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NEW FUNCTION SPACES OF MORREY-CAMPANATO TYPE ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. In the context of spaces of homogeneous type, we introduce and develop some new function spaces of Morrey-Campanato type. The new function spaces are defined by variants of maximal functions associated with generalized approximations to the identity, and they generalize the classical Morrey-Campanato spaces. We show that the John-Nirenberg inequality holds on these spaces. We also establish the endpoint boundedness of fractional integrals.

1. Introduction

The Morrey-Campanato spaces on Euclidean spaces $\mathbb{R}^n$ play an important role in the study of partial differential equation; see [11], [13] and [15]. The concept of spaces of homogeneous type, which is a natural generalization of Euclidean spaces $\mathbb{R}^n$, was introduced in [3]. In this paper, we will study Morrey-Campanato spaces on spaces of homogeneous type. Let χ be a space of homogeneous type equipped with a metric d and measure μ satisfying the doubling property. Following [14], we will say that a locally integral function f is a Morrey-Campanato space $L(\alpha, \chi)$ function ($\alpha > 0$) on χ if

$$\sup_B \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - f_B| \, d\mu(x) < \infty,$$

where the supremum is taken over all balls $B \subset \chi$ and f_B stands for the mean of f over B with respect to μ, that is,

$$f_B = \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y).$$

It is well known that for $\alpha = 0$ the space $L(\alpha, \chi)$ coincides with the $BMO(\chi)$ space. Moreover, $L(\alpha, \chi)$ coincides with $\mathrm{Lip}(\alpha, \chi)$, the Lipschitz integral space, when $0 < \alpha < 1/n$, where n denotes the homogeneous dimension

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of homogeneous space χ. Recently, X. T. Duong and L. X. Yan [7] introduced new function spaces of BMO type that generalize the classical BMO space in the context of spaces of homogeneous type. More precisely, they considered $A_t f(x)$, for certain families of operators A_t with kernel $a_t(x, y)$ with appropriate decay, as an average version of f and used

$$A_{t_B} f(x) = \int_\chi a_{t_B}(x, y) f(y)\, d\mu(y)$$

in place of the mean value f_B in the definition of the classical BMO space, where t_B is scaled to the radius of the ball B. Similarly, D. G. Deng, X. T. Duong and L. X. Yan [4] also gave a new characterization of the Morrey-Campanato spaces on the Euclidean space $\mathbb{R}^n$.

In this paper, motivated by [4] and [7], we introduce new function spaces of Morrey-Campanato type on spaces of homogeneous type. We study and establish important features for these spaces such as the John-Nirenberg inequality on spaces of homogeneous type. Finally, we prove endpoint estimates for new fractional integrals.

In the sequel, C is a positive constant which is independent of the main parameters and not necessary the same at each occurrence.

2. Definition of $\mathrm{Lip}_A(\alpha, \chi)$ and basic properties

2.1. Preliminaries. We briefly recall some basic definitions and facts about spaces of homogeneous type. A quasi-metric d on a set χ is a function from $\chi \times \chi$ to $[0, \infty)$ satisfying the following:

(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x,\ y \in \chi$.
(iii) There exists a constant $C_1 \geq 1$ such that

$$d(x, y) \leq C_1(d(x, z) + d(z, y)), \quad \text{for all } x,\ y,\ z \in \chi.$$

By a result in [14], for any quasi-metric d there exists another quasi-metric d', continuous and equivalent to d, for which every ball is open. So, without loss of generality, the quasi-metric d can be assumed to be continuous and the balls to be open.

A space of homogeneous type (χ, d, μ) is a set χ together with a quasi-metric d and a nonnegative Borel measure μ such that the doubling property

$$\mu(B(x, 2r)) \leq C_2 \mu(B(x, r)) < \infty$$

holds for all $x \in \chi$ and $r > 0$, where the constant $C_2 \geq 1$ is independent of x and r, and $B(x, r) = \{y \in \chi : d(x, y) < r\}$ is the ball with center x and radius r.

Note that the doubling property implies the following strong homogeneity property:

$$\mu(B(x, \lambda r)) \leq C\lambda^n \mu((B(x, r)) \tag{2.1}$$

for some $C,\ n > 0$, uniformly for all $\lambda \geq 1$ and $x \in \chi$. The parameter n is a measure of the dimension of the space. There also exist C and $N,\ 0 \leq N \leq n$, such that

$$\mu(B(y,r)) \leq C\left(1+\frac{d(x,y)}{r}\right)^{N} \mu(B(x,r)) \tag{2.2}$$

uniformly for all $x,\ y \in \chi$ and $r > 0$. See also [7].

As in [7], we will work with a class of integral operators $\{A_t\}_{t>0}$, which plays the role of generalized approximations to the identity. We assume that the operators A_t are defined by kernels $a_t(x,y)$ in the sense that

$$A_t f(x) = \int_{\chi} a_t(x,y) f(y)\, d\mu(y)$$

for every function f that satisfies the growth condition (2.5) below.

We also assume that the kernels $a_t(x,y)$ satisfy the estimate

$$|a_t(x,y)| \leq h_t(x,y)$$

for all $x,\ y \in \chi$, where $h_t(x,y)$ is given by

$$h_t(x,y) = \frac{1}{\mu(B(x,t^{1/m}))} g\left(\frac{d^m(x,y)}{t}\right), \tag{2.3}$$

in which m is a positive constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+2N+(n+N)\alpha+\epsilon} g(r^m) = 0 \tag{2.4}$$

for some $\epsilon > 0$, where N is the power appearing in property (2.2), and n the dimension entering the strong homogeneity property. Here and in the sequel α denotes a positive constant.

We will also use the Hardy-Littlewood maximal operator Mf, which is defined by

$$Mf(x) = \sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls containing x.

2.2. Definition of $\mathrm{Lip}_A(\alpha,\chi)$**.** Let $\{A_t\}_{t>0}$ be a generalized approximation to the identity whose kernels $a_t(x,y)$ satisfy conditions (2.3) and (2.4). For a ball B we will use the notation $2^k B,\ k \geq 0$, to denote the ball having the same center as B and radius $2^k r_B$, and $2^{-1}B$ denotes the empty set $\emptyset$.

Let ϵ be the constant in (2.4) and $0 < \beta < \epsilon$. A function $f \in L^1_{\mathrm{loc}}(\chi)$ is said to be a function of type (x_0,β) centered at $x_0 \in \chi$ if f satisfies

$$\int_{\chi} \frac{|f(x)|}{(1+d(x_0,x))^{2N+(n+N)\alpha+\beta} \mu(B(x_0, 1+d(x_0,x)))}\, d\mu(x) \leq C < \infty. \tag{2.5}$$

We denote by $\mathcal{M}_{x_0,\beta}$ the collection of all function of type (x_0, β). If $f \in \mathcal{M}_{x_0,\beta}$, the norm of f in $\mathcal{M}_{x_0,\beta}$ is defined by

$$\|f\|_{\mathcal{M}_{x_0,\beta}} = \inf\{C \geq 0 : (2.5) \text{ holds}\}.$$

For a fixed $x_0 \in \chi$ it is easy to see that $\mathcal{M}_{x_0,\beta}$ is a Banach space under the norm $\|f\|_{\mathcal{M}_{x_0,\beta}} < \infty$. For any $x_1 \in \chi$, $\mathcal{M}_{x_1,\beta} = \mathcal{M}_{x_0,\beta}$ with equivalent norms. We set

$$\mathcal{M} = \bigcup_{x_0 \in \chi} \bigcup_{\beta : 0 < \beta < \epsilon} \mathcal{M}_{x_0,\beta},$$

where ϵ is the constant in (2.4).

Lemma 2.1. *We have the following properties:*

(i) *If $f \in L(\alpha, \chi)$, then $f \in \mathcal{M}$.*
(ii) *For each $t > 0$ and $f \in \mathcal{M}$ we have $|A_t f(x)| < \infty$ for almost all $x \in \chi$.*
(iii) *For each $t, s > 0$ and $f \in \mathcal{M}$ we have $|A_t(A_s f)(x)| < \infty$ for almost all $x \in \chi$.*

As a consequence, if

$$a_{t+s}(x, y) = \int_\chi a_t(x, z) a_s(z, y)\, d\mu(z),$$

then for any $f \in \mathcal{M}$, $A_{t+s} f(x) = A_t(A_s f)(x)$ for almost all $x \in \chi$, and we say that the class A_t satisfies the semigroup property.

The proof of Lemma 2.1 is similar to that of Lemma 2.3 in [7]. We omit the details.

We now introduce the space $\mathrm{Lip}_A(\alpha, \chi)$ associated with a generalized approximation to the identity $\{A_t\}_{t>0}$.

Definition 2.1. We say that $f \in \mathcal{M}$ is in $\mathrm{Lip}_A(\alpha, \chi)$, the space of functions of Lipschitz type associated with a generalized approximation to the identity $\{A_t\}_{t>0}$, if there exists some C such that for any ball B

$$\sup_B \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - A_{t_B} f(x)|\, d\mu(x) \leq C, \tag{2.6}$$

where $t_B = r_B^m$ and r_B is the radius of the ball B.

The smallest bound C for which (2.6) is satisfied is then taken to be the norm of f in this space and is denoted by $\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}$.

Note that when $\alpha = 0$, $\mathrm{Lip}_A(0, \chi) = BMO_A(\chi)$; see [7].

Next, we give a relation between $\mathrm{Lip}_A(\alpha, \chi)$ and $L(\alpha, \chi)$.

Proposition 2.1. *Assume that for every $t > 0$, $A_t(1) = 1$ almost everywhere, that is, $\int_\chi a_t(x,y)\,d\mu(y) = 1$ for almost all $x \in \chi$. Then, we have $L(\alpha,\chi) \subset \mathrm{Lip}_A(\alpha,\chi)$ and there exists a positive constant $C > 0$ such that*

$$\|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \le C\|f\|_{L(\alpha,\chi)}.$$

However, the converse inequality does not hold in general.

Proof. We fix $f \in L(\alpha,\chi)$, $x_0 \in \chi$ and a ball $B \ni x_0$. Then

$$\begin{aligned}
&\frac{1}{\mu(B)^{1+\alpha}}\int_B |f(x) - A_{t_B}f(x)|\,d\mu(x)\\
&\qquad \le \frac{1}{\mu(B)^{1+\alpha}}\int_B\int_\chi h_{t_B}(x,y)|f(x)-f(y)|\,d\mu(y)\,d\mu(x)\\
&\qquad = \frac{1}{\mu(B)^{1+\alpha}}\int_B\int_{2B} h_{t_B}(x,y)|f(x)-f(y)|\,d\mu(y)\,d\mu(x)\\
&\qquad\quad + \sum_{k=1}^\infty \frac{1}{\mu(B)^{1+\alpha}}\int_B\int_{2^{k+1}B\setminus 2^kB}\\
&\qquad\qquad \times h_{t_B}(x,y)|f(x)-f(y)|\,d\mu(y)\,d\mu(x)\\
&\qquad = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We first estimate I. By the doubling property (2.1), we know that $\mu(B) \le 2^N\mu(B(x,r_B))$ since $x \in B$. For $y \in 2B$ we then have

$$h_{t_B}(x,y) = \frac{g(d^m(x,y)t_B^{-1})}{\mu(B(x,t_B^{1/m}))} \le \frac{g(0)}{\mu(B(x,r_B))} \le \frac{C}{\mu(2B)}.$$

Thus,

$$\begin{aligned}
\mathrm{I} &\le \frac{C}{\mu(B)^{1+\alpha}\mu(2B)}\int_B\int_{2B}|f(x)-f(y)|\,d\mu(y)\,d\mu(x)\\
&\le \frac{C}{\mu(B)^{1+\alpha}\mu(2B)}\int_B\int_{2B}|f(x)-f_{2B}|\,d\mu(y)\,d\mu(x)\\
&\quad + \frac{C}{\mu(B)^{1+\alpha}\mu(2B)}\int_B\int_{2B}|f(y)-f_{2B}|\,d\mu(y)\,d\mu(x)\\
&\le \frac{C}{\mu(B)^{1+\alpha}}\int_B|f(x)-f_{2B}|\,d\mu(x)\\
&\quad + C\frac{\mu(2B)^\alpha}{\mu(B)^\alpha}\frac{1}{\mu(2B)^{1+\alpha}}\int_{2B}|f(y)-f_{2B}|\,d\mu(y)\\
&\le C\|f\|_{L(\alpha,\chi)}.
\end{aligned}$$

Regarding II, for $x \in B$ and $y \in 2^{k+1}B \setminus 2^k B$, we have $d(x,y) \geq 2^{k-1} r_B$. Therefore,

$$h_{t_B} = \frac{g(d^m(x,y) r_B^{-m})}{\mu(B(x,r_B))} \leq C \frac{g(2^{(k-1)m})}{\mu(B)} \leq C \frac{g(2^{(k-1)m}) 2^{(k+1)n}}{\mu(2^{k+1}B)},$$

where we used (2.1). Thus,

$$\mathrm{II} \leq C \sum_{k=1}^{\infty} 2^{kn} \frac{g(2^{(k-1)m})}{\mu(B)^{1+\alpha} \mu(2^{k+1}B)} \int_B \int_{2^{k+1}B} |f(x) - f(y)| \, d\mu(y) \, du(x).$$

We estimate each term as follows:

$$\begin{aligned}
\frac{1}{\mu(B)^{1+\alpha}\mu(2^{k+1}B)} & \int_B \int_{2^{k+1}B} |f(x) - f(y)| \, d\mu(y) \, du(x) \\
& \leq \frac{1}{\mu(B)^{\alpha}\mu(2^{k+1}B)} \int_{2^{k+1}B} |f(y) - f_{2^{k+1}B}| \, d\mu(y) \\
& \quad + \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - f_{2^{k+1}B}| \, d\mu(x) \\
& \leq 2^{k\alpha n} \|f\|_{L(\alpha,\chi)} + \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - f_B| \, d\mu(x) \\
& \quad + \frac{1}{\mu(B)^{\alpha}} |f_B - f_{2B}| + \cdots + \frac{1}{\mu(B)^{\alpha}} |f_{2^k B} - f_{2^{k+1}B}| \\
& \leq 2^{k\alpha n} \|f\|_{L(\alpha,\chi)} + \sum_{l=0}^{k} 2^{ln\alpha} \|f\|_{L(\alpha,\chi)} \\
& \leq C 2^{k\alpha n} \|f\|_{L(\alpha,\chi)}.
\end{aligned}$$

Therefore, by (2.4), we obtain

$$\mathrm{I} \leq C \|f\|_{L(\alpha,\chi)} \sum_{k=0}^{\infty} 2^{k(\alpha+1)n} g(2^{(k-1)m}) \leq C \|f\|_{L(\alpha,\chi)}.$$

Finally, we show that the converse inequality does not hold in general. We consider $\mathbb{R}$ with the Lebesgue measure dx and the approximation of the identity $\{D_t : t > 0\}$ given by the kernel

$$a_t(x,y) = \frac{1}{2t^{1/m}} \chi_{(x - t^{1/m}, x + t^{1/m})}(y).$$

Let us take the function $f(x) = x$. For every $t > 0$, $D_t f(x) = x$ and $\|f\|_{\mathrm{Lip}_A(\alpha,\mathbb{R})} = 0$ for $\alpha > 0$, but $\|f\|_{L(\alpha,\mathbb{R})} = +\infty$ for $0 < \alpha < 1$. Thus, $L(\alpha,\mathbb{R}) \subset \mathrm{Lip}_A(\alpha,\mathbb{R})$ for $0 < \alpha < 1$. □

2.3. Basic properties of $\mathrm{Lip}_A(\alpha,\chi)$**.** In this section, let χ be a space of homogeneous type equipped with a quasi-metric d and a measure μ. We assume that:

(a) $\{A_t\}_{t>0}$ is a generalized approximation to the identity with kernels $a_t(x,y)$ satisfying conditions (2.3) and (2.4).
(b) A_0 is the identity operator and the operators $\{A_t\}_{t>0}$ form a semi-group, that is, for any $t,s>0$ and $f\in\mathcal{M}$, $A_tA_sf(x)=A_{t+s}f(x)$ for almost all $x\in\chi$.

We first prove the following proposition.

Proposition 2.2. *Assume that $\{A_t\}_{t>0}$ satisfies assumptions (a) and (b) above. If $f\in \mathrm{Lip}_A(\alpha,\chi)$ with $\alpha>0$, then for any $t>0$ and $K>1$, we have*

$$|A_tf(x)-A_{Kt}f(x)|\le C\left(K^n\mu(B(x,t^{1/m}))\right)^{\alpha}\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}$$

for almost all $x\in\chi$, where $C>0$ is a constant independent of x and K.

To prove Proposition 2.2, we first recall a result of Christ [2], which gives an analogue of the Euclidean dyadic cubes.

Lemma 2.2. *There exists a collection of open subsets $\{Q^k_\alpha\subset\chi : k\in \mathbb{Z},\ \alpha\in I_k\}$, where I_k denotes some index set depending on k, and constants $\delta\in(0,1)$, $\alpha_0\in(0,1)$, and $0<D<\infty$, such that:*

(i) *$\mu(\chi\setminus\bigcup_\alpha Q^k_\alpha)=0$ for $k\in\mathbb{Z}$.*
(ii) *If $l\ge k$, then either $Q^l_\beta\subset Q^k_\alpha$ or $Q^l_\beta\subset Q^k_\alpha=\emptyset$.*
(iii) *For each (k,α) and each $l<k$ there is a unique β such that $Q^k_\alpha\subset Q^l_\beta$.*
(iv) *The diameter of $(Q_\alpha)^k$ is $\le D\delta^k$.*
(v) *Each Q^k_α contains some ball $B(z^k_\alpha,\alpha_0\delta^k)$.*

Proof of Proposition 2.2. For any $t>0$ we choose s such that $t/4\le s\le t$ with the notation as in Lemma 2.2. First fix l_0 such that $D\delta^{l_0}\le s^{1/m}\le D\delta^{l_0-1}$ and fix a point $x\in\chi$. From conditions (i) and (iv) of Lemma 2.2, we can find a $Q^{l_0}_{\alpha_0}$ such that $x\in Q^{l_0}_{\alpha_0}$ and $Q^{l_0}_{\alpha_0}\subset B(x,D\delta^{l_0})$. For any $k\in\mathbb{N}$ we define M_k by

$$M_k=\{\beta : Q^{l_0}_\beta\bigcap B(x,D\delta^{l_0})\neq\emptyset\}.$$

Again, by (i) and (iv) of Lemma 2.2, we have

$$B(x,D\delta^{l_0-k})\subset\bigcup_{\beta\in M_k}Q^{l_0}_\beta\subset B(x,D\delta^{l_0-(k+k_0)}),$$

where k_0 is an integer such that $\delta^{-k_0}\ge 2C_1$ and C_1 is the constant appearing in the definition of a quasi-metric d.

In [7], X. T. Duong and L. X. Yan proved that there exists a constant $C>0$ independent of k such that the number of open subsets $\{Q^{l_0}_\beta\}_{\beta\in M_k}$ is less than $C\delta^{-k(n+N)}$, that is,

$$m_k=\#\{Q^{l_0}_\beta : \beta\in M_k\}\le C\delta^{-k(n+N)},$$

where N is the power that appeared in property (2.2) and n the "dimension" entering the strong homogeneity property.

We now estimate the term $|A_t f(x) - A_{t+s} f(x)|$ for the case $t/4 \le s \le t$. By property (b) of the semigroup $\{A_t\}_{t>0}$, we can write

$$A_t f(x) - A_{t+s} f(x) = A_t(f - A_s f)(x).$$

Since $f \in \mathrm{Lip}_A(\alpha, \chi)$, we have

$$\begin{aligned}
&|A_t f(x) - A_{t+s} f(x)| \\
&\quad \le \int_\chi h_t(x,y)|f(y) - A_s f(y)|\, d\mu(y) \\
&\quad = \frac{1}{\mu(B(x,t^{1/m}))} \int_\chi g\left(\frac{d^m(x,y)}{t}\right) |f(y) - A_s f(y)|\, d\mu(y) \\
&\quad \le \frac{C}{\mu(B(x,t^{1/m}))} \int_{B(x,t^{1/m})} g\left(\frac{d^m(x,y)}{t}\right) |f(y) - A_s f(y)|\, d\mu(y) \\
&\qquad + \frac{c}{\mu(B(x,t^{1/m}))} \int_{\chi \setminus B(x,t^{1/m})} g\left(\frac{d^m(x,y)}{t}\right) |f(y) - A_s f(y)|\, d\mu(y) \\
&\quad \le C\mu(B(x,t^{1/m}))^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} + \mathrm{I}\,.
\end{aligned}$$

Noting that for any $y \in B(x, D\delta^{l_0-(k+1)}) \setminus B(x, D\delta^{l_0-k})$, we have $d(x,y) \ge D\delta^{l_0-k}$, we obtain

$$\begin{aligned}
\int_{\chi \setminus B(x,t^{1/m})} & g\left(\frac{d^m(x,y)}{t}\right) |f(y) - A_s f(y)|\, d\mu(y) \\
&\le \int_{\chi \setminus B(x,D\delta^{l_0})} g\left(\frac{d^m(x,y)}{t}\right) |f(y) - A_s f(y)|\, d\mu(y) \\
&\le \sum_{k=0}^\infty \int_{B(x,D\delta^{l_0-(k+1)}) \setminus B(x,D\delta^{l_0-k})} g\left(\frac{d^m(x,y)}{t}\right) \\
&\qquad \times |f(y) - A_s f(y)|\, d\mu(y) \\
&\le \sum_{k=0}^\infty g(\delta^{-(k-1)m}/4) \int_{B(x,D\delta^{l_0-(k+1)})} |f(y) - A_s f(y)|\, d\mu(y) \\
&\le \sum_{k=0}^\infty \sum_{\beta \in M_k} g(\delta^{-(k-1)m}/4) \int_{Q_\beta^{l_0}} |f(y) - A_s f(y)|\, d\mu(y).
\end{aligned}$$

Applying (iv) of Lemma 2.2, we get $Q_\beta^{l_0} \subset B(z_\beta^{l_0}, s^{1/m})$. From property (2.2), we have

$$\mu(B(x,s^{1/m}))^{-1} \le C\delta^{-kN} \mu(B(z_\beta^{l_0}, s^{1/m}))$$

for any $\beta \in M_{k+1}$. Thus, using the decay of function g and the estimate $m_k \le C\delta^{-(k+N)}$, we then obtain

$$
\begin{aligned}
\mathrm{I} &\le \sum_{k=0}^{\infty} \sum_{\beta \in M_{k+1}} g(\delta^{-(k-1)m}/4) \int_{Q_\beta^{l_0}} |f(y) - A_s f(y)|\, d\mu(y) \\
&\le \sum_{k=0}^{\infty} m_{k+1} \delta^{-kN} g(\delta^{-(k-1)m}/4) \mu(B(z_\beta^{l_0}, s^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\le \sum_{k=0}^{\infty} \delta^{-k(n+2N)} \delta^{-kN\alpha} g(\delta^{-(k-1)m}/4) \mu(B(x, t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\le C \sum_{k=0}^{\infty} \delta^{-k(n+2N+2N\alpha)} g(\delta^{-km}) \mu(B(x, t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\le C\mu(B(x, t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)}.
\end{aligned}
$$

In the case $0 < s < t/4$ we write

$$A_t f(x) - A_{t+s} f(x) = (A_t f(x) - A_{2t} f(x)) - A_{t+s}(f - A_{t-s} f)(x).$$

Noting that $(t+s)/4 \le t - s < t + s$, the same argument as above applies. In general, for any $K > 1$, we let l be the integer satisfying $2^l \le K < 2^{l+1}$, so that $l \le \log_2 K$. Thus, there exists a constant $C > 0$ independent of t and K such that

$$
\begin{aligned}
&|A_t f(x) - A_{2t} f(x)| \\
&\quad \le \sum_{k=0}^{l-1} |A_{2^k t} f(x) - A_{2^{k+1} t} f(x)| + |A_{2^l t} f(x) - A_{2^{l+1} t} f(x)| \\
&\quad \le \sum_{k=0}^{l-1} \mu(B(x, 2^k t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} + C\mu(B(x, 2^l t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\quad \le \sum_{k=0}^{l-1} 2^{kn\alpha} \mu(B(x, t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} + C 2^{ln\alpha} \mu(B(x, t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\quad \le C K^{n\alpha} \mu(B(x, t^{1/m}))^{\alpha} \|f\|_{\mathrm{Lip}_A(\alpha,\chi)}
\end{aligned}
$$

for all $x \in \chi$. The proof of Proposition 2.2 is complete. □

Using Proposition 2.2, we can prove the following proposition.

PROPOSITION 2.3. *Let m be the positive constant in (2.3). Then there exists a positive constant C such that*

$$\sup_{t>0, x \in \chi} \mu(B(x, t^{1/m}))^{-\alpha} \,|A_t(|f - A_t f|)(x)| \le C\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}.$$

Proof. Assume that $f \in \mathrm{Lip}_A(\alpha, \chi)$. For any fixed $t > 0$ and $x \in \chi$ we choose a ball B centered at x and of radius $r_B = t^{1/m}$. Let $t_{2^k B} = r^m_{2^k B}$. By Proposition 2.2 we have, for all $k \geq 0$,

$$\begin{aligned}
\frac{1}{\mu(2^k B)} &\int_{2^k B} |f(x) - A_t f(x)|\, d\mu(x) \\
&\leq \frac{C}{\mu(2^k B)} \int_{2^k B} \left|f(x) - A_{t_{2^k B}} f(x)\right| d\mu(x) \\
&\quad + C \sup_{x \in 2^k B} \left|A_{t_{2^k B}} f(x) - A_t f(x)\right| \\
&\leq C 2^{nk\alpha} \mu(B(x, t^{1/m}))^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)}.
\end{aligned}$$

From (2.4) we get

$$\begin{aligned}
|A_t(|f - A_t f|)(x)| &\leq C \sum_{k=0}^{\infty} \frac{1}{\mu(B)} \int_{2^k B \setminus 2^{k-1} B} g\left(\frac{d^m(x,y)}{t}\right) \\
&\qquad \times |f(x) - A_t f(x)|\, d\mu(x) \\
&\leq C \sum_{k=0}^{\infty} 2^{kn} g(2^{(k-1)m}) \frac{1}{\mu(2^k B)} \\
&\qquad \times \int_{2^k B} |f(x) - A_t f(x)|\, d\mu(x) \\
&\leq C \sum_{k=0}^{\infty} 2^{kn(1+\alpha)} g(2^{(k-1)m}) \mu(B(x, t^{1/m}))^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\leq C \mu(B(x, t^{1/m}))^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)}.
\end{aligned}$$

Thus, Proposition 2.3 is proved. □

We next show that the average value $A_{t_B} f$ in Definition 2.1 of $\mathrm{Lip}_A(\alpha, \chi)$ can be replaced by other value f^B that satisfies appropriate estimates.

Definition 2.2. Suppose that for a given $f \in \mathcal{M}$ there exists a constant C and a collection of functions $\{f^B(x)\}_B$ (that is, for each ball B, there exists a function $f^B(x)$) such that

$$\sup_B \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - A_{t_B} f(x)|\, d\mu(x) < \infty, \tag{2.7}$$

$$\left|f^{B_2}(x) - f^{B_1}(x)\right| \leq C \left(\frac{r_{B_2}}{r_{B_1}}\right)^{n\alpha} \mu(B(x, r_{B_1}))^\alpha \tag{2.8}$$

for any two balls $B_1 = B(x, r_{B_1}) \subset B_2 = B(x, r_{B_2})$, and for almost all $x \in \chi$,

$$\left|f^B(x) - A_{t_B} f^B(x)\right| \leq C \mu(B(x, r_B))^\alpha, \tag{2.9}$$

where $t_B = r_B^m$. We define

$$\|f\|_{\widetilde{\mathrm{Lip}_A}} = \inf\{C : \ C \ \text{ satisfies (2.7), (2.8) and (2.9)}\},$$

where the infimum is taken over all constants C and the sets of functions $\{f^B(x)\}$ that satisfy (2.7), (2.8) and (2.9).

We have the following equivalence of norms.

Proposition 2.4. *The norms $\|\|_{\mathrm{Lip}_A(\alpha,\chi)}$ and $\|\|_{\widetilde{\mathrm{Lip}_A}(\alpha,\chi)}$ are equivalent.*

Proof. Let $f \in \mathcal{M}$. To see that $\|f\|_{\widetilde{\mathrm{Lip}_A}(\alpha,\chi)} \leq C\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}$, we set $f^B(x) = A_{t_B} f(x)$ for each ball B. Applying Proposition 2.2, the estimates (2.5), (2.7) and (2.8) hold with the constant $C = C_1\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}$.

It remains to prove that, for any fixed ball B centered at x_0 and the radius r_B,

$$\sup_B \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - A_{t_B} f(x)|\, d\mu(x) \leq C\|f\|_{\widetilde{\mathrm{Lip}_A}(\alpha,\chi)},$$

where $t_B = r_B^m$. For any $x \in B$, by (2.8) we have

$$\begin{aligned}
&\left|A_{t_B}(f - f^B)(x)\right| \\
&\qquad \leq \frac{1}{\mu(B(x, t_B^{1/m}))} \int_\chi g\left(\frac{d^m(x,y)}{t_B}\right) |f(y) - f^B(y)|\, d\mu(y) \\
&\qquad \leq C \sum_{k=0}^{\infty} \frac{1}{\mu(B)} \int_{2^k B \setminus 2^{k-1} B} g\left(\frac{d^m(x,y)}{t_B}\right) |f(y) - f^B(y)|\, d\mu(y) \\
&\leq C \sum_{k=0}^{\infty} 2^{kn(1+\alpha)} g(2^{(k-2)m}) \frac{\mu(B)^\alpha}{\mu(2^k B)^{1+\alpha}} \int_{2^k B} |f(y) - f^{2^k B}(y)|\, d\mu(y) \\
&\qquad + C \sum_{k=0}^{\infty} 2^{kn} g(2^{(k-2)m}) \sup_{y \in 2^k B} \left| f^B(y) - f^{2^k B}(y) \right| \\
&\leq C \sum_{k=0}^{\infty} 2^{kn(1+\alpha)} g(2^{(k-2)m}) \mu(B)^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\qquad + C \sum_{k=0}^{\infty} 2^{kn+(n+N)\alpha} g(2^{(k-2)m}) \mu(B)^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
&\leq C\mu(B)^\alpha \|f\|_{\mathrm{Lip}_A(\alpha,\chi)}.
\end{aligned}$$

From (2.7), (2.9), and the above inequality we obtain

$$
\begin{aligned}
&\frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - A_{t_B} f(x)|\, d\mu(x) \\
&\qquad \le \frac{1}{\mu(B)^{1+\alpha}} \int_B |f(x) - f^B(x)|\, d\mu(x) \\
&\qquad\quad + \frac{1}{\mu(B)^{1+\alpha}} \int_B |f^B(x) - A_{t_B} f^B(x)|\, d\mu(x) \\
&\qquad\quad + \frac{1}{\mu(B)^{1+\alpha}} \int_B |A_{t_B}(f - A_{t_B} f)(x)|\, d\mu(x) \\
&\qquad \le C\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}.
\end{aligned}
$$

Thus, the proof of Proposition 2.4 is complete. □

3. A variant of the John-Nirenberg inequality on $\mathrm{Lip}_A(\alpha, \chi)$

We continue to assume that the operators $\{A_t\}_{t>0}$ satisfy properties (a) and (b) in Section 2. In this section, we will prove a variant of the John-Nirenberg inequality for the space $\mathrm{Lip}_A(\alpha, \chi)$ associated with the semigroup $\{A_t\}_{t>0}$ by using Proposition 2.2 and adapting the arguments of pages 1398–1400 in [7].

Theorem 3.1. *If $f \in \mathrm{Lip}_A(\alpha, \chi)$, there exists positive constant c_1 and c_2 such that for every ball B and every $\lambda > 0$, we have*

$$
\begin{aligned}
\mu\{x \in B : |f(x) - A_{t_B} f(x)| > \lambda\}& \\
\le c_1 \mu(B) \exp&\left\{ - \frac{c_2 \lambda}{\|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \mu(B)^{\alpha}} \right\},
\end{aligned}
\tag{3.1}
$$

where $t_B = r_B^m$.

Proof. In order to prove (3.1), it is enough to consider the case $\|f\|_{\mathrm{Lip}_A(\alpha,\chi)} > 0$. We may assume that $\|f\|_{\mathrm{Lip}_A(\alpha,\chi)} = 1$ because inequality (3.1) does not change if we replace f by Cf, where C is a constant. We need to prove that for a fixed $B \subset \chi$,

$$
\mu\{x \in B : |f(x) - A_{t_B} f(x)| > \lambda\} \le c_1 \mu(B) \exp\left\{ - \frac{c_2 \lambda}{\mu(B)^{\alpha}} \right\},
\tag{3.2}
$$

where $t_B = r_B^m$.

Denote by $B = B(x_0, r_B)$ a ball centered at x_0 and of radius r_B. We fix the ball B in χ and set $f_0 = (f - A_{t_B} f)\chi_{10C_1^4 B}$, where C_1 is the constant appearing in the definition of a quasi-metric d in Section 2. By Proposition 2.2 we have

$$
\|f_0\|_{L^1(\chi)} \le \int_{10C_1^4 B} |f(x) - A_{t_B} f(x)|\, d\mu(x) \le C\mu(B)^{1+\alpha}.
$$

Denote by M the Hardy-Littlewood maximal operator. Take $\beta > 1$ and define two sets F and Ω as follows:

$$F = \{x : M(f_0) \le \beta\mu(B)^\alpha\} \ \text{ and } \ \Omega = F^c = \{x : M(f_0) > \beta\mu(B)^\alpha\}.$$

By Theorem 1.3 of Chapter III in [3] there exists a collection of balls $B_{1,1}, B_{1,2}, \dots, B_{1,i}, \dots$, satisfying;

(i) $\bigcup_i B_{1,i} = \Omega$.
(ii) Each point of Ω is contained in at most a fine number L of the balls $B_{1,i}$.
(iii) There exists $c > 1$ such that $cB_{1,i} \bigcap F \neq \emptyset$ for each i.

Property (i) implies that for any $x \in B \setminus (\cup_i B_{1,i})$,

$$|f(x) - A_{t_B} f(x)| = |f_0(x)|\chi_F(x) \le M(f_0)(x)\chi_F(x) \le \beta\mu(B)^\alpha.$$

Since the Hardy-Littlewood maximal operator is of weak type (1,1), it follows from (i) and (ii) that

$$\sum_i \mu(B_{1,i}) \le L\mu(\Omega) \le \frac{C}{\beta\mu(B)^\alpha}\|f_0\|_1 \le \frac{c_3}{\beta}\mu(B)$$

for some $c_3 > 0$.

For any $B_{1,i} \bigcap B \neq \emptyset$ we denote by $B_{1,i} = B(x_{B_{1,i}}, r_{B_{1,i}})$ a ball centered at $x_{B_{1,i}}$ and of radius $r_{B_{1,i}}$. Then we have

$$\mu(B) \le C\left(\frac{r_B}{r_{B_{1,i}}}\right)^n \mu(B(x_0, r_{B_{1,i}})) \le \frac{c_4}{\beta}\left(\frac{r_B}{r_{B_{1,i}}}\right)^{n+N}\mu(B)$$

for some $c_4 > 0$ and n and N as above.

We choose β such that $\beta > \min\{c_4(10C_1)^{n+N}, c_3^2\}$. Obviously, $r_B > 10C_1 r_{B_{1,i}}$. This implies that for any $B_{1,i} \bigcap B \neq \emptyset$ we have $B_{1,i} \subset 2C_1 B$. We now prove that for any $B_{1,i} \bigcap B \neq \emptyset$ there exists a constant c_5 such that

$$|A_{t_{B_{1,i}}} f(x) - A_{t_B} f(x)| \le c_5\beta\mu(B)^\alpha \ \text{ for all } x \in B_{1,i}. \tag{3.3}$$

Using property (b) of the semigroup $\{A_t\}_{t>0}$, we write

$$A_{t_{B_{1,i}}} f(x) - A_{t_B} f(x) = A_{t_{B_{1,i}}}(f - A_{t_B} f)(x) + (A_{t_{B_{1,i}} + t_B} f(x) - A_{t_B} f(x)).$$

Because $t_{B_{1,i}} + t_B$ and t_B have comparable sizes, applying Proposition 2.2 we obtain

$$|A_{t_{B_{1,i}} + t_B} f(x) - A_{t_B} f(x)| \le C\mu(B(x, r_B))^\alpha \le C\beta\mu(B(x_0, r_B))^\alpha \ \text{ for } x \in B_{1,i}.$$

Hence, in order to prove (3.3), we need only to show that

$$|A_{t_{B_{1,i}}}(f - A_{t_B} f)(x)| \le C\mu(B(x_0, r_B))^\alpha \ \text{ for all } x \in B_{1,i}. \tag{3.4}$$

Let q_i be minimal so that $2C_1^2 B \subset 2^{q_i+1} B_{1,i}$ and $2C_1^2 B \bigcap (2^{q_i+1} B_{1,i})^c \neq \emptyset$. For any $z \in 2^{q_i+1} B_{1,i}$ we have $d(x_0, z) \le 10C_1^4 r_B$ and $2^{q_i+1} B_{1,i} \subset 10C_1^4 B$.

Therefore

$$
\begin{aligned}
&|A_{t_{B_{1,i}}}(f - A_{t_B} f)(x)| \\
&\quad \leq C \frac{1}{\mu(B_{1,i})} \int_{\mathcal{X}} g\left(\frac{d^m(x,y)}{t_{B_{1,i}}}\right) |f(y) - A_{t_B} f(y)|\, d\mu(y) \\
&\quad \leq C \frac{1}{\mu(B_{1,i})} \int_{\mathcal{X}} g\left(\frac{d^m(x,y)}{t_{B_{1,i}}}\right) |f(y) - A_{t_B} f(y)|\, d\mu(y) \\
&\quad \leq C \sum_{k=1}^{q_i+1} \frac{1}{\mu(B_{1,i})} \int_{2^k B_{1,i} \setminus 2^{k-1} B_{1,i}} g\left(\frac{d^m(x,y)}{t_{B_{1,i}}}\right) \\
&\qquad\qquad \times |f(y) - A_{t_B} f(y)|\, d\mu(y) \\
&\qquad + C \frac{1}{\mu(B_{1,i})} \int_{\mathcal{X} \setminus 2^{q_i+1} B_{1,i}} g\left(\frac{d^m(x,y)}{t_{B_{1,i}}}\right) \\
&\qquad\qquad \times |f(y) - A_{t_B} f(y)|\, d\mu(y) \\
&\quad = \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

It follows immediately from property (iii) of the balls $B_{1,i}$ that there is a positive constant C independent of k such that

$$\frac{1}{\mu(2^k B_{1,i})} \int_{2^k B_{1,i}} |f_0(x)|\, d\mu(x) \leq C\beta\mu(B)^\alpha.$$

Hence, for $k = 0, 1, 2, \ldots, q_i + 1$ we have

$$
\begin{aligned}
\frac{1}{\mu(2^k B_{1,i})} &\int_{2^k B_{1,i}} |f(x) - A_{t_B} f(x)|\, d\mu(x) \\
&= \frac{1}{\mu(2^k B_{1,i})} \int_{2^k B_{1,i}} |f_0(x)|\, d\mu(x) \leq C\beta\mu(B)^\alpha,
\end{aligned}
\tag{3.5}
$$

since $2^{q_{i+1}} B_{1,i} \subset 10C_1^4 B$. For any $x \in B_{1,i}$ and $y \in 2^k B_{1,i} \setminus 2^{k-1} B_{1,i}$, $k = [\log_2 C_1] + 2, \ldots$ there exists a constant $c_6 > 0$ such that $d(y,x) \geq c_6 2^k r_{B_{1,i}}$. (Here $[\log_2 C_1]$ denotes the integer part of $\log_2 C_1$.) Hence, from (3.5) and (2.4) we get

$$
\begin{aligned}
\mathrm{I} &\leq C \sum_{k=0}^{[\log_2 C_1]+1} 2^{kn} g(0) \frac{1}{\mu(2^k B_{1,i})} \int_{2^k B_{1,i}} |f(x) - A_{t_B} f(x)|\, d\mu(x) \\
&\quad + \sum_{k=[\log_2 C_1]+2}^{q_i+1} 2^{kn} g(c_6^m 2^{km}) \frac{1}{\mu(2^k B_{1,i})} \int_{2^k B_{1,i}} |f(x) - A_{t_B} f(x)|\, d\mu(x) \\
&\leq C\beta\mu(B)^\alpha \sum_{k=0}^{[\log_2 C_1]+1} 2^{kn} g(0) + C\beta\mu(B)^\alpha \sum_{k=[\log_2 C_1]+2}^{q_i+1} 2^{kn} g(c_6^m 2^{km}) \\
&\leq C\beta\mu(B)^\alpha.
\end{aligned}
$$

To estimate II, we let p_i be an integer such that $2^{p_i} r_{B_{1,i}} \le r_B < 2^{p_i+1} r_{B_{1,i}}$. Set $2^{-1}B_{1,i} = \emptyset$. We then have $\mu(B(x_0, r_{B_{1,i}})) \le C2^{p_i N}\mu(B_{1,i})$. For any $x \in B_{1,i}$ and $y \in 2^k B_{1,i} \setminus 2^{k-1}B_{1,i}$, $k = [2\log_2 C_1] + 2, \dots$ there exists a constant $c_7 > 0$ such that $d(y,x) \ge c_7 2^{k+p_i} r_{B_{1,i}}$. Thus,

$$\begin{aligned}
\mathrm{II} \le C & \sum_{k=[2\log_2 C_1]+1}^{\infty} \frac{1}{\mu(B_{1,i})} \int_{2^k B_{1,i}\setminus 2^{k-1}B_{1,i}} g\left(\frac{d^m(x,y)}{t_{B_{1,i}}}\right) \\
&\qquad \times |f(y) - A_{t_B} f(y)|\, d\mu(y) \\
\le C & \sum_{k=[2\log_2 C_1]+1}^{\infty} 2^{p_i N} 2^{(k+p_i)n} g(c_7^m 2^{(k+p_i)m}) \\
&\qquad \times \frac{1}{\mu(2^{k+1}B)} \int_{2^{k+1}B} |f(x) - A_{t_B} f(x)|\, d\mu(x) \\
\le C & \sum_{k=[2\log_2 C_1]+1}^{\infty} 2^{(k+p_i)(n+N+(n+N)\alpha)} g(c_7^m 2^{(k+p_i)m}) \|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \\
\le C & \mu(B)^\alpha \le C\beta\mu(B)^\alpha.
\end{aligned}$$

Combining the above estimates of I and II, we obtain (3.4). Estimate (3.3) then follows.

On each $B_{1,i}$, we again use the decomposition in Theorem 1.3 of Chapter III in [2] of the function

$$f_{1,i}(x) = (f - A_{B_{1,i}} f)(x)\chi_{10C_1^4 B_{1,i}}(x)$$

with same value $\beta\mu(B)^\alpha$. We then obtain a collection of balls $\{B_{2,m}\}$ for any $x \in B_{1,i} \setminus (\bigcup_m B_{2,m})$ such that $|f(x) - A_{t_{B_{1,i}}} f(x)| \le \beta\mu(B)^\alpha$ and

$$\sum_m \mu(B_{2,m}) \le \frac{c_3}{\beta\mu(B)^\alpha}\mu(B_{1,i})^{1+\alpha} \le \frac{c_3}{\beta}\mu(B_{1,i}).$$

Also, for any $B_{2,m} \bigcap B_{1,i} \ne \emptyset$ we have

$$|A_{t_{B_{1,i}}} f(x) - A_{t_{B_{2,m}}} f(x)| \le c_5\beta\mu(B)^\alpha \quad \text{for all } x \in B_{2,m}.$$

Now we combine all families $\{B_{2,m}\}$ corresponding to different $B_{1,i}$'s and still call the resulting family $\{B_{2,m}\}$. Then we have

$$|f(x) - A_{t_B} f(x)| \le |f(x) - A_{t_{B_{1,i}}} f(x)| + |A_{t_B} f(x) - A_{t_{B_{1,i}}} f(x)| \le 2c_5\beta\mu(B)^\alpha$$

for $x \in B \setminus (\bigcup_m B_{2,m})$, and so

$$\sum_m \mu(B_{2,m}) \le \left(\frac{c_3}{\beta}\right)^2 \mu(B).$$

We then obtain for each natural number K a family of balls $\{B_{k,m}\}$ such that outside of their union we have

$$|f(x) - A_{t_B} f(x)| \le K c_5 \beta \mu(B)^\alpha, \quad x \in B \setminus \left(\bigcup_m B_{K,m} \right)$$

and

$$\sum_m \mu(B_{K,m}) \le \left(\frac{c_3}{\beta} \right)^K \mu(B).$$

If $K c_5 \beta \mu(B)^\alpha \le \lambda < (K+1) c_5 \beta \mu(B)^\alpha$ with $K = 1, 2, \ldots$, using the condition $\beta > c_3^2$, we then obtain

$$\begin{aligned} \mu\{x \in B : |f(x) - A_{t_B} f(x)| > \lambda\} &\le \sum_m \mu(B_{K,m}) \le \left(\frac{c_3}{\beta} \right)^K \mu(B) \\ &\le e^{-(K \log_2)/2} \mu(B) \\ &\le \sqrt{\beta} e^{-\frac{\lambda \log \beta}{4 c_5 \beta}} \mu(B). \end{aligned}$$

On the other hand, if $\lambda < c_5 \beta \mu(B)^\alpha$, we have

$$\mu\{x \in B : |f(x) - A_{t_B} f(x)| > \lambda\} \le \mu(B) \le e^{1 - \frac{\lambda}{c_5 \beta \mu(B)^\alpha}} \mu(B).$$

Thus, we obtain (3.2) by choosing

$$c_1 = \max(e, \sqrt{\beta}) \quad \text{and } c_2 = \frac{\min\{(\log \beta)/4, 1\}}{c_5 \beta}.$$

Thus, Theorem 3.1 is proved. □

As a consequence of Theorem 3.1, we obtain the following theorem, which is equivalent to Theorem 3.1.

Theorem 3.2. *Suppose that f is in $\mathrm{Lip}_A(\alpha, \chi)$. There exist positive constants λ and C such that*

$$\sup_B \frac{1}{\mu(B)} \int_B \exp\left\{ \frac{\lambda}{\|f\|_{\mathrm{Lip}_A(\alpha,\chi)} \mu(B)^\alpha} |f(x) - A_{t_B} f(x)| \right\} d\mu(x) \le C,$$

where $t_B = r_B^m$.

Proof. We choose $\lambda = c_2/2$, where c_2 is the constant in Theorem 3.1. We then have

$$
\begin{aligned}
&\int_B \exp\left\{\frac{\lambda}{\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}\mu(B)^\alpha}|f(x)-A_{t_B}f(x)|\right\}d\mu(x)\\
&=\int_0^\infty \mu\left\{x\in B:\exp\left\{\frac{\lambda}{\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}\mu(B)^\alpha}|f(x)-A_{t_B}f(x)|\right\}>t\right\}dt\\
&\le \mu(B)\\
&\qquad+\int_1^\infty \mu\left\{x\in B:|f(x)-A_{t_B}f(x)|>\frac{\log t\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}\mu(B)^\alpha}{\lambda}\right\}dt\\
&\le \mu(B)+c_1\mu(B)\int_1^\infty \exp\left\{-\frac{c_2\log t}{\lambda}\right\}dt\\
&\le \mu(B)+c_1\mu(B)\int_1^\infty t^{-c_2/\lambda}\,dt\\
&\le C\mu(B).
\end{aligned}
$$

Thus, Theorem 3.2 is proved. □

DEFINITION 3.1. Given $p\in[1,\infty)$, we now define the space $\mathrm{Lip}_A^p(\alpha,\chi)$ as follows: $f\in\mathcal{M}$ is in $\mathrm{Lip}_A^p(\alpha,\chi)$ if there exists some constant C such that for any ball B,

$$
\sup_B \frac{1}{\mu(B)^\alpha}\left(\frac{1}{\mu(B)}\int_B |f(x)-A_{t_B}f(x)|^p\right)^{1/p}\mu(x)<\infty, \tag{3.6}
$$

where $t_B=r_B^m$ and r_B is the radius of the ball.

The smallest bound C for which (3.6) is satisfied is then taken to be the norm of f in this space and is denoted by $\|f\|_{\mathrm{Lip}_A^p(\alpha,\chi)}$.

We have the following result.

THEOREM 3.3. *For $1\le p<\infty$ the spaces $\|f\|_{\mathrm{Lip}_A^p(\alpha,\chi)}$ coincide, and the norms $\|\cdot\|_{\mathrm{Lip}_A^p}$ are equivalent for different values of p.*

Proof. For any $f\in\mathcal{M}$, by Hölder's inequality we have $\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}\le C\|f\|_{\mathrm{Lip}_A^p(\alpha,\chi)}$. To obtain the converse inequality, we apply Theorem 3.1. If $f\in\mathrm{Lip}_A(\alpha,\chi)$, then

$$
\begin{aligned}
&\int_B |f(x)-A_{t_B}f(x)|^p\,d\mu(x)\\
&\qquad=p\int_0^\infty \lambda^{p-1}\mu\{x\in B:|f(x)-A_{t_B}f(x)|>\lambda\}\,d\lambda\\
&\qquad\le C_p\int_0^\infty \lambda^{p-1}\exp\left\{-\frac{c_2\lambda}{\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}\mu(B)^\alpha}\right\}d\lambda\mu(B)\\
&\qquad\le C_p\|f\|^p_{\mathrm{Lip}_A(\alpha,\chi)}\mu(B)^{p\alpha}\mu(B).
\end{aligned}
$$

Hence $\|f\|_{\mathrm{Lip}_A^p(\alpha,\chi)}\le C_p\|f\|_{\mathrm{Lip}_A(\alpha,\chi)}$. □

4. Applications

In this section, we consider the $\mathrm{Lip}_A(\alpha,\chi)$-boundedness ($\alpha > 0$) of a fractional integral that is similar to the singular integral introduced in [5]; see also [8]. The fractional integral is defined in the following way:

$$I_\beta f(x) = \int_\chi k(x,y) f(y)\, d\mu(y) \quad \text{for } 0 < \beta < 1,$$

if the kernel $k(x,y)$ satisfies the following two conditions:

(a) There exists a positive constant C_1 such that

$$|k(x,y)| \le C_1 \mu(x, d(x,y))^{\beta-1} \quad \text{for all } x, y \in \chi;$$

(b) There exists a generalized approximation to the identity $\{A_t\}_{t>0}$ satisfying (2.3) and (2.4) such that the operator $(I_\beta - A_t I_\beta)$ has associated kernels $k_t(x,y)$ and

$$|k_t(x,y)| \le C_2 \frac{1}{\mu(B(x,d(x,y)))^{1-\beta}} \frac{t^{\delta/m}}{d^\delta(x,y)}, \quad \text{when } d(x,y) \ge C_3 t^{1/m},$$

for some $C_2,\ C_3,\ \delta > 0$. (In fact, without loss of generality, in what follows we will assume that $C_3 = 1$.)

It is well known that I_β is bounded from $L^p(\chi)$ to $L^q(\chi)$ with $1/q = 1/p - \beta$ and $1 < p < 1/\beta$. See page 91 in [1].

Next we will prove the boundedness of fractional integrals on the space $\mathrm{Lip}_A(\alpha,\chi)$.

THEOREM 4.1. *Let $0 < \beta < 1$, $1/\beta \le p < \infty$, and $\alpha = \beta - 1/p$. Assume that I_β is an operator satisfying the above conditions (a) and (b) with $\delta > n\alpha$. Then there exists a constant C such that*

$$\|I_\beta f\|_{\mathrm{Lip}_A(\alpha,\chi)} \le C \|f\|_{L^p(\chi)}$$

for all $f \in L^1(\chi) \bigcap L^p(\chi)$.

Proof. It suffices to prove that for any ball

$$\frac{1}{\mu(B)^{1+\alpha}} \int_B |I_\beta f(x) - A_{t_B} I_\beta f(x)|\, d\mu(x) \le C \|f\|_{L^p(\chi)},$$

where $t_B = r_B^m$.

Let $f \in L^1(\chi) \bigcap L^p(\chi)$. Since

$$A_{t_B} I_\beta f(x) = \int_\chi a_{t_B}(x,y) I_\beta f(y)\, d\mu(y)$$

and the kernels $a_{t_B}(x,y)$ of A_{t_B} satisfy (2.3) and (2.4), we have

$$|A_{t_B} I_\beta f(x)| \le C M(I_\beta f)(x)$$

for all $x \in \chi$, where M denotes the Hardy-Littlewood maximal operator; see [12] and [7]. Let $f_1 = f\chi_{4C_1B}$ and $f_2 = f - f_1$. We write

$$I_\beta f - A_{t_B} I_\beta f = (I_\beta f_1 - A_{t_B} I_\beta f_1) + (I_\beta - A_{t_B} I_\beta) f_2.$$

We then have

$$\begin{aligned}
&\int_B |I_\beta f(x) - A_t I_\beta f(x)|\, d\mu(x) \\
&\quad \le \int_B |I_\beta f_1(x) - A_{t_B} I_\beta f_1(x)| + |(I_\beta - A_{t_B} I_\beta) f_2(x)|\, d\mu(x) \\
&\quad \le C \int_B M(I_\beta f_1)(x)\, d\mu(x) + \int_B |(I_\beta - A_{t_B} I_\beta) f_2(x)|\, d\mu(x) \\
&\quad = \mathrm{I} + \mathrm{II}\,.
\end{aligned}$$

For I, let $1/q = 1/p_1 - \beta$ and $1 < p_1 < 1/\beta$. Then

$$\begin{aligned}
\mathrm{I} &\le C\mu(B)^{1/q'} \left(\int_\chi M^q(I_\beta f_1)(x)\, d\mu(x) \right)^{1/q} \\
&\le C\mu(B)^{1/q'} \left(\int_\chi |I_\beta f_1(x)|^q\, d\mu(x) \right)^{1/q} \\
&\le C\mu(B)^{1/q'} \left(\int_{4C_1B} |f(x)|^{p_1}\, d\mu(x) \right)^{1/p_1} \\
&\le C\mu(B)^{1+\alpha} \|f\|_{L^p(\chi)},
\end{aligned}$$

where $1/q + 1/q' = 1$.

For II, using (2.1), condition (b), and Hölder's inequality, we have

$$\begin{aligned}
\mathrm{II} &\le \int_B \int_{(4C_1B)^c} |k_t(x,y)||f(y)|\, d\mu(y)\, d\mu(x) \\
&\le C\|f\|_{L^p(\chi)} \int_B \left(\int_{(4C_1B)^c} |k_t(x,y)|^{1/p'}\, d\mu(y) \right)^{1/p'} d\mu(x) \\
&\le C\|f\|_{L^p(\chi)} \int_B \Bigg(\sum_{k=1}^\infty r_B^{\delta p'} \int_{2^{k-1}r_B \le d(x,y) < 2^k r_B} \\
&\qquad \times \mu(B(x,d(x,y)))^{(\beta-1)p'} d(x,y)^{-\delta p'}\, d\mu(y) \Bigg)^{1/p'} d\mu(x) \\
&\le C\mu(B)^{1+\alpha} \|f\|_{L^p(\chi)} \left(\sum_{k=1}^\infty 2^{-k\delta p'} 2^{kn[1+p'(\beta-1)]} \right)^{1/p'} \\
&\le C\mu(B)^{1+\alpha} \|f\|_{L^p(\chi)},
\end{aligned}$$

since $\delta > n(\beta - 1/p)$ and $1/p + 1/p' = 1$.

Thus, the proof of Theorem 4.1 is complete. □

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References

[1] M. Bramanti and M. C. Cerutti, *Commutators of singular integrals and fractional integrals on homogeneous spaces*, Harmonic analysis and operator theory (Caracas, 1994), Contemp. Math., vol. 189, Amer. Math. Soc., Providence, RI, 1995, pp. 81–94. MR 1347007 (96m:42024)

[2] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), 601–628. MR 1096400 (92k:42020)

[3] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Springer-Verlag, Berlin, 1971, Étude de certaines intégrales singulières, Lecture Notes in Mathematics, Vol. 242. MR 0499948 (58 #17690)

[4] D. Deng, X. T. Duong, and L. Yan, *A characterization of the Morrey-Campanato spaces*, Math. Z. **250** (2005), 641–655. MR 2179615 (2006g:42039)

[5] X. T. Duong and A. MacIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Rev. Mat. Iberoamericana **15** (1999), 233–265. MR 1715407 (2001e:42017a)

[6] X. T. Duong and D. W. Robinson, *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*, J. Funct. Anal. **142** (1996), 89–128. MR 1419418 (97j:47056)

[7] X. T. Duong and L. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58** (2005), 1375–1420. MR 2162784 (2006i:26012)

[8] ———, *On commutators of fractional integrals*, Proc. Amer. Math. Soc. **132** (2004), 3549–3557. MR 2084076 (2005e:42046)

[9] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. MR 807149 (87d:42023)

[10] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR 0131498 (24 #A1348)

[11] S. Janson, M. Taibleson, and G. Weiss, *Elementary characterizations of the Morrey-Campanato spaces*, Harmonic analysis (Cortona, 1982), Lecture Notes in Math., vol. 992, Springer, Berlin, 1983, pp. 101–114. MR 729349 (85k:46033)

[12] J. M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math. **161** (2004), 113–145. MR 2033231 (2005b:42016)

[13] C. B. Morrey, Jr., *Partial regularity results for non-linear elliptic systems*, J. Math. Mech. **17** (1967/1968), 649–670. MR 0237947 (38 #6224)

[14] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. **33** (1979), 257–270. MR 546295 (81c:32017a)

[15] J. Peetre, *On the theory of $\mathcal{L}_{p,\ \lambda}$ spaces*, J. Functional Analysis **4** (1969), 71–87. MR 0241965 (39 #3300)

[16] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. MR 1232192 (95c:42002)

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