Illinois Journal of Mathematics
Volume 51, Number 2, Summer 2007, Pages 583–596
S 0019-2082

FINITE RANK COMMUTATORS AND SEMICOMMUTATORS OF TOEPLITZ OPERATORS WITH HARMONIC SYMBOLS

KUNYU GUO, SHUNHUA SUN, AND DECHAO ZHENG

ABSTRACT. In this paper we completely characterize finite rank semicommutator or commutator of two Toeplitz operators with bounded harmonic symbols on the Bergman space. We show that if the product of two Toeplitz operators with bounded harmonic symbols has finite rank, then one of the Toeplitz operators must be zero.

1. Introduction

Let dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. The *Bergman space* L_a^2 is the Hilbert space consisting of the analytic functions on D that are also in $L^2(D, dA)$. For $z \in D$, the *Bergman reproducing kernel* is the function $K_z \in L_a^2$ such that

$$h(z) = \langle h, K_z \rangle$$

for every $h \in L_a^2$. The *normalized Bergman reproducing kernel* k_z is the function $K_z/\|K_z\|_2$. Here the norm $\| \ \|_2$ and the inner product $\langle \ , \ \rangle$ are taken in the space $L^2(D, dA)$.

For $f \in L^\infty(D, dA)$, the *Toeplitz operator* T_f with symbol f is the operator on L_a^2 defined by $T_f h = P(fh)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L_a^2. We denote the semicommutator and commutator of two Toeplitz operators T_f and T_g by

$$(T_f, T_g] = T_{fg} - T_f T_g$$

and

$$[T_f, T_g] = T_f T_g - T_g T_f,$$

Received June 2, 2005; received in final form December 9, 2005.

2000 *Mathematics Subject Classification.* Primary 47B35, 47B47.

The first author was supported by the National Outstanding Youth Secience Foundation of China and SRFDP. The second author was supported in part by the National Natural Science Foundation-10471041 of China. The third author was partially supported by the National Science Foundation.

respectively. Note that if $g \in H^\infty(D)$ (the set of bounded analytic functions on D), then T_g is just the operator of multiplication by g on L_a^2 and hence $(T_f, T_g] = 0$ for any $f \in L^\infty(D, dA)$.

For a bounded operator S on L_a^2, the *Berezin transform* of S is the function $B(S)$ on D defined by

$$B(S)(z) = \langle Sk_z, k_z \rangle.$$

The Berezin transform $B(u)(z)$ of a function $u \in L^\infty(D, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_u. In other words,

$$B(u)(z) = B(T_u)(z) = \int_D u\left(\frac{z-w}{1-\bar{z}w}\right) dA(w).$$

The last equality follows from a change of variable in the definition of the Berezin transform. The above integral formula extends the Berezin transform to $L^1(D, dA)$ and clearly gives

$$B(u)(z) = u(z) \tag{1}$$

for any harmonic function $u \in L^1(D, dA)$.

Let Δ denote the Laplace operator $4\frac{\partial^2}{\partial z \partial \bar{z}}$. A function h on D is harmonic if $\Delta h(z) \equiv 0$ on D. We use $\tilde{\Delta}$ to denote the invariant Laplace operator $(1-|z|^2)^2 4\frac{\partial^2}{\partial z \partial \bar{z}}$. The invariant Laplace operator commutes with the Berezin transform [1], [8], which is useful in studying Toeplitz operators on the Bergman space [1].

An operator A on a Hilbert space H is said to have *finite rank* if the closure of $\operatorname{Ran}(A)$ of the range $A(H)$ of the operator has finite dimension. For a bounded operator A on H, define $\operatorname{rank}(A) = \dim \operatorname{Ran}(A)$. If A has finite rank, then $\operatorname{rank}(A) < \infty$.

In this paper we study the problem for which bounded harmonic functions f, g on the unit disk, the semicommutator $(T_f, T_g]$ or commutator $[T_f, T_g]$ has finite rank on the Bergman space. The analogous problem on the Hardy space has been completely solved in [3], [7]. We will reduce the problem to the problem of when a Toeplitz operator has finite rank. Although the problem on finite rank Toeplitz operators remains open, Ahern and Čučković [1] have shown that for $u \in L^\infty(D)$, if T_u has rank one, then $u = 0$. One naturally conjectures that for $u \in L^\infty(D)$, if T_u has finite rank, then $u = 0$. In this paper, we will show that this conjecture is true provided that u is a finite sum of products of an analytic function and a co-analytic function in $L^2(D, dA)$. Using this result we shall completely characterize finite rank semicommutators and commutators of two Toeplitz operators with bounded harmonic symbols. The zero semicommutator and commutators of two Toeplitz operators with bounded harmonic symbols have been completely characterized in [4] and [13]. In fact, we shall show that if the semicommutator or the commutator of two Toeplitz operators with bounded harmonic symbols has finite rank, then it must be zero. This is not the case on the Hardy space [3], [7]. Moreover, on the Bergman space there exist nonzero compact semicommutators or commutators of two Toeplitz operators with bounded harmonic symbols [11], [13]. We will show that for two bounded harmonic functions $f,\ g$, if the product T_fT_g has finite rank, then either f or g equals 0, which extends the result on zero products of Toeplitz operators in [1].

2. Toeplitz operators

In this section, we study Toeplitz operators with finite rank. Some notation is needed. For a family $\{A_n\}$ of operators on the Hilbert space H and an operator A on H, we say that A_n converges to A in weak operator topology, if for each $x,\ y \in H$,

$$\lim_{n\to}\langle A_n x, y\rangle = \langle Ax, y\rangle.$$

The following result is implicitly contained in Lemma 3.1 in [10]. We include a proof for completeness.

Lemma 1. *Suppose that A_n and A are bounded operators on the Hilbert space H. If A_n converges to A in the weak operator topology, then*

$$\operatorname{rank}(A) \le \liminf_{n\to\infty} \operatorname{rank}(A_n).$$

Proof. Let l denote $\liminf_{n\to\infty}\operatorname{rank}(A_n)$. We only need to consider the case $l < \infty$. We claim that $\operatorname{rank}(A) \le l$. If this is false, we may assume that $\operatorname{rank}(A) \ge l+1$. Thus there are $(l+1)$ elements $\{x_j\}_{j=1}^{l+1}$ in H such that $\{Ax_j\}_{j=1}^{l+1}$ are linearly independent and so

$$\det[\langle Ax_i, Ax_j\rangle]_{(l+1)\times(l+1)} \neq 0,$$

where $\det[\langle Ax_i, Ax_j\rangle]_{(l+1)\times(l+1)}$ denotes the determinant of the $(l+1)\times(l+1)$ matrix $[\langle Ax_i, Ax_j\rangle]_{(l+1)\times(l+1)}$. Since A_n converges to A in the weak operator topology, for each $i,\ j$,

$$\lim_{n\to\infty}\langle A_n x_i, Ax_j\rangle = \langle Ax_i, Ax_j\rangle.$$

This gives

$$\lim_{n\to\infty}\det[\langle A_n x_i, Ax_j\rangle]_{(l+1)\times(l+1)} = \det[\langle Ax_i, Ax_j\rangle]_{(l+1)\times(l+1)}.$$

Thus for some large N,

$$\det[\langle A_N x_i, Ax_j\rangle]_{(l+1)\times(l+1)} \neq 0, \tag{2}$$

but

$$\operatorname{rank}(A_N) \le l. \tag{3}$$

So (3) gives that there are constants c_i with $\sum_{i=1}^{l+1}|c_i| \neq 0$ such that

$$\sum_{i=1}^{l+1} c_i A_N x_i = 0.$$

Hence

$$\mathbf{c}[\langle A_N x_i, Ax_j\rangle]_{(l+1)\times(l+1)} = \mathbf{0}.,$$

where $\mathbf{c} = (c_1, \ldots, c_{l+1})$. This implies

$$\det[\langle A_N x_i, Ax_j\rangle]_{(l+1)\times(l+1)} = 0.$$

This contradicts (2) and completes the proof. □

THEOREM 2. *Suppose that f is in $L^\infty(D)$ and equal to $\sum_{j=1}^{l} f_j(z)\overline{g_j(z)}$ for finitely many functions $f_j(z)$ and $g_j(z)$ analytic on the unit disk D. If T_f has finite rank, then $f = 0$.*

Proof. First we will show that $T_{|f|^2}$ has finite rank. To do so, for each $0 < r < 1$, define $f_r(z) = f(rz)$. Let $g_r = \overline{f_r}$. Since

$$f(z) = \sum_{j=1}^{l} f_j(z)\overline{g_j(z)}$$

for finitely many functions $f_j(z)$ and $g_j(z)$ in L_a^2, we have

$$\begin{aligned} T_{fg_r} &= T_{f\overline{(\sum_{j=1}^{l} f_j(rz)\overline{g_j(rz)})}} \\ &= \sum_{j=1}^{l} T_{f\overline{f_j(rz)}g_j(rz)} \\ &= \sum_{j=1}^{l} T_{\overline{f_j(rz)}} T_f T_{g_j(rz)}. \end{aligned}$$

The last equality follows from the basic properties of Toeplitz operators [2]

$$T_{\bar{h}} T_f = T_{\bar{h}f}$$

and

$$T_f T_h = T_{fh},$$

for $f \in L^\infty(D, dA)$ and $h \in H^\infty(D)$. If T_f has finite rank and $\operatorname{rank}(T_f) = N$, then for each $0 < r < 1$,

$$\operatorname{rank}(T_{fg_r}) \le Nl.$$

Thus

$$\limsup_{r\to 1} \operatorname{rank}(T_{fg_r}) \le Nl.$$

Next we shall show that T_{fg_r} converges to $T_{|f|^2}$ in the weak operator topology. To do this, we observe that for each $z \in D$,

$$|f(z)g_r(z)| = |f(z)f(rz)| \le \|f\|_\infty^2,$$

and

$$\lim_{r\to 1^-} f(z)g_r(z) = |f(z)|^2.$$

By the dominant convergence theorem we have that for $h_1, h_2 \in L^2_a$,

$$\lim_{r\to 1^-} \int_D f(z)g_r(z)h_1(z)\overline{h_2(z)}dA(z) = \int_D |f(z)|^2 h_1(z)\overline{h_2(z)}dA(z),$$

to obtain

$$\begin{aligned}\lim_{r\to 1^-} \langle T_{fg_r}h_1, h_2\rangle &= \lim_{r\to 1^-} \langle fg_r h_1, h_2\rangle \\ &= \lim_{r\to 1^-} \int_D f(z)g_r(z)h_1(z)\overline{h_2(z)}dA(z) \\ &= \int_D |f(z)|^2 h_1(z)\overline{h_2(z)}dA(z) \\ &= \langle T_{|f|^2}h_1, h_2\rangle.\end{aligned}$$

This means that T_{fg_r} converges to $T_{|f|^2}$ in weak operator topology. By Lemma 1, we have that the Toeplitz operator $T_{|f|^2}$ with nonnegative function symbol has finite rank and its rank is at most Nl.

To finish the proof we need to prove that if the Toeplitz operator with nonnegative function symbol has finite rank, it must be zero. This was well known. For completeness, we include a proof here. Since $T_{|f|^2}$ has finite rank, the kernel of $T_{|f|^2}$ contains a nonzero function $h \in L^2_a$. Thus

$$\begin{aligned}0 &= \langle T_{|f|^2}h, h\rangle \\ &= \langle |f|^2 h, h\rangle \\ &= \int_D |f(z)|^2 |h(z)|^2 dA(z),\end{aligned}$$

and so

$$|f(z)|^2|h(z)|^2 = 0$$

for a.e. $z \in D$. Noting that $h(z)$ is in the Bergman space, we conclude that $f = 0$ in $L^\infty(D, dA)$ to complete the proof. □

3. Finite sum of products of Hankel operators

For $f \in L^\infty(D, dA)$, the *Hankel operator* H_f with symbol f is the operator on L^2_a defined by $H_f h = (I - P)(fh)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L^2_a. The relation between Toeplitz operators and Hankel operators is established by the following well-known identity:

$$(T_f, T_g] = H^*_{\bar{f}} H_g.$$

In this section, we shall reduce the problem of when a finite sum of products of two Hankel operators has finite rank to the problem of when a Toeplitz operator has finite rank.

For each bounded harmonic function f on the unit disk, f can be written uniquely as a sum of an analytic function and a co-analytic function on the unit disk D up to a constant. Let f_+ denote the analytic part and f_- the co-analytic part with $f_-(0) = 0$. In fact, both f_+ and $\overline{f_-}$ are in both the Hardy space H^2 and the Bloch space [2], [9].

For bounded harmonic functions f_i and g_i on the unit disk for $i = 1, \dots, k$, define

$$\sigma(f_1, \dots, f_k; g_1, \dots, g_k) = \tilde{\Delta}\left[\sum_{i=1}^{k}(f_i)_-(g_i)_+\right].$$

For two bounded harmonic functions f and g on the unit disk, let $\sigma_{sc}(f, g)$ denote $\sigma(g; f)$ and $\sigma_c(f, g)$ denote $\sigma(f, -g; g, f)$. Easy calculations give

$$\sigma(f_1, \dots, f_k; g_1, \dots, g_k) = (1 - |z|^2)^2 \sum_{i=1}^{k}(f_i)'_-(g_i)'_+, \tag{4}$$

where $(f_i)'_- = \partial_{\bar{z}} f_i$. Hence

$$\begin{aligned}
\sigma_{sc}(f, g) &= \tilde{\Delta}(f_+ g_-) \\
&= (1 - |z|^2)^2(\partial_z f)(\partial_{\bar{z}} g) \\
&= (1 - |z|^2) f'_+(z)(1 - |z|^2) g'_-(z), \\
\sigma_c(f, g) &= \tilde{\Delta}[f_- g_+ - f_+ g_-] \\
&= (1 - |z|^2)^2[(\partial_{\bar{z}} f)(\partial_z g) - (\partial_z f)(\partial_{\bar{z}} g)] \\
&= (1 - |z|^2) f'_-(z)(1 - |z|^2) g'_+(z) - (1 - |z|^2) f'_+(z)(1 - |z|^2) g'_-(z).
\end{aligned}$$

LEMMA 3. *Suppose that f_i and g_i are bounded harmonic functions on the unit disk for $i = 1, \dots, k$. Then $\sigma(f_1, \dots, f_k; g_1, \dots, g_k)$ is in $L^\infty(D, dA)$.*

Proof. Since f_i and g_i are bounded harmonic functions on the unit disk, $(f_i)_+$, $\overline{(f_i)_-}$, $(g_i)_+$ and $\overline{(g_i)_-}$ are in the Bloch space

$$B = \{h : h \text{ analytic on } D,\ \sup_{z \in D}(1 - |z|^2)|h'(z)| < \infty\}$$

(see [2]). (4) gives that $\sigma(f_1, \dots, f_k; g_1, \dots, g_k)$ is in $L^\infty(D, dA)$. □

PROPOSITION 4. *Suppose that f_i and g_i are bounded harmonic functions on D for $i = 1, \dots, k$. If the finite sum $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$ of products of Hankel operators has finite rank, then $T_{\sigma(f_1,\dots,f_k;g_1,\dots,g_k)}$ has finite rank.*

Proof. For these bounded harmonic functions f_i, g_i on the unit disk, write

$$f_i = (f_i)_+ + (f_i)_-$$

and

$$g_i = (g_i)_+ + (g_i)_-,$$

where $(f_i)_+$, $(g_i)_+$, $\overline{(f_i)_-}$, and $\overline{(g_i)_-}$ are in the Hardy space H^2. By Lemma 3, $\sigma(f_1, \dots, f_k; g_1, \dots, g_k)(z)$ is in $L^\infty(D, dA)$. Thus $T_{\sigma(f_1,\dots,f_k;g_1,\dots,g_k)}$ is bounded on the Bergman space L^2_a.

We shall get the Berezin transform of $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$. First we calculate the Berezin transform of $B((T_f, T_g])(z)$ of the semicommutator $(T_f, T_g]$. By the basic properties of Toeplitz operators on the Bergman space [2], [14], we have

$$T_f k_z = (f_+ + f_-(z))k_z,$$

for $z \in D$. Since f is harmonic in the unit disk, we also have

$$B(f)(z) = f(z).$$

For two bounded harmonic functions $f,\ g$ on D, easy calculations give

$$\begin{aligned}
B((T_f, T_g])(z) &= B(T_{fg} - T_f T_g)(z)\\
&= \langle fgk_z, k_z\rangle - \langle (g_+ + g_-(z))k_z, \bar{f}k_z\rangle\\
&= \langle [fg - f(g_+ + g_-(z))]k_z, k_z\rangle\\
&= \langle [f(g_- - g_-(z))]k_z k_z\rangle\\
&= \langle [f_+g_- + f_-g_- - fg_-(z)]k_z, k_z\rangle\\
&= \langle f_+g_-k_z, k_z\rangle + \langle f_-g_-k_z, k_z\rangle - g_-(z)\langle fk_z, k_z\rangle\\
&= B(f_+g_-)(z) + f_-(z)g_-(z) - g_-(z)B(f)(z)\\
&= B(f_+g_-)(z) + f_-(z)g_-(z) - g_-(z)f(z)\\
&= B(f_+g_-)(z) + f_-(z)g_-(z) - g_-(z)(f_+(z) + f_-(z))\\
&= B(f_+g_-)(z) - f_+(z)g_-(z)
\end{aligned}$$

for all $z \in D$. Noting

$$(T_f, T_g] = H^*_{\bar{f}} H_g,$$

we have

$$B(H^*_{\bar{f}} H_g)(z) = B(f_+g_-)(z) - f_+(z)g_-(z).$$

Thus

$$\begin{aligned}
&B\left(\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}\right)(z)\\
&\quad = B\left(\sum_{j=1}^{k} (g_j)_+(f_j)_-\right)(z) - \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z).
\end{aligned}$$

Applying the invariant Laplace operator $\tilde{\Delta}$ to both sides of the above equation gives

$$\begin{aligned}
&\tilde{\Delta} B\left(\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}\right)(z)\\
&\quad = \left[\tilde{\Delta} B\left(\sum_{j=1}^{k} (g_j)_+(f_j)_-\right)\right](z) - \left[\tilde{\Delta} \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z)\right].
\end{aligned}$$

Since the invariant Laplace operator commutes with the Berezin transform (Lemma 1, [1]), we have

$$\begin{aligned} &B(\sigma(f_1,\dots,f_k;g_1,\dots,g_k))(z)\\ &\qquad = (1-|z|^2)^2\left[\sum_{j=1}^{k}(g_j)'_+(z)(f_j)'_-(z)\right] + \tilde{\Delta}B\left(\sum_{j=1}^{k} H^*_{\overline{g_j}}H_{f_j}\right)(z). \end{aligned}$$

In other words, the above equality becomes

$$\begin{aligned} &\langle T_{\sigma(f_1,\dots,f_k;g_1,\dots,g_k)}k_z,k_z\rangle = B(\sigma(f_1,\dots,f_k;g_1,\dots,g_k))(z)\\ &\qquad = (1-|z|^2)^2\left[\sum_{j=1}^{k}(g_j)'_+(z)(f_j)'_-(z)\right] + \tilde{\Delta}B\left(\sum_{j=1}^{k} H^*_{\overline{g_j}}H_{f_j}\right)(z). \end{aligned}$$

For two functions x and y in L^2_a, define the operator $x\otimes y$ of rank one to be

$$(x\otimes y)f = \langle f,y\rangle x$$

for $f\in L^2_a$. Then it is easy to verify that

$$\begin{aligned} B(x\otimes y)(z) &= \langle (x\otimes y)k_z,k_z\rangle\\ &= (1-|z|^2)^2\langle (x\otimes y)K_z,K_z\rangle\\ &= (1-|z|^2)^2\langle K_z,y\rangle\langle x,K_z\rangle\\ &= (1-|z|^2)^2 x(z)\overline{y(z)}, \end{aligned}$$

for $z\in D$. If the semicommutator $\sum_{j=1}^{k} H^*_{\overline{g_j}}H_{f_j}$ has finite rank N, then there exist functions x_j and y_j in L^2_a for $j=1,\dots,N$ such that

$$\sum_{j=1}^{k} H^*_{\overline{g_j}}H_{f_j} = \sum_{j=1}^{N} x_j\otimes y_j.$$

Thus

$$B\left(\sum_{j=1}^{k} H^*_{\overline{g_j}}H_{f_j}\right)(z) = (1-|z|^2)^2\left(\sum_{j=1}^{N} x_j(z)\overline{y_j(z)}\right).$$

Observe

$$(1-|z|^2)^2\left(\sum_{j=1}^{N} x_j(z)\overline{y_j(z)}\right) = \sum_{j=1}^{3N}\hat{x}_j(z)\overline{\hat{y}_j(z)},$$

where $\hat{x}_j$ and $\hat{y}_j$ are in the Bergman space L^2_a. So

$$\begin{aligned} &\langle T_{\sigma(f_1,\dots,f_k;g_1,\dots,g_k)}k_z,k_z\rangle\\ &\qquad = (1-|z|^2)^2\left[\sum_{j=1}^{k}(g_j)'_+(z)(f_j)'_-(z)\right] + (1-|z|^2)^2\left(\sum_{j=1}^{3N}\hat{x}'_j(z)\overline{\hat{y}'_j(z)}\right). \end{aligned}$$

Dividing by $(1-|z|^2)^2$, we obtain

$$(5) \quad \langle T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}K_z, K_z\rangle = \sum_{j=1}^{k}(g_j)'_+(z)(f_j)'_-(z) + \left(\sum_{j=1}^{3N}\hat{x}'_j(z)\overline{\hat{y}'_j(z)}\right).$$

As in [1] we complexify the above identity. Write the left hand side as an integral as in [1] to get

$$\langle T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}K_z, K_z\rangle = \int_D \sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)(\lambda)\frac{1}{|1-\bar{z}\lambda|^4}dA(\lambda).$$

Since the right hand side of (5) and the above integral are real analytic functions of z and $\bar{z}$, we obtain

$$\langle T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}K_w, K_z\rangle = \sum_{j=1}^{k}(g_j)'_+(z)(f_j)'_-(w) + \left(\sum_{j=1}^{3N}\hat{x}'_j(z)\overline{\hat{y}'_j(w)}\right).$$

Differentiating both sides of the above equation l times with respect to $\bar{w}$ and then letting $w=0$ gives

$$(6) \qquad T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}z^l = \sum_{j=1}^{k}a_{lj}(g_j)'_+(z) + \sum_{j=1}^{3N}b_{lj}\hat{x}'_j(z)$$

for some constants a_{lj}, b_{lj}.

Although some of the $(g_j)'_+$ and $\hat{x}'_j$ may not be in L^2_a, we observe that for each $0<r<1$, all of $(g_j)'_+|_{rD}$ for $j=1,\ldots,k$ and $\hat{x}'_j|_{rD}$ for $j=1,\ldots,3N$ are in $L^2_a(rD,dA)$.

We claim that

$T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}$ *has finite rank on the Bergman space* L^2_a.

If this claim is false, we may assume that there are $3N+k+1$ linearly independent functions $\{\phi_\mu\}_{\mu=1}^{3N+k+1}$ in the range of $T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}$. Thus for each $0<r<1$, $\{\phi_\mu|_{rD}\}_{\mu=1}^{3N+k+1}$ are also linearly independent in the space $L^2_a(rD,dA)$. Since analytic polynomials are dense in L^2_a, for each μ, there are analytic polynomials $p_{\mu l}$ such that $T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}p_{\mu l}$ converges to ϕ_μ. Thus $T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}p_{\mu l}$ converges uniformly to ϕ_μ on each compact subset of the unit disk D. Noting that rD is contained in a compact subset of the unit disk, we have

$$\lim_{l\to\infty}\int_{rD}|T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}p_{\mu l}(z)-\phi_\mu(z)|^2dA(z)=0.$$

On the other hand, (6) gives that $T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}p_{\mu l}|_{rD}$ is contained in the subspace spanned by $(g_j)'_+|_{rD}$ and $\hat{x}'_j|_{rD}$ of $L^2_a(rD,dA)$. But the subspace has dimension at most $3N+k$. This contradicts that $\{\phi_\mu|_{rD}\}_{\mu=1}^{3N+k+1}$ are also linearly independent and hence gives that $T_{\sigma(f_1,\ldots,f_k;g_1,\ldots,g_k)}$ has finite rank to complete the proof. □

4. Main results

Now we are ready to state and prove our main results.

Theorem 5. *Suppose that f and g are bounded harmonic functions on the unit disk. The semicommutator $(T_f, T_g]$ has finite rank if and only if either $\bar{f}$ or g is analytic on the unit disk.*

Proof. If either $\bar{f}$ or g is analytic on the unit disk, then $T_f T_g = T_{fg}$ and so the semicommutator $(T_f, T_g]$ equals 0.

If the semicommutator $(T_f, T_g]$ has finite rank, noting

$$(T_f, T_g] = H^*_{\bar{f}} H_g$$

by Proposition 4, the Toeplitz operator $T_{\sigma_{sc}(f,g)}$ has finite rank. Since

$$\begin{aligned}\sigma_{sc}(f,g)(z) &= (1-|z|^2)^2 f'_+(z) g'_-(z) \\ &= f'_+(z)g'_-(z) - 2z f'_+(z) g'_-(z)\bar{z} + z^2 f'_+(z) g'_-(z)\bar{z}^2,\end{aligned}$$

Theorem 2 gives that for $z \in D$,

$$\sigma_{sc}(f,g)(z) = (1-|z|^2)^2 f'_+(z) g'_-(z) \equiv 0.$$

This implies

$$f'_+(z) g'_-(z) \equiv 0$$

on D. Thus either f_+ or g_- is constant on D. So we conclude that either $\bar{f}$ or g is analytic on D to complete the proof. □

Theorem 6. *Suppose that f and g are bounded harmonic functions on the unit disk. The commutator $[T_f, T_g]$ has finite rank if and only if f and g are both analytic on D, or $\bar{f}$ and $\bar{g}$ are both analytic on D, or there are constants c_1, c_2, not both 0, such that $c_1 f + c_2 g$ is constant on D.*

Proof. If f and g are both analytic on D, both T_f and T_g are multiplication operators on the Bergman space and therefore are commuting. Hence the commutator $[T_f, T_g]$ equals 0.

If $\bar{f}$ and $\bar{g}$ are both analytic on D, both T_f and T_g are adjoints of multiplication operators on the Bergman space and therefore are commuting. Hence the commutator $[T_f, T_g]$ equals 0.

If there are constants c_1, c_2, not both 0, such that $c_1 f + c_2 g$ is constant on D, noting that the Toeplitz operator with constant symbol commutes with any bounded operator on the Bergman space, we have that T_f commutes with T_g and thus obtain that the commutator $[T_f, T_g]$ equals 0.

Conversely, if the commutator $[T_f, T_g]$ has finite rank, noting

$$\begin{aligned}[T_f, T_g] &= T_f T_g - T_g T_f \\ &= (T_{gf} - T_g T_f) - (T_{fg} - T_f T_g) \\ &= (T_g, T_f] - (T_f, T_g] \\ &= H^*_{\bar g} H_f - H^*_{\bar f} H_g,\end{aligned}$$

we have that $H^*_{\bar g} H_f - H^*_{\bar f} H_g$ has also finite rank. Lemma 3 gives that $\sigma_c(f,g)$ is bounded on D, and easy calculations give

$$\begin{aligned}\sigma_c(f,g)(z) &= (1-|z|^2)^2[f'_-(z)g'_+(z) - f'_+(z)g'_-(z)] \\ &= f'_-(z)g'_+(z) - f'_+(z)g'_-(z) - 2\bar z f'_-(z)g'_+(z)z \\ &\quad + 2z f'_+(z)g'_-(z)\bar z + \bar z^2 f'_-(z)g'_+(z)z^2 - z^2 f'_+(z)g'_-(z)\bar z^2.\end{aligned}$$

Thus Theorem 2 and Proposition 4 give that $\sigma_c(f,g)(z) \equiv 0$ on the unit disk.

Let $u = g_+ + ig_-$ and $v = if_+ + f_-$. Clearly, u and v are harmonic on D. An easy calculation gives

$$\begin{aligned}\tilde\Delta(uv) &= \tilde\Delta[g_+f_- - f_+g_- + ig_+f_+ + ig_-f_-] \\ &= \tilde\Delta[g_+f_- - f_+g_-] \\ &= (1-|z|^2)^2[f'_-(z)g'_+(z) - f'_+(z)g'_-(z)] \\ &= \sigma_c(f,g)(z).\end{aligned}$$

Thus uv is also harmonic on D. By Lemma 4.2 [6], we have that at least one of the following conditions holds:

(1) u and v are both analytic on D;
(2) $\bar u$ and $\bar v$ are both analytic on D;
(3) there exist complex numbers α, β, not both 0, such that $\alpha u + \beta v$ and $\bar\alpha \bar u - \bar\beta \bar v$ are both analytic on D.

Condition (1) gives that f and g are both analytic on D. Condition (2) gives that $\bar f$ and $\bar g$ are analytic on D. Condition (3) gives that $\alpha(g_+ + ig_-) + \beta(if_+ + f_-)$ and $\bar\alpha(\overline{g_+ + ig_-}) - \bar\beta(\overline{if_+ + f_-})$ are both analytic on D. Thus $\alpha i g_- + \beta f_-$ and $\bar\alpha\overline{g_+} - \overline{\beta i f_+}$ are constants on D, and so $\alpha g_- - \beta i f_-$ and $\alpha g_+ - \beta i f_+$ are constants on D. Hence we conclude that

$$\alpha g - i\beta f = (\alpha g_- - i\beta f_-) + (\alpha g_+ - \beta i f_+)$$

is constant on D. This completes the proof. □

THEOREM 7. *Suppose that f and g are bounded harmonic functions on the unit disk. $T_f T_g$ has finite rank if and only if either f or g equals* 0.

Proof. It is clear that if either f or g equals 0, then $T_f T_g = 0$.

Conversely, if $T_f T_g$ has finite rank, we shall show that either f or g equals 0. An easy calculation gives

$$B(T_f T_g)(z) = B(fg)(z) - B(f_+ g_-)(z) + f_+(z)g_-(z). \tag{7}$$

Applying the invariant Laplace operator $\tilde{\Delta}$ to both sides of the above equation gives

$$[\tilde{\Delta} B(T_f T_g)](z) = \tilde{\Delta} B(fg - f_+ g_-)(z) + \tilde{\Delta}[f_+(z)g_-(z)].$$

Since the invariant Laplace operator commutes with the Berezin transform (Lemma 1, [1]), we have

$$B(\tilde{\Delta}(fg - f_+ g_-))(z) = [\tilde{\Delta} B(T_f T_g)](z) - \tilde{\Delta}[f_+(z)g_-(z)].$$

As in the proof of Proposition 4, the Toeplitz operator $T_{\tilde{\Delta}(fg - f_+ g_-)}$ has finite rank. Theorem 2 gives that $\tilde{\Delta}(fg - f_+ g_-) \equiv 0$. This implies that $fg - f_+ g_-$ is harmonic and $f'_-(z) g'_+(z) = 0$ on D. Thus either f_- or g_+ is constant and hence either f or $\bar{g}$ is analytic on D.

On the other hand, since $fg - f_+ g_-$ is harmonic, (7) gives

$$B(T_f T_g)(z) = f(z)g(z).$$

By the main result of [5],

$$\lim_{|z| \to 1} B(T_f T_g)(z) = 0.$$

Because the radial limits of both f and g exist on the unit circle, we have that $f(z)g(z) \equiv 0$ on the unit circle and therefore either f or g equals 0 on the unit circle. Hence f or g equals 0 on the unit disk. This completes the proof. □

Theorems 5, 6 and 7 suggest the following theorem.

THEOREM 8. *Suppose that f_i and g_i are bounded harmonic functions on D for $i = 1, \dots, k$. The following are equivalent:*

(1) $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j}$ *has finite rank.*
(2) $\sum_{j=1}^{k} H^*_{\overline{g_j}} H_{f_j} = 0$.
(3) $\sigma(f_1, \dots, f_k; g_1, \dots, g_k) \equiv 0$.

Proof. It is clear that (2) implies (1).

First we prove that (1) implies (3). Proposition 4 immediately gives that $T_{\sigma(f_1,\dots,f_k;g_1,\dots,g_k)}$ has finite rank. Theorem 2 gives that

$$\sigma(f_1, \dots, f_k; g_1, \dots, g_k) \equiv 0.$$

To prove that (3) implies (2), we need the following equality obtained in the proof of Proposition 4:

$$B\left(\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j}\right)(z)$$
$$= B\left(\sum_{j=1}^{k} (g_j)_+(f_j)_-\right)(z) - \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z).$$

(3) implies that the function $\sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z)$ is harmonic and hence

$$B\left(\sum_{j=1}^{k} (g_j)_+(f_j)_-\right)(z) = \sum_{j=1}^{k} (g_j)_+(z)(f_j)_-(z).$$

Therefore

$$B\left(\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j}\right)(z) = 0.$$

By the injection of the Berezin transform [12], we conclude that the operator $\sum_{j=1}^{k} H_{\overline{g_j}}^* H_{f_j}$ must equal 0. This completes the proof. □

Acknowledgments. We would like to thank the referee for his comments. The third author would like to thank Xiaoman Chen, Kunyu Guo and Jiaxing Hong for their hospitality while he visited the Institute of Mathematics at Fudan University, where part of this paper was in progress.

Addendum. After this paper was accepted, we were informed by Daniel Luecking that he proved that if a Toeplitz operator with bounded symbol has finite rank, then its symbol must be zero.

References

[1] P. Ahern and Ž. Čučković, *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*, J. Funct. Anal. **187** (2001), 200–210. MR 1867348 (2002h:47040)

[2] S. Axler, *Bergman spaces and their operators*, Surveys of some recent results in operator theory, Vol. I, Pitman Res. Notes Math. Ser., vol. 171, Longman Sci. Tech., Harlow, 1988, pp. 1–50. MR 958569 (90b:47048)

[3] S. Axler, S.-Y. A. Chang, and D. Sarason, *Products of Toeplitz operators*, Integral Equations Operator Theory **1** (1978), 285–309. MR 511973 (80d:47039)

[4] S. Axler and Ž. Čučković, *Commuting Toeplitz operators with harmonic symbols*, Integral Equations Operator Theory **14** (1991), 1–12. MR 1079815 (92f:47018)

[5] S. Axler and D. Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J. **47** (1998), 387–400. MR 1647896 (99i:47045)

[6] ———, *The Berezin transform on the Toeplitz algebra*, Studia Math. **127** (1998), 113–136. MR 1488147 (98m:47030)

[7] X. Ding and D. Zheng, *Finite rank commutator of Toeplitz operators or Hankel operators*, preprint.

[8] M. Engliš, *Berezin transform and the Laplace-Beltrami operator*, St. Petersburg Math. J. **7** (1996), 89–102. MR 1356535 (96k:32049)

[9] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, vol. 96, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981. MR 628971 (83g:30037)

[10] D. C. V. Greene, S. Richter, and C. Sundberg, *The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels*, J. Funct. Anal. **194** (2002), 311–331. MR 1934606 (2003h:46038)

[11] K. Stroethoff, *Essentially commuting Toeplitz operators with harmonic symbols*, Canad. J. Math. **45** (1993), 1080–1093. MR 1239915 (94h:47046)

[12] ______, *The Berezin transform and operators on spaces of analytic functions*, Linear operators (Warsaw, 1994), Banach Center Publ., vol. 38, Polish Acad. Sci., Warsaw, 1997, pp. 361–380. MR 1457018 (98g:47025)

[13] D. Zheng, *Hankel operators and Toeplitz operators on the Bergman space*, J. Funct. Anal. **83** (1989), 98–120. MR 0993443 (91b:47057)

[14] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 139, Marcel Dekker Inc., New York, 1990. MR 1074007 (92c:47031)

Kunyu Guo, Department of Mathematics, Fudan University, Shanghai, 200433, PR China

E-mail address: `kyguo@fudan.edu.cn`

Shunhua Sun, Institute of Mathematics, Jiaxing University, Jiaxing, Zhejiang, 314001, PR China

E-mail address: `shsun@mail.zjxu.edu.cn`

Dechao Zheng, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: `zheng@math.vanderbilt.edu`