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ON THE HIGHER MOMENTS OF THE ERROR TERM IN THE DIVISOR PROBLEM

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Abstract. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem. Our main results are the asymptotic formulas for the integral of the cube and the fourth power of $\Delta(x)$. The exponents that we obtain in the error terms, namely $\beta = 7/5$ and $\gamma = 23/12$, respectively, are new. They improve on the values $\beta = 47/28, \gamma = 45/23$, due to K.-M. Tsang. A result on integrals of $\Delta^3(x)$ and $\Delta^4(x)$ in short intervals is also proved.

1. Introduction and statement of results

For a fixed $k \in \mathbb{N}$, let

$$\Delta_k(x) = \sum_{n\le x} d_k(n) - xP_{k-1}(\log x) \tag{1.1}$$

denote the error term in the (general) Dirichlet divisor problem. Here $d_k(n)$ denotes the number of ways n may be written as a product of k factors (so that $d_2(n) = d(n)$ is the number of divisors of n), and $P_{k-1}(z)$ is a suitable polynomial of degree $k-1$ in z (see, e.g., [6, Chapter 13] for more details). In particular,

$$\Delta_2(x) \equiv \Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1) \tag{1.2}$$

is the error term in the classical Dirichlet divisor problem ($\gamma = -\Gamma'(1) = 0.5772\ldots$ is Euler's constant). A vast literature exists on the estimation of $\Delta_k(x)$ and especially on $\Delta(x)$ (see, e.g., [6] and [12]), both pointwise and in various means. Here we shall be concerned with the third and fourth moment of $\Delta(x)$. We note that the first author in [5] proved a large values estimate for $\Delta(x)$, which yielded the bound

$$\int_1^X \Delta^4(x)\,\mathrm{d}x \ll_\varepsilon X^{2+\varepsilon}, \tag{1.3}$$

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where here and later ε denotes arbitrarily small, positive constants, which are not necessarily the same ones at each occurrence. The asymptotic formula for the fourth moment with an error term was obtained by K.-M. Tsang [13]. He proved that

$$\int_1^X \Delta^4(x)\,\mathrm{d}x = CX^2 + O_\varepsilon(X^{\gamma+\varepsilon}) \tag{1.4}$$

holds with explicitly given $C\,(>0)$ and $\gamma = 45/23 = 1.956\ldots$. Tsang also proved an asymptotic formula for the integral of the cube of $\Delta(x)$, namely

$$\int_1^X \Delta^3(x)\,\mathrm{d}x = BX^{7/4} + O_\varepsilon(X^{\beta+\varepsilon}) \tag{1.5}$$

with explicit $B > 0$ and $\beta = 47/28 = 1.678\ldots$. Later D.R. Heath-Brown [3] established asymptotic formulas for higher moments of $\Delta(x)$, but his method does not produce error terms.

The main aim of this paper is to improve on the values of Tsang's exponents β and γ in (1.5) and (1.4), respectively. The results are

Theorem 1. *We have*

$$\int_1^X \Delta^3(x)\,\mathrm{d}x = BX^{7/4} + O_\varepsilon(X^{\beta+\varepsilon}) \qquad \left(\beta = \frac{7}{5} = 1.4\,,\ B > 0\right). \tag{1.6}$$

Theorem 2. *We have*

$$\int_1^X \Delta^4(x)\,\mathrm{d}x = CX^2 + O_\varepsilon(X^{\gamma+\varepsilon}) \qquad \left(\gamma = \frac{23}{12} = 1.91666\ldots\,,\ C > 0\right). \tag{1.7}$$

The true values of β and γ for which (1.6) and (1.7) hold are hard to determine. However, it is not difficult to show that (1.7) cannot hold with $\gamma < 5/4$. To see this, note first that, for $1 \ll H \ll X$,

$$\begin{aligned}\Delta(X) - \frac{1}{2H}\int_{X-H}^{X+H} \Delta(x)\,\mathrm{d}x &= \frac{1}{2H}\int_{X-H}^{X+H} (\Delta(X) - \Delta(x))\,\mathrm{d}x \\ &\ll \frac{1}{H}\int_{X-H}^{X+H} \Big(\big|\sum_{X\le n\le x} d(n)\big| + O(H\log X)\Big)\,\mathrm{d}x \ \ll_\varepsilon\ HX^\varepsilon,\end{aligned}$$

which by Hölder's inequality and (1.7) gives

$$\begin{aligned}\Delta^4(X) &\ll_\varepsilon \frac{1}{H}\int_{X-H}^{X+H} \Delta^4(x)\,\mathrm{d}x + H^4X^\varepsilon \\ &\ll_\varepsilon X + H^{-1}X^{\gamma+\varepsilon} + H^4X^\varepsilon.\end{aligned}$$

But taking $H = X^{\gamma/5}$ we get

$$\Delta(X) \ll_\varepsilon X^{1/4} + X^{\gamma/5+\varepsilon},$$

which is a contradiction if $\gamma < 5/4$, since $\limsup_{X\to\infty} |\Delta(X)|X^{-1/4} = \infty$ is a classical result of G.H. Hardy [2]. For the cube this procedure does not work directly, since $\Delta^3(x)$ may be negative.

We note that asymptotic formulas for moments of $\Delta(x)$ in short intervals were recently investigated by W.G. Nowak [10], but his results do not imply the asymptotic formulas (1.4) and (1.5). Nowak actually works, for technical reasons, with $\Delta(x^2)$ instead of $\Delta(x)$. With a slight change of notation, his formulas for the cube and the fourth moment may be written as

$$\int_{X-H}^{X+H} \Delta^3(x)\,\mathrm{d}x = (D_1 + o(1))HX^{3/4} \qquad (X^{3/4+\delta} \le H \le \lambda X,\ 0 < \lambda \le 1) \tag{1.8}$$

for any fixed $0 < \delta < 1/4$ and $X \to \infty$, and similarly

$$\int_{X-H}^{X+H} \Delta^4(x)\,\mathrm{d}x = (D_2 + o(1))HX \qquad (X^{3/4+\delta} \le H \le \lambda X,\ 0 < \lambda \le 1), \tag{1.9}$$

where $D_j = D_j(\lambda)$ (> 0) is explicitly given by Nowak for $j = 1, 2$. For example, $D_1 = 7B/2$ if $H = o(X)$ and $D_1 = B\lambda^{-1}((1+\lambda)^{7/4} - (1-\lambda)^{7/4})$ if $H = \lambda X$, where B is the constant appearing in (1.6). We shall improve the range for which (1.8) and (1.9) hold. We avoid "$o(1)$" in (1.9) and formulate our results as follows:

Theorem 3. *For any fixed $0 < \delta < 1/3$ there exists $\kappa > 0$ such that uniformly*

$$\int_X^{X+H} \Delta^3(x)\,\mathrm{d}x = B\left((X+H)^{7/4} - X^{7/4}\right)(1 + O(X^{-\kappa})) \qquad (X^{7/12+\delta} \le H \le X), \tag{1.10}$$

and also

$$\int_X^{X+H} \Delta^4(x)\,\mathrm{d}x = C\left((X+H)^2 - X^2\right)(1 + O(X^{-\kappa})) \qquad (X^{2/3+\delta} \le H \le X), \tag{1.11}$$

where B and C are the constants appearing in Theorem 1 and Theorem 2.

From the proof of Theorem 3 it will be clear that, using the sharpest bound for $\Delta(x)$, we can improve on the exponents $7/12$ and $2/3$ which appear in (1.10) and (1.11), respectively. To gain in clarity we have separated the proofs of Theorems 1 and 2 from that of Theorem 3, since in Theorem 3 the accent is on the range of H, and not the value of κ. Following the method of proof of Theorem 1 and Theorem 2, one can obtain analogous results for the cube and fourth power of two well-known number theoretic error terms. This is given by

Corollary 1. *We have*

$$\begin{aligned} &\int_1^T E^3(t)\,\mathrm{d}t = B_1 T^{7/4} + O_\varepsilon(T^{\beta_1+\varepsilon}),\\ &\int_1^X P^3(x)\,\mathrm{d}x = B_2 X^{7/4} + O_\varepsilon(X^{\beta_2+\varepsilon}), \end{aligned} \tag{1.12}$$

with $\beta_1 = 5/3$, $\beta_2 = 7/5$.

Corollary 2. *We have*

$$\begin{aligned} &\int_1^T E^4(t)\,\mathrm{d}t = C_1 T^{2} + O_\varepsilon(T^{\gamma_1+\varepsilon}),\\ &\int_1^X P^4(x)\,\mathrm{d}x = C_2 X^{2} + O_\varepsilon(X^{\gamma_2+\varepsilon}), \end{aligned} \tag{1.13}$$

with $\gamma_1 = \gamma_2 = 23/12$.

Here B_1, B_2, C_1, C_2 are explicit, positive constants,

$$E(T) \;=\; \int_0^T |\zeta(1/2+it)|^2\,\mathrm{d}t - T\Big(\log\big(\frac{T}{2\pi}\big) + 2\gamma - 1\Big)$$

denotes the error term in the mean square formula for $|\zeta(1/2+it)|$, while

$$P(x) = \sum_{n\le x} r(n) - \pi x \qquad (r(n) = \sum_{n=a^2+b^2;a,b\in\mathbb{Z}} 1)$$

denotes the error term in the circle problem.

Namely we have the explicit, truncated formula (see, e.g., [6] or [12])

$$\Delta(x) = \frac{1}{\pi\sqrt{2}} x^{1/4} \sum_{n\le N} d(n) n^{-3/4}\cos(4\pi\sqrt{nx} - \frac14\pi) + O_\varepsilon(x^{1/2+\varepsilon}N^{-1/2}) \qquad (2\le N\ll x), \tag{1.14}$$

and also the formula (3.1) due to T. Meurman [9]. The basic tool for our results on $\Delta(x)$ are these two formulas, without recourse to the arithmetic structure of $d(n)$, except the trivial bound $d(n) \ll_\varepsilon n^\varepsilon$. For $P(x)$ there also exists an explicit formula, namely

$$(1.15)\quad P(x) = -\frac{1}{\pi}x^{1/4}\sum_{n\le N} r(n)n^{-3/4}\cos(2\pi\sqrt{nx}+\frac{1}{4}\pi) + O_\varepsilon(x^{1/2+\varepsilon}N^{-1/2}) \qquad (2 \le N \ll x).$$

Note that (1.15) is similar to (1.14), and the proof follows on the same lines, e.g., by the method given in E.C. Titchmarsh [12]. A formula for $P(x)$ analogous to (3.1) holds as well. Hence all the results on $\Delta(x)$ that depend only on (1.14) and $d(n) \ll_\varepsilon n^\varepsilon$ have their corresponding analogues for $P(x)$, so that by this principle the second formulas in (1.12) and (1.13) follow.

In what concerns the formulas involving $E(t)$, we note that

$$(1.16)\quad \begin{aligned} \int_0^T E^3(t)\,\mathrm{d}t &= 16\pi^4\int_0^{T/2\pi}(\Delta^*(t))^3\,\mathrm{d}t + O(T^{5/3}\log^{3/2}T),\\ \int_0^T E^4(t)\,\mathrm{d}t &= 32\pi^5\int_0^{T/2\pi}(\Delta^*(t))^4\,\mathrm{d}t + O(T^{23/12}\log^{3/2}T) \end{aligned}$$

holds, as proved by the first author [7, Theorem 2]. In (1.16) we set

$$\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x).$$

Then the arithmetic interpretation of $\Delta^*(x)$ is

$$\frac{1}{2}\sum_{n\le 4x}(-1)^n d(n) = x(\log x + 2\gamma - 1) + \Delta^*(x).$$

One also has (see [6, eq. (15.68)]), for $1 \ll N \ll x$,

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}x^{1/4}\sum_{n\le N}(-1)^n d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\frac{1}{4}\pi) + O_\varepsilon(x^{1/2+\varepsilon}N^{-1/2}),$$

which is completely analogous to (1.14). Hence the analogues of our formulas (1.4) and (1.5) hold for $\Delta^*(x)$, and therefore by (1.14) we easily obtain then the first formulas in (1.10) and (1.11). Note that the exponents β_1 and β_2 in (1.12) are not equal; this reflects the state of art that in (1.16) the first error term has the exponent of T equal to $5/3$, and $7/5 < 5/3$.

It should be remarked that moments of other number-theoretic error terms can be dealt with by our methods. This involves, for example, $A(x) = \sum_{n\le x} a(n)$, where $a(n)$ is the n-th Fourier coefficient of a holomorphic cusp form of weight $\kappa = 2n\,(\ge 12)$, which possesses an explicit formula analogous to (1.14). In general, error terms connected with the coefficients of Dirichlet series belonging to the well-known Selberg class of degree two may be considered.

The plan of the paper is as follows. In Section 2 we shall prove lemmas on the spacing of square roots, which are needed for the proof of our results. In Section 3 we shall prove Theorem 1. Section 4 contains the proof of Theorem 2, while Theorem 3 will be proved in Section 5.

Remarks and acknowledgments. After the first version of this paper was written, with the exponent 7/5 in (1.6), Kai-Man Tsang kindly informed us of the doctoral thesis of his student Yuk-Kam Lau [8]. Lau investigates a slightly more general function than $\Delta(x)$, namely $\Delta_a(x)$, the error term in the asymptotic formula for the summatory function of $\sigma_a(n) = \sum_{d|n} d^a$ for a in a certain range. Lau obtains, for the integral of $\Delta_a^3(x)$ in the case $a = 0$, the same exponent $7/5 + \varepsilon$ as we do, "following the method of Tsang and some refinements suggested by him" ([8, p. 98]). Since Lau's result has not been published in any periodical, and we obtained (1.6) independently of him, we thought appropriate to retain our proof of (1.6) in the final version of the paper.

We are also grateful to T. Meurman who informed us of the papers of W. Zhai [14], who kindly sent us his works and made valuable remarks. Zhai establishes asymptotic formulas for the integrals of $\Delta^j(x)$, when $3 \le j \le 9$. For $j = 3, 4$ in the notation of (1.6) and (1.7) he had $\beta = 3/2$ and $\gamma = 80/41$, which is poorer than what we obtained, although in correspondence Zhai indicated that his methods also can yield $\beta = 7/5$. We thank W.G. Nowak for sending us a preprint of [10], and finally we thank the referee for valuable remarks.

2. Lemmas on the spacing of the square roots

Both in the proof of Theorem 1 and Theorem 2 the basic approach is obvious: the sum approximating $\Delta(x)$ (cf. (1.14)) is raised to the third, respectively fourth power, and the resulting expressions are integrated. In this process sums and differences of square roots will appear in the exponentials. Thus several lemmas on the spacing of the square roots will be needed.

Lemma 1 (O. Robert and P. Sargos [11]). *Let $k \ge 2$ be a fixed integer and $\delta > 0$ be given. Then the number of integers n_1, n_2, n_3, n_4 such that $N < n_1, n_2, n_3, n_4 \le 2N$ and*

$$|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll_\varepsilon N^\varepsilon (N^4\delta + N^2). \tag{2.1}$$

Lemma 2. *If m,n,k are natural numbers such that $\sqrt{m} + \sqrt{n} \ne \sqrt{k}$, then*

$$|\sqrt{m} + \sqrt{n} - \sqrt{k}| \gg (mnk)^{-1/2}. \tag{2.2}$$

If m,n,l,k are natural numbers such that $m \le k$, $n \le k$ and $\sqrt{m} + \sqrt{n} \pm \sqrt{k} \ne \sqrt{l}$, then

$$|\sqrt{m} + \sqrt{n} \pm \sqrt{k} - \sqrt{l}| \gg k^{-2}(mnl)^{-1/2}. \tag{2.3}$$

Proof. These results should be compared with Tsang [13, Lemma 2] and [13, Lemma 3], who had (2.2) and (2.3) with the right-hand sides replaced by $\max(m,n,k)^{-1/2}$ and $\max(m,n,k,l)^{-7/2}$, respectively. Thus our bounds are better when at least one of the integers in question is smaller than the other ones.

To prove (2.2), we note first that if $4x \in \mathbb{N}$ is not a square, then

$$\|2\sqrt{x}\| \gg \frac{1}{\sqrt{x}}, \tag{2.4}$$

where as usual $\|y\|$ is the distance of y to the nearest integer. Namely if $\eta = \|2\sqrt{x}\|$, then $2\sqrt{x} = n \pm \eta$, $n \in \mathbb{N}$, and $0 < \eta \le 1/2$. Then $4x = n^2 \pm 2\eta n + \eta^2$. If $4x$ is not a square, we have

$$1 \le |4x - n^2| \le 2\eta n + \eta^2 \ll \eta\sqrt{x},$$

which implies (2.4).

Now let $\theta := \sqrt{m} + \sqrt{n} - \sqrt{k}$ and $0 < |\theta| < 1/2$. Then we have

$$m + n + 2\sqrt{mn} = (\sqrt{m} + \sqrt{n})^2 = (\sqrt{k} + \theta)^2 = k + 2\theta\sqrt{k} + \theta^2.$$

If $4mn$ is a square, then $2\theta\sqrt{k} + \theta^2$ is an integer, say v. If $v = 0$, then $\theta = 0$, which is impossible. If $v \ne 0$, then $1 \le |v| \le 2|\theta|(\sqrt{k} + 1)$, implying $|\theta| \gg 1/\sqrt{k}$, which is better than (2.2). If $4mn$ is not a square, then we have

$$\theta(2\sqrt{k} + \theta) = P + 2\sqrt{mn}, \quad P = k - m - n \in \mathbb{Z}.$$

Hence, by (2.4),

$$|\theta|\sqrt{k} \gg \|2\sqrt{mn}\| \gg \frac{1}{\sqrt{mn}},$$

giving (2.2).

The proof of (2.3) is on the same lines as the proof of (2.2), only it is a little more involved. It suffices to consider the case of the '+' sign, since both cases are analogous. Hence let

$$\rho := \sqrt{m} + \sqrt{n} - \sqrt{k} - \sqrt{l} \tag{2.5}$$

with $\rho \ne 0$. We can obviously suppose that $|\rho| \le 1/(5\sqrt{k})$. Squaring (2.5) it follows that $2\sqrt{mn} - 2\sqrt{kl} = v + \rho\mu$, where $v = k + l - m - n$ is an integer and $\mu = 2(\sqrt{k} + \sqrt{l}) + \rho$, so that $0 < \mu \le 4\sqrt{k} + \rho < 5\sqrt{k}$. If $v = 0$, then a better result than (2.3) will follow. If $v \ne 0$, another squaring yields

$$-8\sqrt{mnkl} = Q + \rho\nu \qquad (Q \in \mathbb{Z}),$$

with $\nu = 2v\mu + \rho\mu^2$. If $\nu = 0$, then $\rho\mu = 2|v| \geq 2$, which contradicts $\rho \leq 1/(5\sqrt{k})$. Hence $\nu \neq 0$. If $64mnkl$ is a square, then again a better result than (2.3) will follow. If this is not the case, then $|v| = |k+l-m-n| \leq 2k$. Therefore we have, since $0 < \nu \ll k^{3/2}$,

$$k^{3/2}|\rho| \;\gg\; |\rho\nu| \;\geq\; \|\,8\sqrt{mnkl}\,\| \;\gg\; \frac{1}{\sqrt{mnkl}}\,,$$

which gives then (2.3) by an obvious analogue of (2.4). □

Tsang's proof of (1.7) with $\gamma = 45/23$ depended on the following lemma.

Lemma (K.-M. Tsang [13]). *For any real numbers $\alpha \neq 0, \beta$ and $0 < \delta < 1/2$, we have uniformly*

$$\#\left\{ K < k \leq 2K \;:\; \|\beta + \alpha\sqrt{k}\| < \delta \right\} \ll K\delta + |\alpha|^{1/3}K^{1/2} + |\alpha|^{-1/2}K^{3/4}. \tag{2.6}$$

We shall present now a lemma which, in the relevant range needed for the proof of Theorem 2, supersedes (2.6).

Lemma 3. *For real numbers $0 < \delta < 1/2$, β and $\alpha \gg 1$*

$$\#\left\{ K < k \leq 2K \;:\; \|\beta + \alpha\sqrt{k}\| < \delta \right\} \ll_\varepsilon K\delta + |\alpha|^{1/2}K^{1/4+\varepsilon} + K^{1/2+\varepsilon}, \tag{2.7}$$

where the $\ll$-constant depends only on ε.

Proof. We may suppose $\alpha > 0$. Denote by $\mathcal{N}$ the expression on the left-hand side of (2.7). We shall use an elementary idea contained, e.g., in M.N. Huxley [4, Lemma 3.1.1]. This says that the number of integer points close to the function ($\beta + \alpha\sqrt{k}$ in this particular case) is essentially the same as the number of points close to the function that is the inverse of the original function (with appropriate new δ). For our problem, in his notation $T = \alpha, L \asymp K, F(x) = \sqrt{x} + \beta/\alpha, x \asymp K$, $\delta' = \delta\sqrt{K}/\alpha, G(y) = (y - \beta/\alpha)^2$. Then we obtain

$$\mathcal{N} \ll 1 + \delta' + \mathcal{N}' \ll 1 + \delta\sqrt{K} + \mathcal{N}', \tag{2.8}$$

where

$$\mathcal{N}' = \#\left\{ n \asymp \alpha\sqrt{K} \;:\; \|G(n/T)\| < \delta' \right\}.$$

Now we may easily reduce the problem to the estimation of exponential sums. For example, [4, Lemma 5.3.2] gives (setting $Y = \alpha\sqrt{K}$)

$$\mathcal{N}' \ll Y\delta' + YH^{-1} + H^{-1}\sum_{h=1}^{H}\Big|\sum_{\nu\asymp Y} \mathrm{e}\left(hG(\frac{\nu}{T})\right)\Big|,$$

where $\mathrm{e}(z) = \mathrm{e}^{2\pi iz}$, and H is an integer satisfying $H \asymp (\delta')^{-1}$. To the above exponential sum we apply the A-process of van der Corput (see e.g., [6, Lemma 2.5]). For an integer R with $1 \le R \le Y$ we have

$$\mathcal{N}' \ll Y\delta' + YH^{-1} + YR^{-1/2} + Y^{1/2}\left\{(HR)^{-1}\sum_{h=1}^{H}\sum_{r=1}^{R}\Big|\sum_{\nu\asymp Y} \mathrm{e}(2hr\nu\alpha^{-2})\Big|\right\}^{1/2}. \tag{2.9}$$

The sum over ν is a geometric sum, which is estimated (see e.g., [6, Lemma 6.4]) by

$$\Big|\sum_{n\le N}\mathrm{e}(\xi n)\Big| \;\le\; \min\left(N,\, \frac{1}{2\|\xi\|}\right) \qquad (N \ge 1,\, \xi \in \mathbb{R}).$$

In (2.9) we group together the terms with $hr = \mu$, noting that there are at most $d(\mu)$ ($\ll_\varepsilon \mu^\varepsilon$) such pairs for a given μ. We take $R \asymp Y$, and suppose that $\delta' < 1$ (for otherwise the final result is obvious). It follows that the expression in curly brackets in (2.9) is

$$\ll_\varepsilon (HR)^{\varepsilon-1}\sum_{\mu=1}^{2HR}\min\left(\Big\|\frac{\mu}{\alpha^2}\Big\|^{-1},\, Y\right). \tag{2.10}$$

We divide the range of summation in (2.10) into subranges of length $\le \alpha^2$, and note that $\|x\| = \|1-x\|$, and $\|\mu/\alpha^2\| = 0$ if $\mu = \alpha^2$ ($\in \mathbb{N}$). The conditions on H, R and Y imply that $\alpha^2 \ll \delta HR$, thus the expression in (2.10) is

$$\begin{aligned}
&\ll_\varepsilon (HR)^{\varepsilon-1}HR\alpha^{-2}\left(\sum_{1\le\mu\le\frac12\alpha^2}\min\left(\Big\|\frac{\mu}{\alpha^2}\Big\|^{-1},\, Y\right) + Y\right)\\
&\ll_\varepsilon K^\varepsilon\alpha^{-2}\Big(\sum_{\mu\le\frac12\alpha^2}\frac{\alpha^2}{\mu} + \alpha\sqrt{K}\Big)\\
&\ll_\varepsilon K^\varepsilon(\log K + \alpha^{-1}\sqrt{K}).
\end{aligned}$$

It follows then from (2.9) that

$$\begin{aligned}
\mathcal{N}' &\ll_\varepsilon Y\delta' + YH^{-1} + YR^{-1/2} + K^\varepsilon Y^{1/2}(1+\alpha^{-1}\sqrt{K})^{1/2}\\
&\ll_\varepsilon K\delta + \alpha^{1/2}K^{1/4+\varepsilon} + K^{1/2+\varepsilon},
\end{aligned}$$

and (2.7) follows from (2.8) and the above bound. □

Lemma 4. *Let $\mathcal{N}$ denote the number of solutions in integers m,n,k of the inequality*

$$|\sqrt{m} + \sqrt{n} - \sqrt{k}\,| \;\le\; \delta\sqrt{M} \qquad (\delta > 0) \tag{2.11}$$

with $M' < n \le 2M', M < m \le 2M, k \in \mathbb{N}$ and $M' \le M$. Then

$$\mathcal{N} \;\ll_\varepsilon\; M^\varepsilon(M^2M'\delta + (MM')^{1/2}). \tag{2.12}$$

Proof. We have adopted the condition $M' < n < 2M'$ instead of the more natural one $N < n \le 2N$, because we did not want to confound ourselves with the parameter N in (3.4). The result is trivial if $\delta \ge 1/4$. If $\delta < 1/4$, then squaring (2.11) we obtain $k \asymp M$ and also $(\sqrt{m}+\sqrt{n})^2 = k + O(\delta M)$, implying that

$$2\sqrt{mn} = P + O(\delta M), \tag{2.13}$$

where $P = k - m - n$ is an integer. If $\delta \gg 1/M$, then (for a given pair (k,n)) it follows on squaring (2.11), that there are $\ll M\delta$ choices for m, hence the bound in (2.12) is trivial. If $0 < \delta < c/M$ for sufficiently small $c > 0$, then from (2.13) it follows that $\|2\sqrt{r}\| < \delta'$ with $r = mn \ll MM'$ and $\delta' = \delta M$. Since for each r there are at most $d(r)(\ll_\varepsilon r^\varepsilon)$ choices for (m,n), the bound (2.12) follows from Lemma 3 (with $\alpha = 2, \beta = 0$). □

Lemma 5. *Let $\mathcal{N}$ denote the number of solutions in integers m,n,k,l of the inequality*

$$|\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{l}\,| \;\le\; \delta\sqrt{K} \quad (\delta > 0) \tag{2.14}$$

with the conditions $M < m \le 2M$, $M' < n \le 2M'$, $K < k \le 2K$, $L < l \le 2L$, $1 \ll M, M', L \ll K$. Then

$$\mathcal{N} \;\ll_\varepsilon\; KMM'L(\delta + K^{\varepsilon-3/2}) + \sqrt{KMM'L}, \tag{2.15}$$

and also

$$\mathcal{N} \;\ll_\varepsilon\; KMM'L\Big(\delta K^2 + (KMM'L)^{-1/2}\Big)K^\varepsilon. \tag{2.16}$$

Proof. Lemma 5 is the crucial lemma in the proof of Theorem 2. It should be compared to Lemma 1 when $k = 2$, in which case (2.1) provides a sharper bound. However, the variables in Lemma 1 are supposed to be of the same order of magnitude. This condition is not fulfilled here, and it accounts for several technical difficulties. The point of two estimates, (2.15) and (2.16), is that for exceptionally small δ the bound (2.16) supersedes (2.15), and this fact will be used in the proof of Theorem 2.

We begin the proof of (2.15) with some preliminary observations. If $\delta \gg 1$, then (2.15) is trivial. If $c/K \le \delta \ll 1$ for any constant $c > 0$, then from (2.14) we have

$$k = (\sqrt{m}+\sqrt{n}-\sqrt{l})^2 + O(\delta K).$$

This implies that, for a given triplet (ℓ, m, n), there are at most $\ll \delta K$ choices for k, hence trivially there are no more that $\ll MM'\delta KL$ choices for (m,n,k,l), yielding the first bound in (2.15). For $\delta < c/K$ and c small we must have either $M \gg K$ or $M' \gg K$, for otherwise (2.14) is impossible. Suppose that the former holds, so that $M \asymp K$. Thus, as far as the order of magnitude is concerned, the variables m and k play a symmetric rôle. Therefore in what concerns the order of M' and L we may assume, without loss of generality, that $M' \ll L$. When $\delta < c/K$, then if $k = n$ we must have $l = m$, and if $k = m$, then $l = n$, for otherwise (2.14) cannot hold. In these cases the number of solutions is $\ll KM' \ll \sqrt{KMM'L}$, which is accounted for by the last term in (2.15). Henceforth we assume that $k \neq n$, $k \neq m$.

From (2.14) we obtain by squaring

$$k + l + n + 2\sqrt{kl} - 2\sqrt{kn} - 2\sqrt{ln} = m + O(\delta K).$$

This implies, since $\delta K < c$ with small $c > 0$, that

$$\|\beta + \alpha\sqrt{k}\,\| \ll \delta K \quad (\alpha = 2(\sqrt{l} - \sqrt{n}\,) \neq 0,\ \beta = -2\sqrt{ln}\,). \tag{2.17}$$

Denote by $R(l, n)$ the number of solutions (in $K < k \le 2K$) of (2.17). We have to consider two cases when we estimate $R(l, n)$.

(a) *The case when* $\sqrt{l} - \sqrt{n} \gg 1$. In this case we apply Lemma 3 with $1 \ll \alpha = \sqrt{l} - \sqrt{n} \ll \sqrt{K}$ to obtain that

$$\sum_{l,n,\sqrt{l}-\sqrt{n}\gg 1} R(l, n) \ll_\varepsilon M'L(K^2\delta + K^{1/2+\varepsilon}). \tag{2.18}$$

(b) *The case when* $0 < |\sqrt{l} - \sqrt{n}| \ll 1$. In this case we apply [4, Lemma 3.1.2]. In the notation of this lemma we have to take δK as δ, $L = K, T = \alpha = \sqrt{l} - \sqrt{n}\,(\ll 1), M = \sqrt{K}$. Therefore

$$R(l, n) \ll 1 + K^2\delta + K^{1/2}\alpha + K^{3/2}\delta\alpha^{-1}. \tag{2.19}$$

Set $r = |n - l|$, so that $r > 0$, since $\sqrt{l} - \sqrt{n} \neq 0$. If $l > n$, then $l = n + r$ with $r \ll L$, while if $n > r$, then $n = l + r$ with $r \ll M'$. Thus using (2.19) we obtain

$$\begin{aligned}
&\sum_{l,n,0<|\sqrt{l}-\sqrt{n}|\ll 1} R(l, n) \\
&\qquad \ll_\varepsilon L \sum_{1\le r\ll\sqrt{M'}} (1 + K^2\delta + K^{1/2}\alpha + K^{3/2}\delta\sqrt{M'}r^{-1}) \\
&\qquad\quad + M' \sum_{1\le r\ll\sqrt{L}} (1 + K^2\delta + K^{1/2}\alpha + K^{3/2}\delta\sqrt{L}r^{-1}) \\
&\qquad \ll K^2L(M')^{1/2}\delta + K^{1/2}L(M')^{1/2} + LK^{3/2}(M')^{1/2}\delta\log K \\
&\qquad\quad + K^2M'L^{1/2}\delta + K^{1/2}M'L^{1/2} + M'K^{3/2}L^{1/2}\delta\log K \\
&\qquad \ll_\varepsilon M'L(K^2\delta + K^{1/2+\varepsilon}).
\end{aligned} \tag{2.20}$$

Therefore from (2.18) and (2.20) we finally obtain, since $M \asymp K$ and $M' = \min(M, M', L)$,

$$\begin{aligned}\mathcal{N} &\ll KMM'L\delta + \sqrt{KMM'L} + \sum_{l,n} R(l,n) \\ &\ll_\varepsilon KMM'L\delta + \sqrt{KMM'L} + M'LK^2\delta + M'LK^{1/2+\varepsilon} \\ &\ll_\varepsilon KMM'L(\delta + K^{\varepsilon-3/2}) + \sqrt{KMM'L}.\end{aligned}$$

It remains yet to prove the bound (2.16) of Lemma 5. Proceeding from (2.5), as in the proof of (2.3), we have that

$$8\sqrt{mnkl} = Q + O(|\rho|K^{3/2}) \qquad (Q \in \mathbb{Z}),$$

where $\rho\,(\ll \sqrt{K})$ is given by (2.5), which may be written as

$$8\sqrt{j} = Q + O(\delta K^2) \qquad (j = mnkl).$$

The bound (2.16) follows then from Lemma 3 with $\alpha = 8$, since for each given j there are $\ll_\varepsilon K^\varepsilon$ choices of (m, n, k, l). This completes the proof of Lemma 5. □

Lemma 6. *Let $\mathcal{N}$ denote the number of solutions in integers m,n,k,l of the inequality*

$$0 < |\sqrt{m} + \sqrt{n} + \sqrt{k} - \sqrt{l}| \le \delta\sqrt{K} \quad (\delta > 0) \tag{2.21}$$

with the conditions $M < m \le 2M$, $M' \le N \le 2M'$, $K < k \le 2K$, $L < l \le 2L$, $1 \ll M, M' \ll K$. Then

$$\mathcal{N} \ll_\varepsilon KMM'L(\delta + K^{\varepsilon-3/2}), \tag{2.22}$$

and also

$$\mathcal{N} \ll_\varepsilon KMM'L(\delta K^2 + (KMM'L)^{-1/2})K^\varepsilon. \tag{2.23}$$

Proof. The proof is on the same lines as the proof of Lemma 5, only it is less difficult, and the details will be therefore omitted. It is only the case of $|\sqrt{m} + \sqrt{n} + \sqrt{k} - \sqrt{l}|$ that is treated in [13], while the case of the more difficult problem with $|\sqrt{m} + \sqrt{n} - \sqrt{k} - \sqrt{l}|$ is only mentioned on top of p. 76: "By a similar argument, we show that the same estimate holds with $S_4(x)$ in place of $S_5(x)$." Although this is essentially true, it is the proof of the more difficult case that should have been given.

To get back to Lemma 6, note that the condition $1 \ll M, M' \ll K$ may be assumed without loss of generality. However, then from (2.14) we have

$$l = (\sqrt{m} + \sqrt{n} + \sqrt{k})^2 + O(\delta K),$$

since $\delta < c/K$ may me assumed as in the case of Lemma 5. This gives $L \asymp K$, and then the proof is analogous to the proof of Lemma 5, but easier. The term $K\min(M, M', L)$, present in (2.15), is not necessary in (2.22). Finally the proof of (2.23) is completely analogous to the proof of (2.16). $\square$

3. Proof of Theorem 1

For the sake of simplicity and readability we shall retain, both in the proof of Theorem 1 and Theorem 2, the notation of [13]. We shall use a modified form of (1.14)–(1.15), due to T. Meurman [9], which shows that, for most x, the partial sum approximating $\Delta(x)$ has a small error, provided that the length of the sum is sufficiently large. More precisely, [9, Lemma 3] says that, for $Q \gg x \gg 1$,

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\le Q} d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \frac{1}{4}\pi) + F(x) = \frac{1}{\pi\sqrt{2}} \sum\nolimits_Q (x) + F(x), \tag{3.1}$$

where $F(x) \ll x^{-1/4}$ if $\|x\| \gg x^{5/2}Q^{-1/2}$ and otherwise $F(x) \ll_\varepsilon x^\varepsilon$. In (3.1) we take $Q = H^7$. Then we have

$$\int_H^{2H} (\Delta(x) - F(x))^3 \,\mathrm{d}x = \big(\frac{1}{\pi\sqrt{2}}\big)^3 \int_H^{2H} \sum\nolimits_Q (x)^3 \,\mathrm{d}x. \tag{3.2}$$

The left-hand side of (3.2) equals

$$\begin{aligned}
&\int_H^{2H} (\Delta(x) + O(H^{-1/4}))^3 \,\mathrm{d}x - \int_{H,\|x\|\ll 1/H}^{2H} (\Delta(x) + O_\varepsilon(H^\varepsilon))^3 \,\mathrm{d}x \\
&= \int_H^{2H} \Delta^3(x)\,\mathrm{d}x + O\left(H^{-1/4}\int_H^{2H} \Delta^2(x)\,\mathrm{d}x\right) + O_\varepsilon(H^\varepsilon \sup_{H\le x\le 2H} |\Delta(x)|^3) \\
&= \int_H^{2H} \Delta^3(x)\,\mathrm{d}x + O(H^{5/4}),
\end{aligned}$$

since $\Delta(x) \ll x^{1/3}$ and $\int_H^{2H} \Delta^2(x)\,\mathrm{d}x \ll H^{3/2}$ (see [6, Chapter 13]). Now in (3.2) we write

$$\sum\nolimits_Q (x) = \sum\nolimits_N (x) + R_{N,Q}(x) \qquad (H \le x \le 2H,\ 1 \ll N \ll H,\ Q = H^7)$$

with

$$\begin{aligned}
\sum\nolimits_N (x) &= x^{1/4} \sum_{n\le N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \frac{1}{4}\pi), \\
R_{N,y}(x) &= x^{1/4} \sum_{N<n\le y} d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \frac{1}{4}\pi) \\
&\ll_\varepsilon x^\varepsilon \left(1 + x^{1/2}N^{-1/2}\right) \quad (N < y \ll x^A).
\end{aligned} \tag{3.3}$$

The bound for $R_{N,y}(x)$ in (3.3) follows on subtracting (3.1) from (1.14), when we write it once with N and once with y replacing N, and then subtract the resulting expressions. It follows that (3.1) gives

(3.4)
$$\int_H^{2H}\Delta^3(x)\,\mathrm{d}x = \frac{1}{(\pi\sqrt{2})^3}\int_H^{2H}\sum\nolimits_N^3(x)\,\mathrm{d}x + O(H^{5/4})$$
$$+\frac{1}{(\pi\sqrt{2})^3}\int_H^{2H}\Bigl(3\sum\nolimits_N^2(x)R_{N,Q}(x) + 3\sum\nolimits_N(x)R_{N,Q}^2(x) + R_{N,Q}^3(x)\Bigr)\,\mathrm{d}x.$$

Had we used directly (1.14), we would have obtained (3.4), but with the error term $O_\varepsilon(H^{3/2+\varepsilon})$ instead of $O(H^{5/4})$, and this is too large for our purposes.

We shall first evaluate the integral with $\sum_N^3(x)$, and then the remaining integrals. If we used directly the bound (3.3) for $R_{N,y}(x)$, we would obtain the additional error terms (cf. [13]) $O_\varepsilon(H^{2+\varepsilon}N^{-1/2}) + O_\varepsilon(H^{5/2+\varepsilon}N^{-3/2})$, which would be too large to yield the exponent $\beta = 7/5$ in the error term in Theorem 1. Put

$$r = r(m,n,k) := d(m)d(n)d(k)(mnk)^{-3/4} \qquad (1\le m,n,k\le N), \tag{3.5}$$

and $r = 0$ otherwise. Then

$$\begin{aligned}\sum\nolimits_N^3(x) &= \frac{3}{4}\sum rx^{3/4}\cos(4\pi(\sqrt{m}+\sqrt{n}-\sqrt{k})\sqrt{x}-\tfrac{1}{4}\pi)\\ &\quad+\frac{1}{4}\sum rx^{3/4}\cos(4\pi(\sqrt{m}+\sqrt{n}+\sqrt{k})\sqrt{x}-\tfrac{3}{4}\pi)\\ &= S_0(x)+S_1(x)+S_2(x),\end{aligned}$$

say, where

$$\begin{aligned}S_0(x) &:= \frac{3}{4\sqrt{2}}\sum_{\sqrt{m}+\sqrt{n}=\sqrt{k}} rx^{3/4},\\ S_1(x) &:= \frac{3}{4}\sum_{\sqrt{m}+\sqrt{n}\ne\sqrt{k}} rx^{3/4}\cos(4\pi(\sqrt{m}+\sqrt{n}-\sqrt{k})\sqrt{x}-\tfrac{1}{4}\pi)\\ S_2(x) &:= \frac{1}{4}\sum rx^{3/4}\cos(4\pi(\sqrt{m}+\sqrt{n}+\sqrt{k})\sqrt{x}-\tfrac{3}{4}\pi).\end{aligned}$$

Tsang [13] has shown that, with explicit $c_1>0$,

$$\int_H^{2H} S_0(x)\,\mathrm{d}x = \frac{3c_1}{4\sqrt{2}}\int_H^{2H} x^{3/4}\,\mathrm{d}x + O_\varepsilon(H^{7/4+\varepsilon}N^{-1}) \tag{3.6}$$

and, by using the first derivative test, it follows that

$$\int_H^{2H} S_2(x)\,\mathrm{d}x \ll_\varepsilon H^{5/4+\varepsilon}N^{1/4}. \tag{3.7}$$

The most delicate task is the estimation of the integral of $S_1(x)$, where we shall proceed differently than in [13]. In integrating $S_1(x)$ we have that $E \neq 0$, where

$$E \ = \ E(m,n,k) \ := \ \sqrt{m} + \sqrt{n} - \sqrt{k}.$$

But if $E \neq 0$, then by (2.2) (Lemma 2) we have that

$$|E| \ \gg \ (mnk)^{-1/2}. \tag{3.8}$$

Without loss of generality we may assume that $n \le m$, hence it suffices to bound

$$\sum{}^{*} r \int_H^{2H} x^{3/4} \cos(4\pi E\sqrt{x} - \frac{3}{4}\pi)\,\mathrm{d}x, \tag{3.9}$$

where $\sum^*$ denotes summation over $m, n, k \le N$ such that $n \le m$ and (3.8) holds. We further suppose that $M' < n \le 2M'$, $M < m \le 2M$, $K < k \le 2K$, so that $M' \ll M$. Then either $K \ll M$ or $M \ll K$. Suppose first $K \ll M$. We distinguish the following cases when we estimate (3.9).

(a) *The case* $|E| \gg m^{1/2}$. Then the integral in (3.9) is estimated, by the first derivative test, as $\ll H^{5/4}|E|^{-1}$. The corresponding portion of the sum in (3.9) is

$$\begin{aligned} &\ll_\varepsilon H^{5/4+\varepsilon} \sum_{M'<n\le 2M'} \sum_{M<m\le 2M} \sum_{K<k\le 2K} (mnk)^{-3/4} m^{-1/2} \\ &\ll_\varepsilon H^{5/4+\varepsilon}(M'MK)^{1/4}M^{-1/2} \\ &\ll_\varepsilon H^{5/4+\varepsilon}(M')^{1/4} \\ &\ll_\varepsilon H^{5/4+\varepsilon}N^{1/4}. \end{aligned} \tag{3.10}$$

(b) *The case* $|E| \le cm^{1/2}$ *with small* $c > 0$. Then it follows that

$$k = (\sqrt{m} + \sqrt{n})^2 + O(|E|\sqrt{m}) = (\sqrt{m} + \sqrt{n})^2 + O(cm),$$

hence $k \asymp m$ if c is sufficiently small. We use the first derivative test and Lemma 4 (with $\delta = |E|\sqrt{M}$) to obtain that the contribution will be

$$\begin{aligned} &\ll_\varepsilon H^{5/4+\varepsilon} \max_{M'\le M\ll N, |E|\gg 1/\sqrt{M^2M'}} (M^2M')^{-3/4}|E|^{-1}(M^{3/2}M'|E| \\ &\qquad\qquad + (MM')^{1/2}) \\ &\ll_\varepsilon H^{5/4+\varepsilon}N^{1/4} + H^{5/4+\varepsilon} \max_{M'\le M\ll N} (M^2M')^{-3/4}(M^2M')^{1/2}(MM')^{1/2}) \\ &\ll_\varepsilon H^{5/4+\varepsilon}N^{1/4}. \end{aligned}$$

If $M \ll K$, we repeat the above argument according to the cases when $|E| \gg k^{1/2}$ and $|E| \le ck^{1/2}$. The bound corresponding to (3.10) will be, in this case,

$$\ll_\varepsilon H^{5/4+\varepsilon}(M'MK)^{1/4}K^{-1/2} \ll_\varepsilon H^{5/4+\varepsilon}(M')^{1/4} \ll_\varepsilon H^{5/4+\varepsilon}N^{1/4}.$$

It remains to estimate the integrals in (3.4) with $R_{N,Q}(x)$, all of which will contribute to the error terms. Namely when we expand each of the three terms, we shall obtain sums analogous to $S_0(x), S_1(x), S_2(x)$, but at least one variable, say k, will satisfy $k > N$. Should it happen that a sum of two roots equals a third root, then this can happen only if $\sqrt{m} + \sqrt{n} = \sqrt{k}$. But then the argument of [13], which yielded (3.6), clearly shows that the contribution of such sums must be $\ll_\varepsilon H^{7/4+\varepsilon}N^{-1}$. In the case of the integrals of $\sum_N(x)R^2_{N,Q}(x)$ and $\sum^2_N(x)R_{N,Q}(x)$, the analogues of the sums $S_1(x)$ and $S_2(x)$ are estimated as $\ll_\varepsilon H^{5/4+\varepsilon}N^{1/4}$ by the above method of proof, since we have $M' \ll N$, and this is the crucial condition both in case (a) and case (b). However, when we integrate $R^3_{N,Q}(x)$, this approach does not work, since all variables are $\gg N$. Instead we use $R_{N,Q}(x) \ll_\varepsilon H^{1/2+\varepsilon}N^{-1/2}$, and the first derivative test to obtain that

(3.11)
$$\begin{aligned}
&\int_H^{2H} R^3_{N,Q}(x)\,\mathrm{d}x \ll_\varepsilon H^{1/2+\varepsilon}N^{-1/2}\int_H^{2H} R^2_{N,Q}(x)\,\mathrm{d}x\\
&\ll_\varepsilon H^{1/2+\varepsilon}N^{-1/2}\int_H^{2H} H^{1/2}\Big|\sum_{N<n\le H^7} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\frac{1}{4}\pi)\Big|^2\,\mathrm{d}x\\
&\ll_\varepsilon H^{1/2+\varepsilon}N^{-1/2}\Bigg(H^{3/2}\sum_{n>N} d^2(n)n^{-3/2} +\\
&\qquad\qquad + H\sum_{N<m\ne n<H^7}\frac{d(m)d(n)}{(mn)^{3/4}|\sqrt{m}-\sqrt{n}|}\Bigg)\\
&\ll_\varepsilon H^{1/2+\varepsilon}N^{-1/2}(H^{3/2}N^{-1/2}+H)\\
&\ll_\varepsilon H^{2+\varepsilon}N^{-1},
\end{aligned}$$

since $N \ll H$. Thus gathering all the estimates we arrive at

$$\begin{aligned}
\text{(3.12)}\qquad \int_H^{2H}\Delta^3(x)\,\mathrm{d}x = \frac{3c_1}{4\sqrt{2}}\int_H^{2H} x^{3/4}\,\mathrm{d}x + O_\varepsilon(H^{2+\varepsilon}N^{-1})\\
+ O_\varepsilon(H^{5/4+\varepsilon}N^{1/4}).
\end{aligned}$$

The proof is completed when we take $N = H^{3/5}$, then $H = X/2, X/2^2, \ldots$ and sum all the resulting expressions. Note that it is only in (3.11) that the error term $H^{2+\varepsilon}N^{-1}$ appears, while before we had $H^{7/4+\varepsilon}N^{-1}$. If we could obtain (3.12) with the latter error term instead of $H^{2+\varepsilon}N^{-1}$, we would then obtain the exponent $\beta = 27/20$ in (1.6). This would be then the limit of the present method.

4. Proof of Theorem 2

We pass now to the proof of Theorem 2. As in the proof of Theorem 1, our approach differs from Tsang's in the treatment of the critical sums S_4, S_5 below, and additional saving comes since we use Lemma 5 and Lemma 6 instead of Tsang's (2.7). Although slight improvements of his result could be obtained by using the theory of exponent pairs to estimate the exponential sum appearing on the right-hand side of [13, eq. (4.9)], these improvements cannot attain the strength of our Lemma 5 and Lemma 6.

We start from (1.14) to obtain

$$
\begin{aligned}
(4.1)\quad \int_H^{2H} \Delta^4(x)\,\mathrm{d}x &= (\pi\sqrt{2})^{-4}\int_H^{2H} {\sum}_N(x)^4\,\mathrm{d}x \\
&\quad + O_\varepsilon\left(H^{1/2+\varepsilon}N^{-1/2}\int_H^{2H} |{\sum}_N(x)|^3\,\mathrm{d}x\right) + O_\varepsilon(H^{3+\varepsilon}N^{-2}) \\
&= (\pi\sqrt{2})^{-4}\int_H^{2H} {\sum}_N(x)^4\,\mathrm{d}x + O_\varepsilon(H^{9/4+\varepsilon}N^{-1/2}) + O_\varepsilon(H^{3+\varepsilon}N^{-2}),
\end{aligned}
$$

with $\sum_N(x)$ given by (3.3). Here, unlike in (3.4), we used the crude estimate $R_{N,H}(x) \ll_\varepsilon x^{1/2+\varepsilon}N^{-1/2}$ and the bound $\int_H^{2H} |\Delta^3(x)|\,\mathrm{d}x \ll_\varepsilon H^{7/4+\varepsilon}$ (see [5] or [6]). The error terms in (4.1) suffice for (1.7) with $\gamma = 23/12$, although they could be improved by a technique similar to the one used in the proof of Theorem 1.

To evaluate the integral with $\sum_N^4(x)$ in (4.1), we proceed similarly as in Tsang [13]. With

$$
\begin{aligned}
r_1 = r_1(m,n,k,l) \;:=\; &(mnkl)^{-3/4}d(m)d(n)d(k)d(l),\\
&m,n,k,l \;\le\; N;\; m,n,k,l \in \mathbb{N},
\end{aligned}
$$

and $r_1 \equiv 0$ otherwise, we have

$$
\begin{aligned}
{\sum}_N^4(x) &= S_3(x) + S_4(x) + S_5(x) + S_6(x),\\
S_3(x) &= \frac{3}{2}\sum_{\sqrt{m}+\sqrt{n}=\sqrt{k}+\sqrt{l}} r_1 x,\\
S_4(x) &= \frac{3}{8}\sum_{\sqrt{m}+\sqrt{n}\neq\sqrt{k}+\sqrt{l}} r_1 x\cos(4\pi(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{l})\sqrt{x}\,),\\
S_5(x) &= \frac{1}{2}\sum r_1 x\sin(4\pi(\sqrt{m}+\sqrt{n}+\sqrt{k}-\sqrt{l})\sqrt{x}\,),\\
S_6(x) &= -\frac{1}{8}\sum r_1 x\cos(4\pi(\sqrt{m}+\sqrt{n}+\sqrt{k}+\sqrt{l})\sqrt{x}\,).
\end{aligned}
$$

As in [13] it follows that, with suitable $c_2 > 0$, we have

$$\int_H^{2H} S_3(x)\,\mathrm{d}x = \frac{3}{8}c_2 \int_H^{2H} x\,\mathrm{d}x + O_\varepsilon(H^2N^{-1/4+\varepsilon}).$$

By using the first derivative test we obtain

$$\int_H^{2H} S_6(x)\,\mathrm{d}x \ll_\varepsilon H^{3/2+\varepsilon}N^{1/2}.$$

It remains to consider the integrals with $S_4(x)$ and $S_5(x)$. To do this, set

$$\Delta_\pm = \Delta_\pm(m,n,k,l) := 4\pi(\sqrt{m} + \sqrt{n} \pm \sqrt{k} - \sqrt{l}).$$

We may assume that $(M \ll K, M' \ll K)$

$$M < m \le 2M,\ M' < n \le 2M',\ K < k \le K' \le 2K,\ L < l \le L' \le 2L \tag{4.2}$$

and that $\Delta_\pm \ne 0$. In the case of Δ_- the condition $\Delta_- \ne 0$ is assumed in $S_4(x)$, and if $\Delta_+ = 0$, then $S_5(x)$ vanishes identically. If $\Delta_\pm \ne 0$, then by (2.3) we have

$$|\Delta_\pm| \gg k^{-2}(mnl)^{-1/2} \qquad (m \le k,\, n \le k). \tag{4.3}$$

We suppose that

$$|\Delta_\pm| \asymp \delta\sqrt{K}, \tag{4.4}$$

and (similarly as in the proof of Lemma 5) we may assume that $0 < \delta \ll 1/K$. Namely, if $\delta \gg 1$, then the number of solutions of (4.4) is trivially $\ll MM'KL\delta$, hence by the first derivative test and trivial estimation the relevant portions of $S_4(x)$ and $S_5(x)$ are bounded by

$$H^{3/2+\varepsilon}|\Delta_\pm|^{-1}(MM'KL)^{-3/4}MM'KL\delta \ll_\varepsilon H^{3/2+\varepsilon}N^{1/2}, \tag{4.5}$$

since $K \ll N$. If $1/K \ll \delta \ll 1$, then $|\Delta_\pm| \gg K^{-1/2}$, and moreover we have

$$(\sqrt{m} + \sqrt{n} - \sqrt{l})^2 = (\pm\sqrt{k})^2 + O(\delta\sqrt{kK}) = k + O(\delta K),$$

implying that, for a given triplet (m,n,l), there are at most $\ll \delta K$ choices for k. Hence trivially there are no more than $\ll MM'\delta KL$ choices for (m,n,k,l), and this gives us again the bound in (4.5). Then we obtain, squaring the defining relation of $\Delta_\pm$, that $M \asymp K$.

To bound the integrals of $S_4(x)$ and $S_5(x)$ it will be sufficient to estimate

$$\int_H^{2H} \sideset{}{^*}\sum_{m,n,k,l} r_1 x\,\mathrm{e}(\Delta_\pm\sqrt{x}\,)\,\mathrm{d}x, \tag{4.6}$$

where $\sum^*$ means that (4.2), (4.3) and the conditions of Lemma 5 (or Lemma 6) hold with $0 < \delta \ll 1/K$.

We shall consider the case $\Delta = \Delta_-$ in detail, since the case of Δ_+ is analogous, but less difficult, due to the absence of the last term in (2.15) of

Lemma 5 in the bound (2.22) of Lemma 6. We shall estimate the integral in (4.6) trivially, or by the first derivative test to obtain

$$\int_H^{2H} x\mathrm{e}(\Delta\sqrt{x}\,)\,\mathrm{d}x \ \ll\ H^2\min\left(1,\ \frac{1}{|\Delta|\sqrt{H}}\right). \tag{4.7}$$

After that, we estimate the remaining portion of the sum by Lemma 5. Depending on which term in (2.15) (or (2.16)) dominates, we may use either of the bounds in (4.7) as we please. We consider two cases.

(a) *The case when* $K \gg H^{1/6}$. Then by using (2.15) (with $\delta \asymp |\Delta|K^{-1/2}$) and (4.7) we see that the relevant portion of (4.6) is

$$\begin{aligned}
&\ll_\varepsilon H^{2+\varepsilon}\max_{M\asymp K, M'\ll K, H^{1/6}\ll K\ll N}\ \sideset{}{^*}\sum_{m,n,k,l} r_1\min\Big(1,\ \frac{1}{|\Delta|\sqrt{H}}\Big)\\
&\ll_\varepsilon H^{2+\varepsilon}\max_{M\asymp K, K\gg H^{1/6}}\big\{(KMM'L)^{1/4}((KH)^{-1/2}+K^{-3/2})\\
&\quad +(KMM'L)^{-3/4}K(M'L)^{1/2}\big\}\\
&\ll_\varepsilon H^{2+\varepsilon}N^{1/2}+\max_{K\gg H^{1/6}} H^{2+\varepsilon}K^{-1/2}\\
&\ll_\varepsilon H^{2+\varepsilon}N^{1/2}+H^{23/12+\varepsilon}.
\end{aligned} \tag{4.8}$$

(b) *The case when* $K \ll H^{1/6}$. Then by using (2.16) (with $\delta \asymp |\Delta|K^{-1/2}$) and (4.7) we see that the relevant portion of (4.6) is

$$\begin{aligned}
&\ll_\varepsilon H^{2+\varepsilon}\max_{M\asymp K, M'\ll K, K\ll H^{1/6}}\ \sideset{}{^*}\sum_{m,n,k,l} r_1\min\Big(1,\ \frac{1}{|\Delta|\sqrt{H}}\Big)\\
&\ll_\varepsilon H^{2+\varepsilon}\max_{M\asymp K, K\ll H^{1/6},(4.3)}\Big\{(MM'KL)^{1/4}(K^{-1/2}K^2H^{-1/2}\\
&\qquad\qquad + \frac{(MM'KL)^{-1/2}}{|\Delta|\sqrt{H}}\Big\}\\
&\ll_\varepsilon H^{2+\varepsilon}\max_{K\ll H^{1/6},(4.3)}\Big(H^{-1/2}K^{5/2}+H^{-1/2}|\Delta|^{-1}(KMM'L)^{-1/4}\Big)\\
&\ll_\varepsilon H^{2+\varepsilon}\Big(H^{-1/12}+\max_{K\ll H^{1/6},(4.3)} H^{-1/2}|\Delta|^{-1}K^{-1/2}(M'L)^{-1/4}\Big).
\end{aligned} \tag{4.9}$$

But as $\Delta \neq 0$, (4.2) and (4.3) give

$$|\Delta|^{-1} \ \ll\ K^{5/2}(M'L)^{1/2},$$

and therefore the contribution of (4.9) will be

$$\ll_\varepsilon H^{2+\varepsilon}\Big(H^{-1/12}+\max_{K\ll H^{1/6}} K^{5/2}H^{-1/2}\Big) \ll_\varepsilon H^{23/12+\varepsilon}.$$

Therefore putting together all the estimates, we obtain

$$\begin{aligned}(4.10)\qquad \int_H^{2H} \Delta^4(x)\,\mathrm{d}x = \frac{3}{8}c_2\int_H^{2H} x\,\mathrm{d}x &+ O_\varepsilon(H^{2+\varepsilon}N^{-1/4})\\ &+ O_\varepsilon(H^{3/2+\varepsilon}N^{1/2}) + O_\varepsilon(H^{9/4+\varepsilon}N^{-1/2})\\ &+ O_\varepsilon(H^{3+\varepsilon}N^{-2}) + O_\varepsilon(H^{23/12+\varepsilon}).\end{aligned}$$

The choice $N = H^{3/4}$ yields the assertion of Theorem 2, when we take $H = X/2, X/2^2, \ldots$ and sum the resulting expressions. The limit of the method is the exponent $\gamma = 11/6$ in (1.7). This would follow, with $N = H^{2/3}$, if all the error terms in (4.10) could be made to be $O_\varepsilon(H^{2+\varepsilon}N^{-1/4}) + O_\varepsilon(H^{3/2+\varepsilon}N^{1/2})$.

5. Proof of Theorem 3

The starting point for both (1.10) and (1.11) is the formula

$$(5.1)\quad \pi\sqrt{2}\Delta(x) = \sum\nolimits_N(x) + R_{N,X}(x) + O_\varepsilon(x^\varepsilon) \quad (X \le x \le X+H, N \ll X),$$

which follows on combining (1.14) and (3.3), and where the parameter N will be suitably chosen. There are many details in the proof similar to the proof of Theorem 1 and Theorem 2, so that we may be fairly brief. From (5.1) we obtain

$$\begin{aligned}(5.2)\quad \int_X^{X+H} \Delta^3(x)\,\mathrm{d}x = (\pi\sqrt{2})^{-3}\int_X^{X+H} &\sum\nolimits_N^3(x)\,\mathrm{d}x\\ &+ O(\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + HX^{2/3}),\end{aligned}$$

say, where we used (5.6) with $\theta < 1/3$ and we set

$$\mathcal{R}_1 := \Big|\int_X^{X+H} \sum\nolimits_N^2(x)R_{N,X}(x)\,\mathrm{d}x\Big|,$$

$$\mathcal{R}_2 := \int_X^{X+H} \Big|\sum\nolimits_N(x)\Big|R_{N,X}^2(x)\,\mathrm{d}x,\ \mathcal{R}_3 := \int_X^{X+H} |R_{N,X}^3(x)|\,\mathrm{d}x.$$

Since $X^{7/12+\delta} \le H \le X$ it follows (e.g., by the technique that gives (13.48) of [6]) that the contribution of the term $O_\varepsilon(x^\varepsilon)$ in (5.1) is absorbed by the error term $HX^{2/3}$ in (5.2). Analogously as in the proof of Theorem 1 we obtain

$$\begin{aligned}(5.3)\qquad \int_X^{X+H} \sum\nolimits_N^3(x)\,\mathrm{d}x = B&\left((X+H)^{7/4} - X^{7/4}\right)\\ &+ O_\varepsilon(X^{5/4+\varepsilon}N^{1/4}) + O_\varepsilon(HX^{3/4+\varepsilon}N^{-1}).\end{aligned}$$

Note that trivial estimation gives

$$(5.4)\qquad \sum\nolimits_N(x) \ll_\varepsilon X^{1/4+\varepsilon}N^{1/4} \qquad (X \le x \le X+H),$$

hence from (5.1) we obtain

$$R_{N,X}(x) \ll_\varepsilon X^{\theta+\varepsilon} + X^{1/4+\varepsilon}N^{1/4} \qquad (X \le x \le X+H), \tag{5.5}$$

where θ is such a constant ($1/4 \le \theta \le 1/3$) for which one has

$$\Delta(x) \ll_\varepsilon x^{\theta+\varepsilon}. \tag{5.6}$$

We have, by using the estimation of (3.11) and (5.4),

$$\begin{aligned}\mathcal{R}_2 &\ll \max_{X\le x\le X+H}\Big|\sum\nolimits_N(x)\Big|\int_X^{X+H} R^2_{N,X}(x)\,\mathrm{d}x \\ &\ll_\varepsilon X^{5/4+\varepsilon}N^{1/4} + HX^{3/4+\varepsilon}N^{-1/4}.\end{aligned} \tag{5.7}$$

In a similar way, by using (5.5), it follows that

$$\begin{aligned}\mathcal{R}_3 &\ll \max_{X\le x\le X+H}\Big|R_{N,X}(x)\Big|\int_X^{X+H} R^2_{N,X}(x)\,\mathrm{d}x \\ &\ll_\varepsilon X^{\varepsilon}(X^{5/4}N^{1/4} + HX^{3/4}N^{-1/4} + X^{1+\theta} + HX^{1/2+\theta}N^{-1/2}).\end{aligned} \tag{5.8}$$

We develop the integrand in the expression for $\mathcal{R}_1$ (dividing first sums in dyadic intervals of summation), noting that in the exponential we shall obtain expressions of the form

$$\sigma := \pm\sqrt{m} \pm \sqrt{n_1} \pm \sqrt{n_2} \qquad (N < m \le X,\ n_1 \le N,\ n_2 \le N).$$

If $n_1,\ n_2 \le N/8$ or $m > 8N$, then by the first derivative test the contribution to $\mathcal{R}_1$ is clearly $\ll_\varepsilon X^{5/4+\varepsilon}N^{1/4}$. If $\sigma := \sqrt{m}+\sqrt{n_1}-\sqrt{n_2}$ (> 0, since $m > N$ and $n_1, n_2 \le N$), then the contribution to $\mathcal{R}_1$ is

$$\ll_\varepsilon X^{5/4+\varepsilon}\sum_{n_1,n_2\le N}\ \sum_{N<m\le X}(n_1n_2m)^{-3/4}(\sqrt{m}+\sqrt{n_1}-\sqrt{n_2}\,)^{-1}.$$

The contribution of triplets (m, n_1, n_2) for which $\sigma \gg \sqrt{m}$ is $\ll_\varepsilon X^{5/4+\varepsilon}N^{1/4}$. Suppose now that $\sigma \asymp \eta\sqrt{m}$, with $\eta > 0$ sufficiently small. This is possible only if $n_2 \asymp N$ and $m \asymp N$. For a fixed n_1, set $m = N + h, n_2 = N - k$ with $h, k \in \mathbb{N}$. Then $\sigma \ll \eta\sqrt{m}$ implies

$$\sqrt{N+h} + \sqrt{N-k} \ll \eta\sqrt{N},$$

hence $h + k \ll \eta N$. Thus there are $\ll \eta N^2$ choices for (m, n_2), and the contribution of such σ is again $\ll_\varepsilon X^{5/4+\varepsilon}N^{1/4}$. Replacing η by $2^{-j}\eta$ ($j \in \mathbb{N}$) and noting that there are $O(\log X)$ values of j by (2.2) of Lemma 2, it follows that $\mathcal{R}_1$ is clearly $\ll_\varepsilon X^{5/4+\varepsilon}N^{1/4}$. In case when $\sigma = \sqrt{n_1}+\sqrt{n_2}-\sqrt{m}$ ($N/8 \le n_1, n_2 \le N,\ N < m \le 8N$) we can have $\sigma = 0$. The contribution of such triplets (m, n_1, n_2) is easily seen to be $\ll_e HX^{3/4+\varepsilon}N^{-1}$. If $\sigma \ne 0$ in this case, we again get a contribution which is $\ll_\varepsilon X^{5/4+\varepsilon}N^{1/4}$. The contributions of other possible choices of signs in σ are treated similarly, so that we finally have

$$\mathcal{R}_1 \ll_\varepsilon X^{5/4+\varepsilon}N^{1/4} + HX^{3/4+\varepsilon 7}N^{-1}. \tag{5.9}$$

If we now choose (recall that $\theta \ge 1/4$ must hold; see, e.g., [6, Chapter [13])

$$N = X^{4\theta-1+\varepsilon}, \tag{5.10}$$

then from (5.2), (5.3) and (5.7)–(5.10) we obtain

$$\int_X^{X+H} \Delta^3(x)\,\mathrm{d}x = B\left((X+H)^{7/4} - X^{7/4}\right) + \mathcal{R}, \tag{5.11}$$

with

$$\mathcal{R} \ll_\varepsilon X^{1+\theta+\varepsilon} + HX^{3/4-(\theta-1/4-\varepsilon)}. \tag{5.12}$$

The classical value $\theta = 1/3$ gives rise to the exponent $7/12$ in (1.10), while (5.11) and (5.12) show that actually $7/12$ can be replaced by $1/4+\theta$. Thus the conjectural $\theta = 1/4$ would replace $7/12$ by $1/2$ in (1.10), and the value $\theta \le 23/73$ (see M.N. Huxley [4]) yields the constant $165/292 = 0.56506\ldots < 7/12 = 0.58333\ldots$.

We pass now to the proof of (1.11). We need the following lemma.

Lemma 7. *For any given* $0 < \kappa < 1/2$ *and* $X^{2\kappa} \le N \ll X$ *we have*

$$\int_X^{X+H} R^4_{N,X}(x)\,\mathrm{d}x \ll_\kappa X^{5/3+\kappa} + HX^{1-\kappa}. \tag{5.13}$$

Proof. With $M < M' \le 2M$ we have

$$\begin{aligned}\int_X^{X+H} R^4_{N,X}(x)\,\mathrm{d}x &\ll \log X \max_{N\le M\ll X} \int_X^{X+H} R^4_{M,M'}(x)\,\mathrm{d}x \\ &\ll_\varepsilon X^{1+\varepsilon} \max_{N\le M\ll X} \int_{X-H}^{X+H} \varphi(x)\Big|\sum_{M<n\le M'} d(n)n^{-3/4}\mathrm{e}^{4\pi i\sqrt{nx}}\Big|^4\,\mathrm{d}x,\end{aligned} \tag{5.14}$$

where $\varphi(x)$ is a smooth, non-negative function supported in $[X-2H, X+2H]$ such that $\varphi(x) = 1$ when $x \in [X-H, X+H]$, and $\varphi^{(r)}(x) \ll_r H^{-r}$ for $r = 0, 1, 2, \ldots$. The fourth power of the above sum equals

$$\sum_{M<n_1,n_2,n_3,n_4<M'} d(n_1)d(n_2)d(n_3)d(n_4)(n_1n_2n_3n_4)^{-3/4}\mathrm{e}^{Di\sqrt{x}},$$

where $\mathcal{D} = \mathcal{D}(n_1,n_2,n_3,n_4) := 4\pi(\sqrt{n_1}+\sqrt{n_2}-\sqrt{n_3}-\sqrt{n_4}\,)$. We perform then, in the last integral in (5.14), a large number of integrations by parts, taking into account that $\varphi^{(r)}(x) \ll_r H^{-r}$. It transpires that only the contribution of quadruples (n_1,n_2,n_3,n_4) for which $|\mathcal{D}| \le X^{1/2+\varepsilon}H^{-1}$ will be non-negligible. This is estimated, by Lemma 1 (with $k = 2, \delta \asymp X^{1/2+\varepsilon}H^{-1}M^{-1/2}$) and trivial estimation, as

$$\int_{X-H}^{X+H}\varphi(x)\Big|\sum_{M<n\le M'} d(n)n^{-3/4}\mathrm{e}^{4\pi i\sqrt{nx}}\Big|^4\,\mathrm{d}x \tag{5.15}$$
$$\ll_\varepsilon HM^{-3}(M^4X^{1/2+\varepsilon}H^{-1}M^{-1/2}+M^2)$$
$$\ll_\varepsilon HX^{1/2+\varepsilon}M^{1/2}+HM^{-1}.$$

On the other hand, by (3.3),

$$R_{M,M'}(x)\ll_\varepsilon x^{1/2+\varepsilon}M^{-1/2}\qquad (N\le M\ll X). \tag{5.16}$$

This gives

$$\int_X^{X+H} R^4_{M,M'}(x)\,\mathrm{d}x\ll_\varepsilon X^{1+\varepsilon}M^{-1}\int_X^{X+H}R^2_{M,M'}(x)\,\mathrm{d}x \tag{5.17}$$
$$\ll_\varepsilon X^{2+\varepsilon}M^{-1}+HX^{3/2+\varepsilon}M^{-3/2},$$

by the argument used in deriving (3.11).

If $N\le M\le X^{1/3+\kappa}$, then (5.14)-(5.15) give

$$\int_X^{X+H} R^4_{M,M'}(x)\,\mathrm{d}x\ll_\varepsilon X^{5/3+\varepsilon+\kappa/2}+HX^{1+\varepsilon-2\kappa},$$

while for $M>X^{1/3+\kappa}$ and $N\ge X^{2\kappa}$ we infer from (5.17) that

$$\int_X^{X+H} R^4_{M,M'}(x)\,\mathrm{d}x\ll_\varepsilon X^{5/3+\varepsilon-\kappa}+HX^{1+\varepsilon-3\kappa/2}.$$

The bound in (5.13) follows on combining the last two bounds in conjunction with (5.14), and taking, e.g., $\varepsilon=\kappa/4$ sufficiently small. $\square$

It is now not difficult to obtain (1.11). From (5.1) we have

$$\int_X^{X+H}\Delta^4(x)\,\mathrm{d}x=(\pi\sqrt{2})^{-4}\int_X^{X+H}{\sum}_N^4(x)\,\mathrm{d}x \tag{5.18}$$
$$+O\Big(\sum_{j=1}^4 I_j\Big)+O_\varepsilon(HX^{3\theta}),$$

with θ given by (5.6) (for our purposes any $\theta<1/3$ clearly suffices) and

$$I_j:=\int_X^{X+H}|R_{N,X}(x)|^j|{\sum}_N(x)|^{4-j}\,\mathrm{d}x. \tag{5.19}$$

By the arguments of Tsang [13] (see also our discussion after (4.1)) we have

$$\int_X^{X+H}{\sum}_N^4(x)\,\mathrm{d}x=(\pi\sqrt{2})^4C((X+H)^2-X^2) \tag{5.20}$$
$$+O_\varepsilon(HX^{1+\varepsilon}N^{-1/4})+O(X^{3/2+\varepsilon}N^{9/2}),$$

and in particular, with $N = X^\kappa$ and sufficiently small $\kappa > 0$,

$$\int_X^{X+H} \sum\nolimits_N^4(x)\,\mathrm{d}x \ll HX \qquad (X^{1/2+\delta} \le H \ll X). \tag{5.21}$$

Using Lemma 7, (5.21) and Hölder's inequality for integrals we obtain, for $j = 1, 2, 3, 4$,

$$\begin{aligned} I_j &\le \Bigl(\int_X^{X+H} R_{N,X}^4(x)\,\mathrm{d}x\Bigr)^{j/4}\Bigl(\int_X^{X+H} \sum\nolimits_N^4(x)\,\mathrm{d}x\Bigr)^{(4-j)/4} \\ &\ll (X^{5/3+\kappa/2} + HX^{1-\kappa/2})^{j/4}(HX)^{1-j/4} \\ &\ll H^{1-j/4}X^{1+j/6+\kappa j/8} + HX^{1-\kappa j/8}. \end{aligned} \tag{5.22}$$

If $H \ge X^{2/3+\delta}$ with a fixed $\delta > 0$, then taking $0 < \kappa < \delta$ we obtain from (5.22)

$$I_j \ \ll\ HX^{1-\kappa/8} \qquad (j = 1, 2, 3, 4). \tag{5.23}$$

Inserting (5.20) and (5.23) in (5.18) we obtain (1.11), with $\kappa/8$ replacing κ.

There are possibilities to extend the range of H for which (1.11) holds. For example, instead of (5.16) we may write explicitly

$$R_{M,M'}(x) \ll x^{1/4}\Bigl|\sum_{M<n\le M'} d(n)n^{-3/4}\mathrm{e}^{4\pi i\sqrt{nx}}\Bigr|, \tag{5.24}$$

taking account that the sum in (5.24) reduces to a double exponential sum on writing $d(n) = \sum_{m_1m_2=n} 1$. Estimating the terms $m_1 = m_2$ trivially, we see that

$$R_{M,M'}(x) \ll x^{1/4}M^{-1/4}\sum_{1\le m_2\le M'}\Bigl|\sum_{m_1\in I(m_2)} \mathrm{e}^{4\pi i\sqrt{m_1m_2x}}\Bigr| + M^{1/8}x^{1/8},$$

where m_1 runs over an interval $I(m_2)$, which is contained in an interval of length $\ll M/m_2$. Since M is of the order of magnitude close to $X^{1/3}$, $m_1 \gg X^{1/6}$, and the fifth derivative of $\sqrt{m_1m_2x}$ with respect to m_1 is sufficiently small, it means that the fifth derivative test (see Graham-Kolesnik [1, Th. 2.8] with $q = 3$) can be applied. This will produce a non-trivial estimation for $R_{M,M'}(x)$ which, in the range relevant for our problem, will lead to a better value than $2/3$ in (1.11).

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