complete foliated vector fields generate the whole tangent space (cf. [18]). In fact,

$$\mathfrak{X}(M, \mathcal{F})/\Gamma\mathcal{F} \cong \mathfrak{g}, \tag{15}$$

where $\mathfrak{g}$ is the Lie algebra of G, realized as global foliated vector fields on M, and one takes vector fields tangent to $\mathcal{F}$ to generate $T_x\mathcal{F}$ at each point. We will call such a foliation a G-Lie foliation or $\mathfrak{g}$-Lie foliation.

The inclusion $J \colon {}_{\mathrm{diff}}AS_{\mathcal{F}} \to {}_{\mathrm{cont}}AS_{\mathcal{F}}$ induces a spectral sequence homomorphism

$$J_r^{p,q} \colon E_{r,\mathrm{diff}}^{p,q}(\mathcal{F}) \longrightarrow E_{r,\mathrm{cont}}^{p,q}(\mathcal{F}) . \tag{16}$$

We now prove that J is a *quasi-isomorphism*, i.e., that J induces isomorphisms of the terms E_r for $r \geq 2$.

PROPOSITION 2. *Let $\mathcal{F}$ be a Lie foliation on a compact manifold M. The inclusion $J \colon {}_{\mathrm{diff}}AS^*_{\mathcal{F}} \to_{\mathrm{cont}} AS^*_{\mathcal{F}}$ induces an isomorphism*

$$J_2 : E_{2,\mathrm{diff}}(\mathcal{F}) \cong E_{2,\mathrm{cont}}(\mathcal{F}) .$$

Proof. We construct morphisms

$$s \colon {}_{\mathrm{cont}}AS^i(M) \to {}_{\mathrm{diff}}AS^i(M) , \quad h \colon {}_{\mathrm{cont}}AS(M)^i \to {}_{\mathrm{cont}}AS^{i-1}(M) \tag{17}$$

such that

$$
\begin{cases} s(F^p{}_{\mathrm{cont}}AS(M)) \,\subset\, F^p{}_{\mathrm{diff}}AS(M)\,, \\ h(F^p{}_{\mathrm{cont}}AS(M)) \,\subset\, F^{p-1}{}_{\mathrm{cont}}AS(M)\,, \\ h({}_{\mathrm{diff}}AS(M)) \,\subset\, {}_{\mathrm{diff}}AS(M)\,, \end{cases} \tag{18}
$$

and satisfying the relation

$$
\delta h \,+\, h\delta \,=\, \pm(1-s)\,. \tag{19}
$$

So s will be a cochain morphism compatible with the filtrations, which defines a morphism of spectral sequences, and h is a homotopy between the identity and s, which decreases the filtration degree at most by one. Then

$$
s_r\colon\, E_{r,\mathrm{cont}}(\mathcal{F}) \,\longrightarrow\, E_{r,\mathrm{diff}}(\mathcal{F})
$$

is an isomorphism for $r \geq 2$ (cf. [7]).

In order to define s and h let us fix a finite dimensional vector space V of foliated vector fields such that $V(x) = T_xM$ for all $x \in M$. A vector space of foliated vector fields satisfying such a property will be called *transitive*. Since M is compact and $\mathcal{F}$ is transversally complete, we can always find a transitive finite dimensional vector space. We choose a Riemannian metric on V with volume element dX, and a smooth function ρ on V supported in a compact neighborhood of 0. We define s by

$$
\begin{aligned} &(s\varphi)(x_0,\ldots,x_i) \\ &\quad = \int_V \cdots \int_V \varphi(\phi^*_{X_0}(x_0),\ldots,\phi^*_{X_i}(x_i)) \cdot \rho(X_0) \ldots \cdot \rho(X_i) \cdot dX_0 \ldots dX_i \end{aligned} \tag{20}
$$

where $\phi_{tX} : M \longrightarrow M$ denotes the flow of the vector field $X \in V$ and ϕ_X is the diffeomorphism corresponding to $t = 1$. The function ρ can be normalized by $\int_V \rho(X)\, dX = 1$, and h is defined by

$$
\begin{aligned} &(h\varphi)(x_1,\ldots,x_i) \\ &\quad = \sum_{j=1}^{i}(-1)^j \int_V \cdots \int_V \varphi(x_1,\ldots,x_j,\phi^*_{X_j}(x_j),\ldots,\phi^*_{X_i}(x_i)) \cdot \rho(X_j)\cdot \\ &\qquad\qquad \ldots \cdot \rho(X_i) \cdot dX_j \ldots dX_i \end{aligned}
$$

□

See [25] and [16] for similar constructions. One could also conclude that $E_2(AS_{\mathcal{F}})$ is finite dimensional, but this will be a consequence of Theorem 8 below.

Example 3. In general there is no isomorphism between the E_1 terms. The torus T^2 foliated by lines of constant irrational slope provides a counterexample. As it is equal to the de Rham foliated cohomology, $H^1(T^2, {}_{\mathrm{diff}}AS^0_{\mathcal{F}})$ has either infinite dimension or dimension one, depending upon whether the irrational slope is Liouville or diophantine, while $H^1(T^2, {}_{\mathrm{cont}}AS^0_{\mathcal{F}})$ has always infinite dimension (cf. [20]).

To compare the differentiable Alexander-Spanier spectral sequence with the de Rham spectral sequence, we start by computing the term E_2 for a G-Lie foliation.

We consider first the de Rham spectral sequence of a Lie foliation. The sheaves of basic forms for a Lie foliation with Lie algebra $\mathfrak{g}$ are

$$A^p_{\mathcal{F}} \cong \underset{\sim}{\textstyle\bigwedge^p}\mathfrak{g}^* \otimes A^0_{\mathcal{F}}\,,$$

where $\underset{\sim}{\bigwedge^p}\mathfrak{g}^*$ denotes the constant sheaf over M with stack the vector space $\bigwedge^p \mathfrak{g}^*$. Then, by the universal coefficient theorem, we have

$$E^{p,q}_{1,dR}(\mathcal{F}) \cong \bigwedge^p \mathfrak{g}^* \otimes H^q_{\mathcal{F}}\,,$$

where $H^q_{\mathcal{F}}$ denotes the foliated cohomology (11).

The Lie algebra $\mathfrak{X}(M,\mathcal{F})$ acts over $A^*(M)$ by the Lie derivative,

$$(X,\alpha) \longrightarrow \mathcal{L}_X\alpha\,.$$

This action is compatible with the exterior derivative and the filtration, so it defines an action over $H^i_{\mathcal{F}}$. Since $\Gamma\mathcal{F}$ acts on $H^i_{\mathcal{F}}$ as the identity, we get, by (15), an action of $\mathfrak{g}$ on $H^i_{\mathcal{F}}$. If $X_1,\ldots,X_k$ is a basis for a realization of $\mathfrak{g}$ on M and $\omega^1,\ldots,\omega^k$ are dual 1-forms, the differential d_1 is

$$d_1(\eta\otimes[\alpha]) = d_{\mathfrak{g}}\eta\otimes[\alpha] + (-1)^p\sum_{j=1}^k \eta\wedge\omega^j\otimes\theta_j[\alpha]\,, \tag{21}$$

where $d_{\mathfrak{g}}$ is the differential in $\bigwedge\mathfrak{g}^*$ and $\theta_j[\alpha] = [\mathcal{L}_{X_j}\alpha]$. So we have proved (cf. [15]) that for a $\mathfrak{g}$-Lie foliation $\mathcal{F}$ there is an isomorphism

$$E^{p,q}_{2,dR}(\mathcal{F}) \cong H^p(\mathfrak{g},\, H^q_{\mathcal{F}})\,.$$

We need a finer result. With the notation introduced in (12), $H^q_{\mathcal{F}}$ is the cohomology of the complex $(A^{0,q}(M), d_{\mathcal{F}})$. With the C^∞ topology, Im $d_{\mathcal{F}}$ is not always a closed subspace of Ker $d_{\mathcal{F}}$, and the spaces $H^q_{\mathcal{F}}$ are not necessarily Hausdorff. But to compute the E_2 term of the spectral sequence of a Lie foliation one can use $\overline{H}^q_{\mathcal{F}}$, the *reduced foliated cohomology*, the quotient space of $H^q_{\mathcal{F}}$ over the closure of its trivial subspace; i.e., to compute the E_2-term of the spectral sequence, one can use $\overline{E}^{p,q}_{1,dR}(\mathcal{F})$, the quotient space of the E_1-term over the closure of 0. This fact was proved in [16]. This result is also true for the differentiable Alexander-Spanier spectral sequence, with the same proof, by using the compact operator

$$s\colon {}_{\mathrm{diff}}AS^i(M) \longrightarrow {}_{\mathrm{diff}}AS^i(M)$$

defined in (17). So we get the following expression for the second term of the de Rham spectral sequence of a Lie foliation:

Proposition 3. *For a $\mathfrak{g}$-Lie foliation $\mathcal{F}$ there is an isomorphism*

$$E_{2,dR}^{p,q}(\mathcal{F}) \cong H^p(\mathfrak{g}, \overline{H}_{\mathcal{F}}^q).$$

We now derive a similar expression for the differentiable Alexander-Spanier spectral sequence of a Lie foliation. To begin with, we recall a definition by Hu [12]. Let E be a left G-module. Let $(C^p(G,E),\delta)$ be the complex of homogeneous cochains of G over E. A p-dimensional cochain $\varphi \in C^p(G,E)$ is a map

$$\varphi\colon G\times \overset{p+1}{\cdots} \times G \longrightarrow E$$

satisfying the following homogeneity condition:

$$\varphi(gx_0,\dots,gx_p) = g\varphi(x_0,\dots,x_p).$$

We assume that φ is smooth. (The same cohomology is obtained if one assumes that φ is continuous.) The differential δ is defined by (1). A cochain $\varphi \in C^p(G,E)$ is called *locally trivial* if there is a neighborhood U of e in G such that $\varphi(x_0,\dots,x_p) = 0$ whenever all $x_0,\dots,x_p$ are in U. The locally trivial cochains form a subcomplex of $C^*(G,E)$. Let $\big(G_\square^*(G,E),\delta\big)$ be the quotient complex. Its cohomology $H_\square(G,E)$ is, by definition, the *reduced cohomology* of G.

Corresponding to the $\mathfrak{g}$-action, there is an action of G over $H_{\mathcal{F}}^i$. The action can be defined geometrically, by lifting paths of G into $\tilde{M}$, or algebraically, as we do in the following proposition.

Proposition 4. *Let $\mathcal{F}$ be a $\mathfrak{g}$-Lie foliation with dense leaves. There exists an isomorphism*

$$E_{2,\mathrm{diff}}^{p,q}(\mathcal{F}) \cong H_\square^p(G,\overline{H}_{\mathcal{F}}^q).$$

Proof. To compute $H^q(M, {}_{\mathrm{diff}}AS_{\mathcal{F}}^p)$ one can use of the following resolution of ${}_{\mathrm{diff}}AS_{\mathcal{F}}^p$:

$${}_{\mathrm{diff}}AS_{\mathcal{F}}^p \otimes A_M^0 \xrightarrow{id\otimes d_{\mathcal{F}}} {}_{\mathrm{diff}}AS_{\mathcal{F}}^p \otimes A_M^{0,1} \xrightarrow{id\otimes d_{\mathcal{F}}} {}_{\mathrm{diff}}AS_{\mathcal{F}}^p \otimes A_M^{0,2} \longrightarrow \cdots$$

We have

$$\Gamma({}_{\mathrm{diff}}AS_{\mathcal{F}}^p \otimes A^{0,q}) \cong {}_{\mathrm{diff}}AS_{\mathcal{F}}^p(M)\hat{\otimes}A^{0,q}(M),$$

the complete tensor product of ${}_{\mathrm{diff}}AS_{\mathcal{F}}^p(M)$ and $A^{0,q}(M)$. As $A^{0,q}(M)$ are nuclear spaces, from the exact sequence

$$0 \to \overline{\mathrm{Im}\, d_{\mathcal{F}}} \longrightarrow \mathrm{Ker}\, d_{\mathcal{F}} \longrightarrow \overline{H}_{\mathcal{F}} \to 0$$

we deduce the isomorphism

$$\overline{E}_{1,\mathrm{diff}}^{p,q}(\mathcal{F}) \cong {}_{\mathrm{diff}}AS_{\mathcal{F}}^p(M)\hat{\otimes}\overline{H}_{\mathcal{F}}^q.$$

The $d_1^{0,q}$ differential defines an action of G over $\overline{H}^q_{\mathcal{F}}$, given by

$$g\cdot[\alpha] \,=\, d_1^{0,q}[\alpha](e,g)\,.$$

Finally, as the leaves of $\mathcal{F}$ are dense, via the lift of cochains to $\tilde{M}$, we get

$${}_{\mathrm{diff}}AS^p_{\mathcal{F}}(M)\hat{\otimes}\overline{H}^q_{\mathcal{F}} \,\cong\, C^p_{\square}(G,\overline{H}^q_{\mathcal{F}})\,. \qquad \square$$

The spectral sequence homomorphism (14) induced by Λ defines, in particular, a differential complex map

$$\Lambda_1\colon\ (C^p_{\square}(G,\overline{H}^q_{\mathcal{F}})\,,\,d_1) \,\longrightarrow\, (\textstyle\bigwedge^p\mathfrak{g}^*\otimes\overline{H}^q_{\mathcal{F}}\,,\,d_1)\,.$$

Then the G-module structure of $\overline{H}^q_{\mathcal{F}}$ corresponds, by Λ, to the $\mathfrak{g}$-action on $\overline{H}^q_{\mathcal{F}}$.

We need the following result:

Theorem 5. *Let E be a G-module. Assume that E is a Hausdorff locally convex complete topological vector space. There is an isomorphism*

$$H^p_{\square}(G,E) \,\cong\, H^p(\mathfrak{g},E)\,.$$

If E is finite dimensional (a very unusual property for $\overline{H}^q_{\mathcal{F}}$), this is a theorem by Świerczkowski [28]. The same proof works in this case; all one needs is a Poincaré lemma for E-valued forms. We indicate here a slightly different proof.

Proof. To begin with, we define a double complex

$$C^{p,q} \,=\, C^p_{\square}(G,\,\underset{\sim e}{A^q}(G,E))\,, \tag{22}$$

where $\underset{\sim e}{A^q}(G,E)$ is the space of germs at e of de Rham forms on G with values on E, and $C^p_{\square}(G,\,\underset{\sim e}{A^q}(G,E))$ is the space of homogeneous cochains of G with values on $\underset{\sim e}{A^q}(G,E)$, modulo the locally trivial ones. If the vector space (22) is written as $C^p_{\square}(G,\,C^\infty_{\square}(G,\,\Lambda^q\mathfrak{g}^*\otimes E)$, an element φ will be a map

$$\varphi:\ G\times \overset{p+1}{\cdots} \times G \,\longrightarrow\, C^\infty_{\square}(G,\,\Lambda^q\mathfrak{g}^*\otimes E)$$

satisfying

$$\varphi(gx_0,\ldots,gx_p)(gy) \,=\, g\cdot\varphi(x_0,\ldots,x_p)(y)\,, \tag{23}$$

where on the right-hand side we have the G-action on E.

The two differentiation operators d_1 and d_2 of degree $(1,0)$ and $(0,1)$, respectively, are defined as follows. Let $d_1=\delta$, the usual differentiation as defined in (1). The operator d_2 can be given by

$$d_2(\eta) \,=\, d_G\eta \,+\, \sum_{j=1}^k \omega^j\wedge\theta_j\circ\eta\,,$$

where d_G is the exterior derivative on G, $\omega^1, \dots, \omega^k$ is a basis of the left invariant 1-forms on G, $\xi_1, \dots, \xi_k$ is a basis of $\mathfrak{g}$ dual to $\{\omega^i\}$, and θ_j stands for the action of $\xi_j \in \mathfrak{g}$ on E. It is straightforward to check the identities $d_1^2 = 0 = d_2^2$ and $d_1 d_2 + d_2 d_1 = 0$.

The two spectral sequences associated to this double complex, $I_r^{p,q}$ and $II_r^{p,q}$, converge to the same graded space and satisfy

$$I_1^{p,q} \;=\; 0 \;=\; II_1^{q,p} \qquad \text{if} \quad q \neq 0\,.$$

It will follow that $I_2^{p,0} \cong II_2^{0,p}$, which is the statement of the proposition. In fact, the space $I_1^{p,q}$ may be identified with $H^q(C^{p,\star}, d_2)$. Then the Poincaré lemma implies $I_1^{p,q} = 0$, $q > 0$, and $I_1^{p,0} \cong C^p_{\square}(G, E)$. So

$$I_2^{p,0} \;\cong\; H^p_{\square}(G, E)\,. \tag{24}$$

On the other hand, $II_1^{p,q} \cong H^p(C^{\star,q}, d_1)$. The complex $(C^{\star,q}, d_1)$ admits the homotopy $h\colon C^{p,q} \to C^{p-1,q}$ defined by

$$h(\varphi)(x_0, \dots, x_{p-1})(y) \;=\; \varphi(y, x_0, \dots, x_{p-1})(y)\,. \tag{25}$$

For $p > 0$, $h\delta\varphi + \delta h\varphi = \varphi$, and $II_1^{p,q} = 0$. For $p = 0$, $h\delta\varphi = \varphi - c(\varphi)$, with the constant chain $c(\varphi)$ defined by

$$c(\varphi)(x)(y) \;=\; y \cdot [\varphi(e)(e)]\,.$$

If we introduce the transposed double complex ${}^tC$, as usual, to compute $II_r^{p,q}$, we get $II_1^{p,0} \cong \bigwedge^p \mathfrak{g} \otimes E$ and

$$II_2^{p,0} \;\cong\; H^p(\mathfrak{g}, E)\,. \tag{26}$$

□

Finally, Theorem 5 and Propositions 3 and 4 prove the following result:

Proposition 6. *Let $\mathcal{F}$ be a Lie foliation. There exists an isomorphism*

$$E_{2,\mathrm{diff}}(\mathcal{F}) \;\cong\; E_{2,dR}(\mathcal{F})\,.$$

We are now interested in getting an explicit isomorphism. To do that, we recall first the definition of *Riemannian foliation*.

A foliation can be defined by local submersions on some open subsets of $\mathbb{R}^k$. We can assume that these subsets are open balls with center the origin. For a Riemannian foliation there is a Riemannian metric on M, a *bundle-like* metric (cf. [21]), such that these submersions are Riemannian submersions.

For Riemannian foliations one can construct a map

$$\Phi\colon\, A^p_{\mathcal{F}} \;\longrightarrow\; {}_{\mathrm{diff}}AS^p_{\mathcal{F}}$$

between differential sheaves that defines a morphism of spectral sequences and which will be an inverse of the homomorphism Λ, defined in (14), from E_2 onwards.

Let us fix a bundle-like metric on M. Let $f\colon U \to \mathcal{O}$ be a Riemannian submersion defining $\mathcal{F}$ in U. If η is a basic p-form, then $\eta\;|_U = f^*\gamma$, with

$\gamma \in A^p(\mathcal{O})$. Let $x \in U$ and satisfying $\chi_{_U} : U \to \mathbb{R}$ be a smooth function with support in U and satisfying $\chi_{_U} \equiv 1$ on a neighborhood of x. Given $\eta \in A^p_{\mathcal{F}}(U)$, we define $\Phi(\eta)$ by

$$\Phi(\eta)(x_0, \dots, x_p) \,=\, \chi_{_U}(x_0) \cdots \chi_{_U}(x_p) \int_{\Delta[\overline{x}_0, \dots, \overline{x}_p]} \gamma \,,$$

where $\overline{x}_0, \dots, \overline{x}_p$ are the images of $x_0, \dots, x_p$ in $\mathbb{R}^k$ and $\Delta[\overline{x}_0, \dots, \overline{x}_p]$ is the simplex defined by these points. The germ of $\Phi(\eta)$ at x does not depend on the function $\chi_{_U}$ either on the open neighborhood U, and so we have a well defined map between the sheaves. To check that it is a homomorphism of complexes one uses the Stokes' theorem for chains and the fact that the boundary of $\Delta[\overline{x}_0, \dots, \overline{x}_p]$ can be expressed as follows:

$$\partial \Delta[\overline{x}_0, \dots, \overline{x}_p] \,=\, \sum_{j=0}^{p} (-1)^j \Delta[\overline{x}_0, \dots, \overline{x}_{j-1}, \overline{x}_{j+1}, \dots, \overline{x}_p] \,.$$

Finally, we have

$$\Lambda \circ \Phi \,=\, \mathrm{id}_{A_{\mathcal{F}}} \,.$$

The computation can be done in $\mathcal{O} \subset \mathbb{R}^k$ and it is the same as in [8].

If $S^*_{\mathcal{F}}$ denotes the kernel of Λ, we have a split exact sequence

$$0 \longrightarrow S^p_{\mathcal{F}} \longrightarrow {}_{\mathrm{diff}}AS^p_{\mathcal{F}} \stackrel{\Lambda}{\longrightarrow} A^p_{\mathcal{F}} \longrightarrow 0 \,.$$

Thus

$${}_{\mathrm{diff}}AS^p_{\mathcal{F}} \,\cong\, A^p_{\mathcal{F}} \oplus S^p_{\mathcal{F}}$$

and

$$E^{p,q}_{2,\mathrm{diff}}(\mathcal{F}) \,\cong\, E^{p,q}_{2,dR}(\mathcal{F}) \oplus E^{p,q}_2(S_{\mathcal{F}}) \,,$$

where $E^{p,q}_2(S_{\mathcal{F}})$ is the spectral sequence associated to the exact sequence of sheaves $(S^*_{\mathcal{F}}, \delta)$, which converge to 0.

As a consequence, we deduce that the isomorphism in Proposition 6 between $E^{p,q}_{2,\mathrm{diff}}(\mathcal{F})$ and $E^{p,q}_{2,dR}(\mathcal{F})$ is induced by Λ.

5. Riemannian foliations

To prove the next theorem, we will use the description of the structure of a Riemannian foliation given by Molino [18] to reduce the question to the case, already proved, of Lie foliations with dense leaves. To begin with, we consider a particular type of Riemannian foliation.

PROPOSITION 7. *Let $\mathcal{F}_0$ be a Lie foliation on a closed manifold N, let M be a bundle of fiber N with structure group* $\mathrm{Aut}(N, \mathcal{F}_0)$, *and let $\mathcal{F}$ be the foliation induced on M. For this foliation $\mathcal{F}$ the homomorphisms J and Λ are quasi-isomorphisms.*

Proof. To prove the proposition we will use the following well known theorem (see [5, Theorem IV.2.2.]):

THEOREM. *Let $h\colon \mathcal{L}^\star \to \mathcal{M}^\star$ be a homomorphism of differential sheaves on a topological space B and assume that both $\mathcal{L}^\star$ and $\mathcal{M}^\star$ are bounded below. Also assume that the induced map $h^*\colon \mathcal{H}^q(\mathcal{L}^*) \to \mathcal{H}^q(\mathcal{M}^*)$ of derived sheaves is an isomorphism for all q and that*

$$H^*(H^q(B;\mathcal{L}^*)) = 0 = H^*(H^q(B;\mathcal{M}^*))$$

for $q > 0$. Then the induced map $H^n(\mathcal{L}^*(B)) \to H^n(\mathcal{M}^*(B))$ is an isomorphism for all n.

Let $p\colon M \to B$ be the bundle of fiber N. To prove that Λ is a quasi-isomorphism, for $j \geq 0$ fixed, we consider the differential sheaves defined as follows: for each open subset U of B, set

$$\mathcal{L}_j^*(U) = H^j(p^{-1}U, {}_{\mathrm{diff}}AS^*_{\mathcal{F}} \mid p^{-1}U)$$

and

$$\mathcal{M}_j^*(U) = H^j(p^{-1}U, A^*_{\mathcal{F}} \mid p^{-1}U)$$

The sheaf $\mathcal{L}_0^0$ is soft, i.e., every section defined on a closed set can be extended to B. Then all sheaves $\mathcal{L}_j^*$ are soft, as they are modules over $\mathcal{L}_0^0$. The same holds for $\mathcal{M}_j^*$. Since these sheaves are soft, they are acyclic, and thus

$$H^q(B, \mathcal{L}_j^*) = 0 = H^q(B, \mathcal{M}_j^*)$$

for $q > 0$.

We prove now that the homomorphism $\Lambda^*\colon \mathcal{H}(\mathcal{L}^*) \to \mathcal{H}(\mathcal{M}^*)$ induced by Λ is an isomorphism. In fact, as a consequence of the theorem for Lie foliations, this homomorphism will be an isomorphism over each stack of the sheaves.

We compute the stack of $H^*(\mathcal{L}_j^*)$ at a point $x \in B$. Let U be a contractible open subset of B such that $p^{-1}(U) \cong U \times N$.

N and $p^{-1}(U)$ are homotopically equivalent, by foliated 2-homotopies. Then

$$E_{2,\mathrm{diff}}^{p,q}(\mathcal{L}_{p^{-1}(U)}) \cong E_{2,\mathrm{diff}}^{p,q}(\mathcal{F}_0)\,, \tag{27}$$

and, finally,

$$H^p(\mathcal{L}_q^*)_x = E_{2,\mathrm{diff}}^{p,q}(\mathcal{F}_0)\,. \tag{28}$$

Analogously, $H^p(\mathcal{M}_q^*)_x = E_{2,dR}^{p,q}(\mathcal{F}_0)$, and the proof that Λ is a quasi-isomorphism is completed.

The proof for J is similar. □

Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Let Q be the principal bundle of transverse frames of $\mathcal{F}$. Molino defined a lifting foliation $\tilde{\mathcal{F}}$ on Q, with the same dimension as $\mathcal{F}$. For such a foliation, the closures of the leaves are the fibers of a locally trivial fibration, the *basic fibration.* The foliation induced in each fiber is a Lie foliation, so we are under the hypotheses of Proposition 7.

THEOREM 8. *For a Riemannian foliation $\mathcal{F}$ on a compact manifold M the homomorphism J and Λ are quasi-isomorphisms.*

Proof. Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold. We assume, for simplicity, that it is transversally oriented. Let Q be the $SO(k)$-principal bundle of transverse frames of $\mathcal{F}$ equipped with the lifting foliation $\tilde{\mathcal{F}}$. We use the notation $E_2(\mathcal{F})$ for the de Rham and the continuous or differentiable Alexander-Spanier spectral sequences of $\mathcal{F}$. Associated to the action of $SO(k)$ on Q there are, for each q, spectral sequences

$$E_2^{r,s}(q) \,=\, E_2^{s,q}(\mathcal{F}) \otimes H^r(SO(k), \mathbb{R}) \,\Rightarrow\, E_2^{r+s,q}(\tilde{\mathcal{F}})\,.$$

Now, J and Λ induce homomorphisms between the spectral sequences $E_2(q)$, corresponding to the differentiable Alexander-Spanier and to the continuous Alexander-Spanier and de Rham cohomologies, respectively. These homomorphisms are isomorphisms over $E_2^{s,q}(\tilde{\mathcal{F}})$ and over $H^r(SO(k))$. Now the result follows by the Zeeman's comparison theorem. □

COROLLARY 9. *The de Rham spectral sequence $E_{r,dR}(\mathcal{F})$ of a Riemannian foliation is a topological invariant for $r \geq 2$.*

For the basic cohomology, $E_{2,dR}^{p,0}(\mathcal{F})$, this result was proved by El Kacimi and Nicolau [9]. In the general case, it was also proven in [1], by a different method.

For an arbitrary foliation Corollary 9 is not true.

EXAMPLE 4. We consider foliations of codimension 1, without compact leaves. There are well known examples of such foliations in the torus that are topologically equivalent, but have different de Rham spectral sequences (cf. [22], [2]). All these foliations, if they are transversally orientable, can be defined by a nonsingular closed 1-form, but to do that it is sometimes necessary to change the smooth structure of M. (This is a well known theorem by Sacksteder; cf. [6].) This change does not modify the continuous cohomology, but it certainly changes the de Rham spectral sequence: $E_{2,dR}^{1,0}(\mathcal{F})$ is isomorphic to $\mathbb{R}$, in the new smooth structure, and vanishes in the old one. But for a codimension one foliation we always have

$$E_{r,\mathrm{diff}}^{0,q}(\mathcal{F}) \,\cong\, E_{r,dR}^{0,q}(\mathcal{F})$$

for $r \geq 1$, and

$$E_{r,\mathrm{diff}}^{1,0}(\mathcal{F}) \,\cong\, E_{r,dR}^{1,0}(\mathcal{F})\,,$$

for $r \geq 2$. In fact, for codimension one, the sequence

$$0 \longrightarrow S^*_{\mathcal{F}} \longrightarrow AS^*_{\mathcal{F}} \longrightarrow A^*_{\mathcal{F}} \longrightarrow 0$$

always splits. So these foliations provide examples, where continuous and differentiable Alexander-Spanier spectral sequences are different.

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