

EVEN KAKUTANI EQUIVALENCE AND THE LOOSE BLOCK INDEPENDENCE PROPERTY FOR POSITIVE ENTROPY \mathbb{Z}^d ACTIONS

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ABSTRACT. In this paper we define the loose block independence property for positive entropy \mathbb{Z}^d actions and extend some of the classical results to higher dimensions. In particular, we prove that two loose block independent actions are even Kakutani equivalent if and only if they have the same entropy. We also prove that for $d > 1$ the ergodic, isometric extensions of the positive entropy loose block independent \mathbb{Z}^d actions are also loose block independent.

1. Introduction

Even Kakutani equivalence is one of the standard examples of restricted orbit equivalence. The Ambrose-Kakutani Theorem states that a free ergodic measure preserving \mathbb{R} action can be represented as a flow under a function [1]. Two transformations S and T acting on Lebesgue spaces (X, \mathcal{M}, μ) and (Y, \mathcal{G}, ν) are said to be even Kakutani equivalent if they arise as sections of equal frequency in two such representations of a fixed \mathbb{R} action [14]. In higher dimensions, the Katok Representation Theorem is the analog of the Ambrose-Kakutani Theorem and even Kakutani equivalence can be defined in precisely the same way for \mathbb{Z}^d actions, for $d > 1$ (see [2] and [11]).

Two even Kakutani equivalent \mathbb{Z} actions are related via an orbit equivalence which is order preserving on a subset of positive measure (see, for example, [14]). In higher dimensions del Junco and Rudolph have shown that the equivalence relation can still be cast as a restricted orbit equivalence [2]. However, due to the higher dimensional geometry of the orbits the characterization is more complicated.

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In one dimension it is known that two loosely Bernoulli transformations are even Kakutani equivalent if and only if they have equal entropy [14]. The one-dimensional definition of the loosely Bernoulli property involves a form of independence from the past. As the notion of a past is not so canonical in higher dimensions (see, for example, [8] and [18]), it is not clear how to extend the definition of loosely Bernoulli to higher dimensions. In this paper we define instead a “block independence” property called loose block independence (LBI), and we prove the following results:

THEOREM 1.1. *Let T be a free, ergodic, measure preserving \mathbb{Z}^d action on (X, \mathcal{M}, μ) . Then T is LBI if and only if T is even Kakutani equivalent to a Bernoulli shift or a square rank action.*

(A rank one action is said to be square if the generating towers have the shape of rectangles of bounded eccentricity; see [5].)

We note that in the case of zero entropy, since the past determines the future, the one dimensional definition of the loosely Bernoulli property is significantly simpler and can readily be extended to higher dimensions (see, for example, [5]). We show here that this definition is consistent with the loose block independence property. We prove:

THEOREM 1.2. *Let T be a Bernoulli \mathbb{Z}^d action on (X, \mathcal{M}, μ) and S a zero entropy loosely Bernoulli \mathbb{Z}^d action on (Y, \mathcal{F}, ν) . Then $T \times S$ is an LBI action on $(X \times Y, \mathcal{M} \times \mathcal{F}, \mu \times \nu)$.*

Finally, we show:

THEOREM 1.3. *Let (X, μ, T) be a free, ergodic, measure preserving, and LBI \mathbb{Z}^d action, G the group of isometries of a compact metric space C , and $h : X \times \mathbb{Z}^d \rightarrow G$ a T cocycle. If T^h is ergodic then T^h is also LBI.*

In the case $d = 1$ the first two results are due to Ornstein, Rudolph and Weiss [14], and the third result is due to Rudolph [17]. Their proofs are at times similar, and at times different in flavor to our proofs, due to the higher dimensional geometry of the orbits of \mathbb{Z}^d actions. We will point out the similarities and the differences between these arguments throughout the paper.

Our results are perhaps best summarized in the context of the theory of restricted orbit equivalence developed by Kammeyer and Rudolph [10]. When an equivalence relation on measure preserving, ergodic \mathbb{Z}^d actions is cast as a restricted orbit equivalence, it is possible to find a family of distinguished actions for which there is an intrinsic and computable complete invariant; see [10] and [16]. In particular, Kammeyer and Rudolph developed a general theory which associates to each restricted orbit equivalence class a size m and a metric \overline{m} on processes. They defined an invariant for the equivalence

class called m -entropy and the notion of \overline{m} -finitely determined actions, and showed that if two \mathbb{Z}^d actions S and T are \overline{m} -finitely determined and have equal m -entropy then they are m -equivalent. They also showed that a size m is either entropy preserving, that is, m -entropy is the usual entropy, or it is entropy free (see [10]).

In this context the one dimensional theory of even Kakutani equivalence is complete. Feldman introduced the size (\overline{f}) associated with even Kakutani equivalence [3]. Ornstein, Rudolph and Weiss proved the equivalence relation for this size, and showed that the \overline{f} -finitely determined family of transformations are exactly the loosely Bernoulli actions [14]. Some of this theory was developed independently by Katok and Sataev ([12], [19]). We refer the reader to [14] for a history of these developments and more specific references.

In the higher dimensional case Hasfura [4] generalized \overline{f} to higher dimensions and (since his work also predated [10]) proved the equivalence relation for this size in higher dimensions. What remained to be done was to give a definition of loosely Bernoulli that would allow the extension of the one dimensional theory.

The notion of block independence was first introduced by Shields, who defined almost block independence as an alternative characterization of Bernoulli transformations [20]. Rahe and Swanson adapted Shields' definition to the \overline{f} size and gave an alternate characterization of the loosely Bernoulli transformations [15]. In [18] Rudolph and Schmidt defined almost block independence in higher dimensions to characterize higher dimensional actions which are isomorphic to Bernoulli actions. Our definition of loose block independence is an adaptation of their work using the \overline{f} -metric in place of the \overline{d} metric, and it is a direct higher dimensional generalization of the Rahe and Swanson definition.

The outline of this paper is as follows. We begin in Section 2 by introducing some basic notation and terminology that we will need throughout the paper, including the definition of even Kakutani equivalence. In Section 3 we review the definition of the size \overline{f} in higher dimensions, introduce the \overline{f} -block independence property, and discuss its connection to zero entropy loosely Bernoulli actions. In Section 5 we prove one of the main results of the paper: that the \overline{f} -finitely determined actions are exactly the loose block independent ones. Theorem 1.1 follows from the results of this section. In Section 6 we give examples of loose block independent actions and we prove Theorems 1.2 and 1.3.

2. Basic definitions

Let (X, \mathcal{M}, μ) be a Lebesgue probability space and let T be a free, ergodic, measure preserving \mathbb{Z}^d action on X . That is, if $x \in X$ and $\vec{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d$ then $T_{\vec{v}}(x) = T_{\vec{e}_1}^{v_1} \circ \dots \circ T_{\vec{e}_d}^{v_d}(x)$, where the $T_{\vec{e}_i}$ are d commuting, measure preserving transformations on X called the generators of T , and the vectors

$\{\vec{e}_i\}_{i=1}^d$ are the canonical basis of \mathbb{Z}^d . We denote by T^n the action of the subgroup $(n\mathbb{Z})^d$, that is, the action generated by the maps $T_{\vec{e}_i}^n, i = 1, \dots, d$. For $n \in \mathbb{N}$, let $B_n = \{\vec{v} \in \mathbb{Z}^d : 0 \leq v_i < n, 1 \leq i \leq d\}$. For $\vec{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d$ we set $\|\vec{v}\| = \max_{i=1, \dots, d} |v_i|$.

Given a \mathbb{Z}^d action T acting on (X, \mathcal{M}, μ) , we define the orbit relation of T , denoted by $R_T \subset X \times X$, by $(x, y) \in R_T$ if and only if there exists a vector $\vec{v} \in \mathbb{Z}^d$ such that $T_{\vec{v}}x = y$. The T -ordering function $\vec{T} : R_T \rightarrow \mathbb{Z}^d$ is then defined as $\vec{T}(x, y) = \vec{v}$ if and only if $T_{\vec{v}}x = y$.

2.1. Even Kakutani equivalence. As was mentioned in the introduction, two \mathbb{Z} actions T and S acting on (X, \mathcal{M}, μ) and (Y, \mathcal{G}, ν) , respectively, are even Kakutani equivalent if and only if there are sets $A \subset X$ and $B \subset Y$ of equal measure such that S_A and T_B are isomorphic (see, for example, [14]). In terms of restricted orbit equivalence, this translates into the existence of an orbit equivalence between T and S which is order preserving on the set A .

The following definition of higher dimensional even Kakutani equivalence is due to del Junco and was studied in [2].

DEFINITION 2.1. *Two \mathbb{Z}^d actions T and S acting on (X, \mathcal{M}, μ) and (Y, \mathcal{G}, ν) , respectively, are even Kakutani equivalent if there is an orbit equivalence $\phi : X \rightarrow Y$ between T and S such that, given $\epsilon > 0$, there is a constant $N > 0$ and a set $A \subset X$ with $\mu A > 1 - \epsilon$ such that, for all $x, y \in A$ and on the same orbit, if $\|\vec{T}(x, y)\| > N$ then*

$$\|\vec{T}(x, y) - \vec{S}(\phi x, \phi y)\| < \epsilon \|\vec{T}(x, y)\|.$$

The two definitions coincide in the case $d = 1$.

Fix a \mathbb{Z}^d action T on (X, \mathcal{M}, μ) . In [4] Hasfura provides a mechanism for obtaining an action \hat{T} which is even Kakutani equivalent to T by performing a sequence of orbit equivalences. He defines a relation $\overset{\epsilon}{\approx}$, which we will use. To define the relation we begin by defining the size of a permutation of B_n . This idea is defined and extended in [4] and [10].

DEFINITION 2.2. *Let $\pi : B_n \rightarrow B_n$ be a permutation of the indices of B_n . We say π is of size ϵ , and denote this by $m(\pi) < \epsilon$, if there exists a subset S of B_n satisfying*

- (1) $|S| > (1 - \epsilon)|B_n|$, where $|S|$ is the cardinality of the set S ,
- (2) $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \epsilon\|\vec{u} - \vec{v}\|$ for every $\vec{u}, \vec{v} \in S$.

The set S is said to be an ϵ -set of π .

We next consider permutations Π of \mathbb{Z}^d . We denote the restriction of Π to $A \subset \mathbb{Z}^d$ by $\Pi|_A$.

DEFINITION 2.3. A permutation Π of \mathbb{Z}^d is called an ϵ -permutation if there is a $J \in \mathbb{Z}^+$ and a collection $\mathcal{B} = \{B_J + \vec{v}_i\}_{i=1}^\infty$, $\vec{v}_i \in \mathbb{Z}^d$, of disjoint translates of B_J such that $\limsup_n |\mathcal{B}|/|B_n| < \epsilon$, $m(\Pi|_{\{B_J + \vec{v}_i\}}) < \epsilon$ for all i , and such that $\Pi|_{\mathcal{B}^c}$ is the identity.

We are finally ready to define the relation $\overset{\epsilon}{\approx}$.

DEFINITION 2.4. Two \mathbb{Z}^d actions T and S acting on (X, \mathcal{M}, μ) and (Y, \mathcal{G}, ν) , respectively, are said to differ by an ϵ -cocycle if they are orbit equivalent via an orbit equivalence $\phi : X \rightarrow Y$ such that for all $x \in X$ the function $\alpha_x : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ defined by

$$\alpha_x(\vec{v}) = \vec{S}(\phi x, \phi(T^{\vec{v}}x))$$

is an ϵ -permutation of \mathbb{Z}^d .

The geometry of the orbit equivalence defined above is designed to ensure that a sequence of such equivalences applied to an action T will converge to a new action \hat{T} which is even Kakutani equivalent to T .

PROPOSITION 2.5 (Proposition 1 of [4]). Given a \mathbb{Z}^d action T_0 , we can find a sequence ϵ_i , $i \geq 1$, such that if $T_{i-1} \overset{\epsilon_i}{\approx} T_i$ for all i , then the sequence $\{T_i\}$ converges to an action \hat{T} ; furthermore T_0 is even Kakutani equivalent to \hat{T} .

2.2. Processes and process distributions. Let T be a \mathbb{Z}^d action on (X, \mathcal{M}, μ) . If P is a measurable finite partition on X with alphabet $A = \{1, \dots, h\}$, then by (T, P) we mean the usual process associated with T and the partition P . We call μ the distribution of the process, that is, (T, P) is the process on $A^{\mathbb{Z}^d}$ with distribution μ .

For each element $x \in X$ we define its P_n -name $P_n(x) : B_n \rightarrow P$ by $P_n(x)(\vec{v}) = i$ if $T_{\vec{v}}(x) \in p_i$. To simplify our notation we will call an atom of $\bigvee_{\vec{v} \in B_n} T_{\vec{v}}P$ of positive measure an n -name. For an n -name ω we denote the symbol occurring in position \vec{u} in ω by $\omega(\vec{u})$. We denote by μ_n the measure induced by μ on n -names.

Given partitions P and Q on (X, \mathcal{M}, μ) with the same alphabet A , we let P_n^ω be the set of points whose P_n -name is ω and similarly for Q_n^ω . We set

$$|\text{dist } P_n - \text{dist } Q_n| = \sum_{\omega \in A^{B_n}} |\mu_n(P_n^\omega) - \mu_n(Q_n^\omega)|$$

and

$$d(P_n, Q_n) = \sum_{\omega \in A^{B_n}} \mu_n(P_n^\omega \Delta Q_n^\omega),$$

where Δ denotes the symmetric difference of two sets.

We denote the metric entropy of the action T by $h(T)$, and we let $h(T, P)$ denote the entropy of the process corresponding to the partition P .

Throughout the paper we will use the idea of decomposing a name into a grid. While geometrically the idea is self explanatory, we need to introduce some notation for ease of exposition.

DEFINITION 2.6. For $k > n$ we define an n -grid of B_k to be a subdivision of B_k into n -boxes as follows. For the first grid let

$$r_{\vec{0}} = \{(m_1n, m_2n, \dots, m_dn) : 0 \leq m_i \leq \left\lfloor \frac{k}{n} \right\rfloor, i = 1, \dots, d\},$$

and set $R_{\vec{0}} = \{B_n + \vec{u} : \vec{u} \in r_{\vec{0}}\} \cap B_k$. We will call $R_{\vec{0}}$ an n -grid of B_k , the translates of B_n will be called the grid boxes, and the vectors $\vec{u} \in r_{\vec{0}}$ will be called the base points of the grid.

We obtain all n -grids of B_k by translating the grid $R_{\vec{0}}$ by all vectors $\vec{v} \in B_n$. We denote the grid starting at \vec{v} by $R_{\vec{v}} = (R_{\vec{0}} + \vec{v}) \cap B_k$ and its base points by $r_{\vec{v}} = (r_{\vec{0}} + \vec{v}) \cap B_k$.

Our first use of a grid will be to define the independent blocking of a measure.

DEFINITION 2.7. Let (T, P) be an ergodic process with distribution μ and alphabet A . For $n \in \mathbb{N}$, the independent n -blocking of μ is the measure on $A^{\mathbb{Z}^d}$ defined by $\Pi\mu_n$. That is, for $\omega \in A^{B_{jn}}$ with $j \in \mathbb{N}$, we set

$$\mu^n(\omega) = \prod_{\vec{u} \in r_{\vec{0}}} \mu_n(\omega(\vec{u} + B_n)).$$

If $\omega \in A^R$, where $R \subset \mathbb{Z}^d$ does not have shape B_{jn} for some $j \in \mathbb{N}$, then we choose the smallest j such that $B_{jn} \supset R$ and set

$$\mu^n(\omega) = \sum_{\eta \in A^{B_{jn} \setminus R}} \mu^n(\omega * \eta),$$

where $*$ denotes concatenation. That is, $\mu^n(\omega)$ is obtained by summing the μ^n measures of all possible extensions of ω into a jn -block, for the smallest possible j .

The independent n -blocking of (T, P) , denoted by $(T, P)_n$, is the process with alphabet A and distribution μ^n .

The measure μ^n is usually not preserved by T , but it is preserved under T^n . The following result is immediate.

LEMMA 2.8. Consider a process (T, P) with distribution μ , and let μ^n be the independent n -blocking of μ . Then the process (T^n, P_n) with distribution μ^n is a Bernoulli process.

3. The \bar{f} metric

In this section we give the definition of the \bar{f} metric and recall some properties and well-known technical results, which we will need in our later arguments.

DEFINITION 3.1. *Let A be a finite alphabet. Given $\eta, \xi \in A^{B_n}$, we define the distance $\bar{f}_n(\eta, \xi)$ as the infimum of all $\epsilon > 0$ such that there exists a permutation π of B_n satisfying*

- (i) $m(\pi) < \epsilon$, and
- (ii) $\bar{d}(\eta \circ \pi, \xi) < \epsilon$.

Here $\bar{d}(\cdot, \cdot)$ denotes the Hamming metric, which gives the proportion of locations of B_n on which the two names disagree.

We next define the \bar{f} distance between processes.

DEFINITION 3.2. *Given two processes (T, P) and (S, Q) with alphabet A , we define*

$$\bar{f}((T, P), (S, Q)) = \limsup_{n \rightarrow \infty} \bar{f}_n((T, P), (S, Q)),$$

where $\bar{f}_n((T, P), (S, Q))$ is the infimum of all $\epsilon > 0$ such that there exists a measure ρ on $A^{B_n} \times A^{B_n}$ satisfying

- (i) ρ has marginals μ_n and ν_n , and
- (ii) $\rho\{(\eta, \xi) \in A^{B_n} \times A^{B_n} : \bar{f}_n(\eta, \xi) < \epsilon\} > 1 - \epsilon$.

As the following lemma shows, \bar{f} does not satisfy a triangle inequality and thus is not a metric; nonetheless, \bar{f} has traditionally been called a metric.

LEMMA 3.3 (Lemma 4 in [4]). *Let (T, P) , (S, Q) and (U, R) be processes with the same alphabet. Then*

- (1) $\bar{f}((T, P), (S, Q)) = 0 \Rightarrow (T, P) = (S, Q)$;
- (2) $\bar{f}((T, P), (U, R)) \leq 2[\bar{f}((T, P), (S, Q)) + \bar{f}((S, Q), (U, R))]$.

Once the \bar{f} metric is in place, defining the \bar{f} -finitely determined processes is standard. Intuitively, these are processes for which a close approximation in entropy and distribution is sufficient to guarantee a close approximation in \bar{f} .

DEFINITION 3.4. *A process (T, P) is called \bar{f} -finitely determined (\bar{f} -FD) if for every $\epsilon > 0$ there exist $\delta > 0$ and $n \in \mathbb{N}$ such that if (S, Q) is any other processes satisfying*

$$|\text{dist } P_n - \text{dist } Q_n| < \delta \quad \text{and} \quad |h(T, P) - h(S, Q)| < \delta,$$

then it also satisfies

$$\bar{f}((T, P), (S, Q)) < \epsilon.$$

DEFINITION 3.5. *A measure preserving, free, ergodic \mathbb{Z}^d action T on (X, \mathcal{M}, μ) is said to be \bar{f} -FD if (T, P) is \bar{f} -FD for all measurable partitions P of X .*

We note that it is sufficient to show that (T, P) is \bar{f} -FD when P is a generating partition [4].

One of the basic tools of restricted orbit equivalence theory is the so-called *Copying Lemma*. We state here a version of the Copying Lemma that is essentially due to Hasfura [4].

PROPOSITION 3.6. (Lemma 7 of [4]) *Let T_1 and S_1 be \mathbb{Z}^d actions acting on the spaces (X, \mathcal{M}, μ) and (Y, \mathcal{G}, ν) , respectively, with $h(T_1) \geq h(S_1) > 0$, or $h(T_1) = h(S_1) = 0$. Suppose that*

- (1) S_1 is \bar{f} -FD;
- (2) $\{Q_i\}$ is an increasing, generating sequence of partitions on Y .

Then, given $\epsilon > 0$ there is a $\delta > 0$ such that if \tilde{Q}_1 is a partition of X satisfying

$$(1) \quad \bar{f}((T_1, \tilde{Q}_1), (S_1, Q_1)) < \delta,$$

then for any $\delta' > 0$ we can find a partition \tilde{Q}_2 and an action T_2 on X such that

- (1) $T_1 \stackrel{\epsilon}{\approx} T_2$,
- (2) $d(\tilde{Q}'_2, Q_1) < \epsilon$,
- (3) $\bar{f}((T_2, \tilde{Q}_2), (S_1, Q_2)) < \delta'$,

where \tilde{Q}'_2 is the partition obtained by lumping the atoms of \tilde{Q}_2 corresponding to those atoms of Q_2 lumped to obtain Q_1 .

The only difference between this statement and that in [4] is that we allow for the possibility that T_1 has more entropy than S_1 , and that they may both have zero entropy. Both require trivial modifications of the argument found in [4]. For the sake of completeness, we indicate these modifications below.

Proof. Suppose that $h(T_1) > h(S_1) > 0$. By the higher dimensional generalization of Proposition 3.4 in [14], if δ in (1) is small enough, then

$$|h(T_1, \tilde{Q}_1) - h(S_1, Q_1)| < \epsilon^2.$$

We then use the higher dimensional generalization of Proposition 4.4 in [14] to assume without loss of generality that $h(T_1, \tilde{Q}_1) < h(S_1)$. We can now find $j \geq 2$ such that $h(S_1, Q_j) > h(T_1, \tilde{Q}_1)$. The rest of the argument follows [4] verbatim.

If $h(T_1) = h(S_1) = 0$ the result follows from the argument in [4] stripped of all entropy considerations, since every factor of both actions will have 0 entropy. □

4. Loose block independence property

We now introduce an independence property for processes, which we will show to be equivalent to the \bar{f} -FD property.

DEFINITION 4.1. *A process (T, P) is loose block independent (LBI) if, for every $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that*

$$\bar{f}((T, P), (T, P)_n) < \epsilon$$

for every $n \geq m$.

As usual, we will say a \mathbb{Z}^d action T is LBI if (T, P) is LBI for all finite, measurable partitions P .

To motivate Definition 4.1 we remark first that, as was discussed earlier, this definition is an exact analog of the one dimensional loose block independence property defined in [15]. We also note that Rudolph and Schmidt have defined the analogous notion (ABI) for the \bar{d} metric and shown it to be equivalent to the Bernoulli property.

In addition, we note that it is easy to define loosely Bernoulli directly for zero entropy processes in higher dimensions. This is done in [5], where the authors show that finite rank actions with a certain geometry are loosely Bernoulli. We give the definition of zero entropy loosely Bernoulli below, and we prove a result which shows that Definition 4.1 is a natural extension of the zero entropy loosely Bernoulli property to the positive entropy case.

DEFINITION 4.2. *A zero entropy, ergodic process (T, P) is loosely Bernoulli if for any $\epsilon > 0$ there exists an integer N_ϵ such that for any $n \geq N_\epsilon$, there exists a set $W \subset \bigvee_{\vec{v} \in B_n} T_{\vec{v}}P$ such that $\mu(W) > 1 - \epsilon$ and for ω and ω' in W ,*

$$\bar{f}_n(\omega, \omega') < \epsilon.$$

PROPOSITION 4.3. *A zero entropy process (T, P) is loosely Bernoulli if and only if it is LBI.*

Proof. First assume that (T, P) is a zero entropy loosely Bernoulli process. Fix $\epsilon > 0$ and let $\epsilon' = \epsilon/10$. By Lemma 4.15 from [10] there exists $\delta < \epsilon'$ such that if $\pi : B_m \rightarrow B_m$ satisfies $m(\pi) < \delta$, then for every \vec{v} in the δ -set of π , $\|\pi\vec{v} - \vec{v}\| < (\epsilon')^2 m$. By Definition 4.2 we can find $M \in \mathbb{N}$ so that for all $m \geq M$, there is a set W of (T, P, m) -names such that for every $\omega, \omega' \in W$, $\bar{f}(\omega, \omega') < \delta$ and $\mu(W) > 1 - \delta$. Fix such an m .

By Lemma 2.8 we can find $k > 0$ such that for every $n > km$, the set

$$(2) \quad U_n = \left\{ \omega \in A^{B_n} : \frac{|\{\vec{u} \in r_{\vec{0}} : \omega(\vec{u} + B_m) \in W\}|}{|r_{\vec{0}}|} > 1 - 2\epsilon' \right\}$$

has μ^m measure larger than $1 - \epsilon'$.

We can similarly use the ergodicity of the process (T, P) to find n_1 such that for all $n \geq n_1$ the set

$$(3) \quad V_n = \left\{ \omega \in A^{B_n} : \frac{|\{\vec{v} \in B_n : \omega(\vec{v} + B_m) \in W\}|}{|B_n|} > 1 - 2\epsilon' \right\}$$

has μ measure larger than $1 - \epsilon'$.

Now choose $N = \max\{km, n_1\}$, so that $m/N < \epsilon'/2d$. Note that for every $n \geq N$,

$$\mu^m \times \mu(U_n \times V_n) > (1 - \epsilon')^2 > 1 - \epsilon.$$

What remains to be shown is that every $(\omega, \omega') \in U_n \times V_n$ satisfies $\bar{f}_n(\omega, \omega') < \epsilon$.

Note that for $(\omega, \omega') \in U_n \times V_n$, condition (3) implies that there exists $R_{\vec{v}}$, an m -grid of ω' , such that

$$\frac{|\{\vec{u} \in r_{\vec{v}} : \omega'(\vec{u} + B_m) \in W\}|}{|r_{\vec{v}}|} > 1 - 2\epsilon'.$$

Now consider this grid $R_{\vec{v}}$ of ω' and the m -grid $R_{\vec{0}}$ of ω . By our choice of N the number of indices of B_n which are within m of the boundary of B_n are less than $\epsilon'|B_n|$. Thus, without loss of generality, we can assume that $R_{\vec{v}} = R_{\vec{0}}$.

If $\vec{t} \in \vec{r}_{\vec{0}}$ is such that $\omega(\vec{t} + B_m)$ and $\omega'(\vec{t} + B_m)$ are both in W we call $\vec{t} + B_m$ a good grid box and \vec{t} the base point of a good grid box. For such a \vec{t} we let $\pi_{\vec{t}}$ denote the permutation achieving the δ match between $\omega(\vec{t} + B_m)$ and $\omega'(\vec{t} + B_m)$. Finally, let $S_{\vec{t}}$ denote the δ set of $\pi_{\vec{t}}$.

We define a permutation π between ω and ω' as follows. We apply $\pi_{\vec{t}}$ to the pair of good grid boxes based at \vec{t} , and the identity everywhere else. For each such \vec{t} we let $\bar{S}_{\vec{t}}$ denote the indices in $S_{\vec{t}}$ which are farther than $\epsilon'n$ from the boundary of their grid box, and let S denote the union of all the sets $\bar{S}_{\vec{t}}$.

Then $|S| > (1 - 6\epsilon')|B_n| > (1 - \epsilon)|B_n|$. If $\vec{u}, \vec{v} \in B_n$, then either \vec{u} and \vec{v} lie in the same grid box, in which case

$$\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \delta\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|$$

trivially, or $\|\vec{u} - \vec{v}\| > \epsilon'm$, and we have

$$\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| \leq 2(\epsilon')^2m = \frac{2(\epsilon')^2m}{\|\vec{u} - \vec{v}\|}\|\vec{u} - \vec{v}\| < \epsilon'\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|.$$

To see that we have $\bar{d}(\omega, \omega' \circ \pi) > (1 - \delta)(1 - 5\epsilon') > 1 - \epsilon$, we note that we are guaranteed to have matched all but δ of each good grid box in ω . Since all but a proportion $4\epsilon'$ of the grid boxes are good we have matched a proportion $> (1 - \delta)(1 - 4\epsilon')$ of ω . In the case when the grids are not aligned, i.e., when $R_{\vec{v}} \neq R_{\vec{0}}$, we would have translated one grid by $-\vec{v}$ to align them, sacrificing those points which are within m of the boundary of B_n . As discussed earlier these indices form a proportion less than ϵ' of B_n . Thus we have matched a proportion $> (1 - \delta)(1 - 5\epsilon') > 1 - \epsilon$ of ω as needed.

Now suppose that (T, P) is a zero entropy LBI process with distribution μ . We will show in Theorem 5.1 it is \bar{f} -FD. To show that it is loosely Bernoulli, we will first use the \bar{f} -FD property to find a loosely Bernoulli process (S, Q) \bar{f} -close to (T, P) . We will then use the set of names from (S, Q) which are close to one another in the \bar{f} metric to find a similar set for (T, P) . The details are as follows.

Fix $\epsilon > 0$ and let $\epsilon' = \epsilon/12$. Pick n and δ according to Definition 3.4, using this value of ϵ' . Consider an ergodic, zero entropy, loosely Bernoulli system (X, ν, S) . We know from [5] that such a system exists. Use the Rohlin Lemma to find a ν -measurable partition Q of X such that $|\text{dist } P_n - \text{dist } Q_n| < \delta$. Then, by Definition 3.4, we have $\bar{f}((S, Q), (T, P)) < \epsilon'$.

There is then a joining ρ of the two processes and a number M_1 such that for all $m \geq M_1$ the set W of pairs of m -names which satisfy $\bar{f}(\omega, \omega') < \epsilon'$ has ρ -measure larger than $1 - \epsilon'$. Since (S, Q) is a zero entropy loosely Bernoulli process, we can find M_2 such that for all $m \geq M_2$ there is a set W_S of (S, Q, m) -names with $\nu(W_S) > 1 - \epsilon'$ and for all $\omega, \omega' \in W_S$ we have $\bar{f}(\omega, \omega') < \epsilon'$.

Now let $N = \max \{M_1, M_2\}$. For every $n \geq N$, we can set $W_T = \pi_1(W \cap \pi_2^{-1}W_S)$. Since ρ is a joining, we must have $\mu(W_T) > 1 - 2\epsilon' > 1 - \epsilon$. Further, for any $\eta, \eta' \in W_T$ we pick $\omega \in \pi_2(\pi_1^{-1}(\eta) \cap [W \cap \pi_2^{-1}W_S])$ and $\omega' \in \pi_2(\pi_1^{-1}(\eta') \cap [W \cap \pi_2^{-1}W_S])$. Then ω and ω' are in W_S and the pairs $(\eta, \omega), (\eta', \omega')$ are in W , so we have

$$\bar{f}(\eta, \eta') \leq 4[\bar{f}(\eta, \omega) + \bar{f}(\omega, \omega') + \bar{f}(\omega', \eta')] < 12\epsilon' = \epsilon.$$

Hence (T, P) is LB. □

5. \bar{f} -FD is equivalent to LBI

In this section we show that the sets of \bar{f} -FD processes and LBI processes are the same.

THEOREM 5.1. *The process (T, P) is \bar{f} -FD if and only if it is LBI.*

An easy corollary of Theorem 5.1 is that the \bar{f} -BI property is closed under even Kakutani equivalence for all dimensions.

COROLLARY 5.2. *Let T and S be two \mathbb{Z}^d actions acting on Lebesgue spaces (X, \mathcal{M}, μ) and (Y, \mathcal{G}, ν) , respectively. Suppose T is an LBI action and T and S are even Kakutani equivalent. Then S is LBI.*

Proof. This follows directly from Theorem 5.1 above and Theorem 1 in [4]. □

To prove Theorem 5.1 we break up the argument into a series of propositions, whose proofs we give at the end of this section. The sufficiency of the LBI property is established by the following two propositions.

PROPOSITION 5.3. *If (T, P) is LBI then it is the \bar{f} limit of Bernoulli processes.*

PROPOSITION 5.4. *The \bar{f} -FD processes are closed in the \bar{f} metric.*

The Bernoulli processes are \bar{d} -FD [13] and the \bar{d} metric dominates \bar{f} . Hence the Bernoulli processes are \bar{f} -FD. Propositions 5.3 and 5.4 thus show that LBI processes are \bar{f} -FD.

PROPOSITION 5.5. *If (T, P) is \bar{f} -FD then it is the \bar{f} limit of LBI processes.*

PROPOSITION 5.6. *The LBI processes are closed in the \bar{f} metric.*

Clearly these two proposition together yield that an \bar{f} -FD process must be LBI.

We now turn to proving the propositions.

Proof of Proposition 5.3. We let (T, P) be an LBI process and we show that there is a sequence (S_j, Q_j) , $j \geq 1$, of Bernoulli processes with $\bar{f}((T, P), (S_j, Q_j)) < 1/j$, for all $j \geq 1$. Since Bernoulli processes are ABI, they are also LBI, and we will then have the required result.

This is essentially Lemma 2.8 in [18], where Rudolph and Schmidt prove that ABI processes are the \bar{d} -limit of Bernoulli processes. We use exactly the same approach with minor modifications in the technical parts of the arguments to accommodate the differences between the \bar{f} and \bar{d} metrics. For brevity, we give only an outline of the points of the argument in [18] that are identical in our case, but we provide all details for those parts of the argument that are different.

Fix j and use the Geometric Lemma in [10] (Lemma 4.15) to find $\delta < 1/(100j)$ such that if π is a permutation on B_L with $m(\pi) < \delta$ then for all \vec{v} in the δ -set of π we have

$$(4) \quad \|\pi\vec{v} - \vec{v}\| < \frac{1}{(100j)^2(2d)^2}L.$$

Since (T, P) is LBI, we can find $m \in \mathbb{N}$ such that for all $K \geq m$, $\bar{f}((T, P), (T, P)_K) < \delta$. Fix such a K and choose L , an integer multiple of K , large enough that $\bar{f}_L((T, P), (T, P)_K) < \delta$ and $|B_{L+K}| < (1 + \frac{1}{100j})|B_L|$. Set $M = L + K$.

We then define a Bernoulli process (S_j, Q_j) exactly as in [18]. The key property of this process is that the associated distribution μ_j is obtained as an integral $\int_{x \in \{0,1\}^{\mathbb{Z}^d}} \mu^{(x)} d\nu(x)$, where ν is the $(1/2, 1/2)$ Bernoulli measure

on $\{0, 1\}^{\mathbb{Z}^d}$. For ν -a.e. x in $\{0, 1\}^{\mathbb{Z}^d}$ there exists a set $\mathcal{R}(x) = \cup_{\vec{m}} B_M + \vec{m}$ so that the measure $\mu^{(x)}$ is the product of μ_M on each translate of B_M in \mathcal{R} , is independent on \mathcal{R} and \mathcal{R}^c , and on \mathcal{R}^c is a point mass on some symbol a from the alphabet of the process. We also have

$$(5) \quad \limsup_{N \rightarrow \infty} \frac{|\mathcal{R}(x) \cap B_N|}{|B_N|} \geq 1 - \frac{1}{100j}.$$

We denote the process associated to each $\mu^{(x)}$ by $(S_j, Q_j)^{(x)}$.

Continuing to follow the argument in [18], we note that every translate of B_M , say $B_M + \vec{m}$, has a unique translate of B_L , say $B_L + \vec{n}(\vec{m})$, such that $B_L + \vec{n}(\vec{m}) \subset B_M + \vec{m}$ and $B_L + \vec{n}(\vec{m})$ can be written as a disjoint union of grid boxes from the k -grid $R_{\vec{0}}$ of B_M .

Again arguing exactly as in [18], we note that $\bar{f}_{B_L + \vec{n}(\vec{m})}((T, P), (T, P)_K) < \delta$ for all $\vec{n}(\vec{m})$. Let $\lambda_{\vec{n}(\vec{m})}$ be a measure on $A^{B_L + \vec{n}(\vec{m})} \times A^{B_L + \vec{n}(\vec{m})}$ satisfying Definition 3.2 with this δ .

We now show that $\limsup_{N \rightarrow \infty} \bar{f}_N((T, P)_K, (S_j, Q_j)^{(x)}) < 3/(4j)$. Take N large and consider its K -grid $R_{\vec{0}}$. We define a measure Λ on $A^{B_N} \times A^{B_N}$ as the product of the measures $\lambda_{\vec{n}(\vec{m})}$ on each $B_L + \vec{n}(\vec{m})$ and μ^K times the point mass on the symbol a at all other positions. As in [18] this measure will project correctly to μ^K and $\mu^{(x)}$, and it puts all but δ of its weight on pairs of N -names η and ζ with the property that

$$(6) \quad \bar{f}\left(\eta(B_L + \vec{n}(\vec{m})), \zeta(B_L + \vec{n}(\vec{m}))\right) < \delta.$$

We claim that for such a pair η and ζ we can find a permutation π of B_N so that $m(\pi) < 3/(4j)$ and $\bar{d}(\eta \circ \pi, \zeta) < 3/(4j)$. We define π by taking it to be the permutation $\pi_{\vec{n}(\vec{m})}$ achieving (6) on each $B_L + \vec{n}(\vec{m})$ and the identity off these sets. Then we will have matched all but $\delta < 1/(100j)$ of the L -names in each $B_L + \vec{n}(\vec{m})$. Further, by (5) we know that if N is chosen large enough the portion of indices of an N -name not in a $B_L + \vec{n}(\vec{m})$ is less than $1/(10j)$. Thus, after applying the permutation, the \bar{d} -distance between the two N -names is less than $3/(4j)$.

To compute $m(\pi)$, let S be the subset of B_N consisting of all indices except the complement in $B_L + \vec{n}(\vec{m})$ of the δ -sets of each $\pi_{\vec{n}(\vec{m})}$, the part of B_N not covered by the sets $B_L + \vec{n}(\vec{m})$, and the portion of the indices in each $B_L + \vec{n}(\vec{m})$ which lie a distance less than $\frac{L}{2d \times 100j}$ of the boundary. Then

$$|S| > |B_N| - \delta|B_N| - \frac{1}{10j}|B_N| - \left(2d \times \frac{1}{2d \times 100j}\right)|B_N| > \left(1 - \frac{3}{4j}\right)|B_N|.$$

Let $\vec{u}, \vec{v} \in S$. If \vec{u} and \vec{v} are in the same set $B_L + \vec{n}(\vec{m})$, then

$$\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \delta\|\vec{u} - \vec{v}\| < \frac{3}{4j}\|\vec{u} - \vec{v}\|$$

follows immediately by our choice of $\pi_{B_L + \bar{n}(\bar{m})}$. If \vec{u} and \vec{v} lie in distinct translates of B_L , then $\|\vec{u} - \vec{v}\| > \frac{2}{2d \times 100j} L$. By our choice of δ we have

$$\begin{aligned} \|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| &< 2 \frac{1}{(100j)^2 (2d)^2} L = 2 \frac{1}{(100j)^2 (2d)^2} L \frac{\|\vec{u} - \vec{v}\|}{\|\vec{u} - \vec{v}\|} \\ &< \frac{3}{4j} \|\vec{u} - \vec{v}\|. \end{aligned}$$

Thus $m(\pi) < 3/(4j)$ and hence $\limsup_{N \rightarrow \infty} \bar{f}_N(\mu^{\mathcal{B}^\kappa}, \mu^{(x)}) < 3/(4j)$.

The conclusion of the proof now follows exactly as in [18], substituting the weaker triangle inequality for the \bar{f} metric from Lemma 3.3. \square

Proof of Proposition 5.4. To show that the \bar{f} -FD processes are closed in the \bar{f} metric we begin with a sequence S_n of \mathbb{Z}^d actions acting on a space (X, \mathcal{M}, μ) and a sequence Q_n of measurable partitions on X such that each (S_n, Q_n) is an \bar{f} -FD process. We show that if (S, Q) is the \bar{f} -limit of the (S_n, Q_n) , then (S, Q) is \bar{f} -FD.

We first consider the case where $h(S_n, Q_n) > 0$ for all n . Let $n = 1$ and suppose S_1 acts on (X, \mathcal{F}, μ) where $\mathcal{F} = \bigvee_{\vec{v} \in \mathbb{Z}^d} S_1^{\vec{v}} Q_1$. This is, then, an \bar{f} -FD action with Q_1 a generating partition. We set $Q^i = \bigvee_{\vec{v} \in B_i} S^{\vec{v}} Q_1$ for $i > 0$, and define Q^0 to be the trivial partition. Now choose T acting on (Y, \mathcal{G}, ν) to be an \bar{f} -FD action with $h(T) > \sup_n h(S_n, Q_n)$, and let P_0 be the trivial partition on Y . Then we have $\bar{f}((T, P_0), (S_1, Q^0)) = 0$.

Applying Proposition 3.6 repeatedly, we obtain sequences $\{P_i\}$ of partitions on Y and $\{T_i\}$ of \mathbb{Z}^d actions on Y such that

$$\bar{f}((T_i, P_i), (S_1, Q^i)) \rightarrow 0,$$

the P_i converge to a partition \hat{P} on Y , the T_i converge to a \mathbb{Z}^d action \hat{T} such that \hat{T} is even Kakutani equivalent to T , and such that (\hat{T}, \hat{P}) and (S_1, Q_1) are isomorphic as processes. In addition, since Kakutani equivalence preserves entropy and the \bar{f} -FD property, \hat{T} is \bar{f} -FD and $h(\hat{T}) > h(S_n, Q_n)$ for all n , and (S_1, Q_1) is a factor of \hat{T} .

For the next step in our proof we choose a sequence ϵ_k of real numbers decreasing to zero at the rate determined by Proposition 2.5 applied to \hat{T} . Without loss of generality we suppose that for all $k \in \mathbb{N}$ and all integers $n, m \geq k$,

$$\bar{f}((S_n, Q_n), (S_m, Q_m)) < \delta(\epsilon_k),$$

where $\delta(\epsilon_k)$ is obtained by applying Proposition 3.6 to the pair S_n and S_m with $\epsilon = \epsilon_k$. We then apply Proposition 3.6 repeatedly to obtain $\hat{T}_k \stackrel{\epsilon_k}{\approx} \hat{T}_{k-1}$ and a partition P_k such that $d(\hat{P}_{k-1}, P_k) < \epsilon_k$ and $\bar{f}((\hat{T}_k, P_k), (S_k, Q_k)) < \epsilon_k$ for each $k \geq 2$.

We thus obtain $\hat{T}_* = \lim \hat{T}_n$, which is even Kakutani equivalent to \hat{T} , and a partition $P_* = \lim P_n$ such that $\bar{f}((\hat{T}_*, P_*), (S, Q)) = 0$, i.e., (S, Q) and

(\hat{T}_*, P_*) are isomorphic as processes. Since every factor of an \bar{f} -FD action is itself \bar{f} -FD (see Proposition 3 in [4]), it follows that (S, Q) is \bar{f} -FD.

Now suppose the sequence (S_n, Q_n) is such that $h(S_n, Q_n) = 0$ for all n . In this case we choose T to be a zero entropy \bar{f} -FD action on (Y, \mathcal{G}, ν) , and we fix ϵ_k and $\delta(\epsilon_k)$ as before. Since (S_1, Q_1) is \bar{f} -FD we choose $\delta > 0$ such that Definition 3.4 is satisfied for $\epsilon = \delta(\epsilon_1)$. Using the Ergodic Theorem and the Rohlin Lemma we can find a partition P_1 on Y so that $|\text{dist}(P_1)_n - \text{dist}(Q_1)_n| < \delta$. Hence we obtain $\bar{f}((T, P_1), (S_1, Q_1)) < \delta(\epsilon_1)$. We now apply Proposition 3.6 to conclude the proof as above. \square

Proof of Proposition 5.5. To show that every \bar{f} -FD process is the limit of LBI processes, we use the standard result that given (T, P) , n and $\delta > 0$, there is a Bernoulli process (S, Q) such that $|\text{dist}(P_n) - \text{dist}(Q_n)| < \delta$ and $|h(S, Q) - h(T, P)| < \delta$. Thus if (T, P) is \bar{f} -FD, for sufficiently small δ we have $\bar{f}((T, P), (S, Q)) < \epsilon$. \square

Proof of Proposition 5.6. Here we show that if (T, P) is the \bar{f} limit of LBI processes then (T, P) itself is LBI, i.e., for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for $n \geq N$, $\bar{f}((T, P), (T, P)_n) < \epsilon$. We will do this by first finding an LBI process (S, Q) which is \bar{f} close to (T, P) . Then we will apply Lemma 3.3, the “triangle inequality” for \bar{f} to conclude

$$\begin{aligned} \bar{f}((T, P), (T, P)_n) &\leq 4[\bar{f}((T, P), (S, Q)) \\ &\quad + \bar{f}((S, Q), (S, Q)_n) + \bar{f}((S, Q)_n, (T, P)_n)]. \end{aligned}$$

By choosing (S, Q) and n judiciously, the first two terms on the right can be made small. We then use the fact that (T, P) and (S, Q) are \bar{f} close to show that the last term is also small. The details are as follows.

Fix $\epsilon > 0$. Using Lemma 4.15 from [10], we choose $\delta \leq \epsilon/100$ such that if $\pi : B_n \rightarrow B_n$ satisfies $m(\pi) < \delta$, then for every \vec{v} in the δ -set of π , $\|\pi\vec{v} - \vec{v}\| < (\epsilon^2/50)n$. Let (S, Q) be an LBI process such that $\bar{f}((T, P), (S, Q)) < \delta$. Next, take $N \in \mathbb{N}$ so large that for $n \geq N$ we have

$$\bar{f}((S, Q), (S, Q)_n) < \frac{\epsilon}{12} \quad \text{and} \quad \bar{f}_n((T, Q), (S, Q)) < \delta.$$

Fix such an n . We then have a joining ρ between (T, P) and (S, Q) such that if $W = \{(\omega_1, \omega_2) \in A^{B_n} \times A^{B_n} : \bar{f}(\omega_1, \omega_2) < \delta\}$ then $\rho(W) > 1 - \delta > 1 - \epsilon/50$. It is easy to see that ρ^n is a joining of the independently blocked processes $(T, P)_n$ and $(S, Q)_n$. We will show that for sufficiently large m , ρ^n gives large measure to pairs of \bar{f} close m -names from the two blocked processes.

Take $M \in \mathbb{N}$ to be large enough so that for $m \geq M$ we have $n/m < \epsilon/(d100)$ and by Lemma 2.8 the set

$$U = \left\{ (\eta_1, \eta_2) \in A^{B_m} \times A^{B_m} : \frac{|\{\vec{t} \in r_{\vec{0}} : (\eta_1(\vec{t} + B_n), \eta_2(\vec{t} + B_n)) \in W\}|}{|r_{\vec{0}}|} > 1 - \frac{\epsilon}{25} \right\}$$

has measure $\rho^n(U) > 1 - \epsilon/12$.

The set U consists of pairs of words such that the subwords occurring in corresponding grid boxes are \bar{f} -close for all but $\epsilon/25$ of the grid boxes. Call such a grid box *good*. We now want to compute $\bar{f}(\eta_1, \eta_2)$, for $(\eta_1, \eta_2) \in U$.

First note that if $\vec{t} \in r_{\vec{0}}$ is the base point of a good grid box in η_1 , then there is a permutation $\pi_{\vec{t}}$ achieving the \bar{f} match between this grid box and its counterpart in η_2 . Let $S_{\vec{t}}$ be the δ set of $\pi_{\vec{t}}$. Note also that $\pi_{\vec{t}}$ matches all but $\epsilon/50$ of the indices in the two boxes $\eta_1(\vec{t} + B_n)$ and $\eta_2(\vec{t} + B_n)$, and that $\|\pi_{\vec{t}}\vec{u} - \pi_{\vec{t}}\vec{v} - (\vec{u} - \vec{v})\| < \delta\|\vec{u} - \vec{v}\|$ for $\vec{u}, \vec{v} \in S_{\vec{t}}$. Now define π on B_m as follows: in a good grid box use the associated permutation $\pi_{\vec{t}}$, otherwise use the identity. Finally, let $\bar{S}_{\vec{t}}$ be the indices in $S_{\vec{t}}$ which are at a distance greater than $(\epsilon/50)n$ from the boundary of B_n .

Let S be the union of the sets $\bar{S}_{\vec{t}}$ from all the good grid boxes. Then S includes all of B_m except the following:

- the indices within $\epsilon/50$ of the boundary of each good grid box, which is less than $(\epsilon/50)|B_m|$;
- another $(\epsilon/50)|B_n|$ from each good grid box, which is less than $(\epsilon/50)|B_m|$;
- the indices in the bad grid boxes, which is less than $(\epsilon/25)|B_m|$;
- the indices which are in partial grid boxes, which is less than $(\epsilon/100)|B_m|$, by our choice of m .

Thus $|S| > (1 - \epsilon/2)|B_m|$.

For $\vec{u}, \vec{v} \in S$, either \vec{u} and \vec{v} lie in the same grid box or they lie in different good grid boxes. In the first case we have $\pi = \pi_{\vec{t}}$ for some \vec{t} , so we automatically have

$$\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \delta\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|.$$

In the second case, by our choice of δ , we have

$$\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| \leq 2\frac{(\epsilon')^2}{50}m = \frac{2(\epsilon')^2}{50}m \|\vec{u} - \vec{v}\| < \frac{\epsilon}{2}\|\vec{u} - \vec{v}\|,$$

as needed. Thus $m(\pi) < \epsilon/2$.

Next we need to compute $\bar{d}(\eta_1 \circ \pi, \eta_2)$. This is similar to the computation of $|S|$: after π is applied, we know that what is left unmatched in a good grid

box is less than $(\epsilon/25)|B_m|$, and the amount left unmatched in the bad grid boxes and the partial grid boxes is less than $(\epsilon/25 + \epsilon/100)|B_m|$, so we have $\bar{d}(\eta_1 \circ \pi, \eta_2) < \epsilon/2$.

This yields that $\bar{f}(\eta_1, \eta_2) < \epsilon/12$ for any $(\eta_1, \eta_2) \in U$. Thus we have $\bar{f}_m((T, P)_n, (S, Q)_n) < \epsilon/12$ for all sufficiently large m , as needed. \square

6. Examples of LBI \mathbb{Z}^d actions

It is known that all finite rank actions with a particular tower geometry are LB; see [5] and [6]. In addition, if T is a zero entropy LB action, any ergodic, isometric extension of T is also LB [7]. On the positive entropy side we know, of course, that Bernoulli actions are LBI. In this section we will show that, in addition, the direct product of a zero entropy LB action with a Bernoulli action is also LBI and that ergodic, isometric extensions of positive entropy LBI actions are LBI.

Even in one dimension the product of two LB actions is not necessarily LB; a counter-example is constructed in [14]. However, in [14] the authors prove that the product of a Bernoulli transformation with a zero entropy LB transformation is necessarily LB. Their proof relies heavily on the fact that for a zero entropy process, conditioning on a past name of the process determines the future of the name. Since our work here was designed to bypass conditioning on the past, our proof has a different flavor. We show that such a product action must be LBI by explicitly constructing a joining satisfying Definition 4.1.

Proof of Theorem 1.2. Recall that T is a Bernoulli action on (X, μ) and S a zero entropy loosely Bernoulli action on (Y, ν) . Let Q be a generating partition for S with alphabet A . Without loss of generality suppose that T is isomorphic to the shift action on $A^{\mathbb{Z}^d}$ with Bernoulli measure μ . We let P be the time zero partition for T and we fix $\epsilon > 0$.

By Proposition 4.3, (S, Q) is LBI. Hence there exists an $N > 0$ such that for all $n \geq N$, $\bar{f}((S, Q), (S, Q)_n) < \epsilon$. Fix such an n . By the definition of the \bar{f} metric and the LBI property we can find $M > 0$ such that for all $m \geq M$

$$(7) \quad \bar{f}_m((S, Q), (S, Q)_n) < \epsilon.$$

We fix such an m and recall that, by the proof of Proposition 4.3, this \bar{f} distance is achieved by the product measure $\nu \times \nu^n$.

To show that $S \times T$ is LBI, we will show that for such a pair n and m

$$(8) \quad \bar{f}_m\left(\left((S, Q) \times (T, P)\right), \left((S, Q)_n \times (T, P)_n\right)\right) < \epsilon.$$

We first remark that since (T, P) is Bernoulli, we have $(T, P)_n = (T, P)$. Further, given any atom η of P_m and a permutation π of B_m , $\eta \circ \pi$ and $\eta \circ \pi^{-1}$ are also atoms of P_m and $\mu(\eta \circ \pi) = \mu(\eta \circ \pi^{-1}) = \mu(\eta)$. We use these properties

of (T, P) to define a joining ρ of $((S, Q) \times (T, P)) \times ((S, Q)_n \times (T, P)_n)$ as follows:

By (7) we can find a set $G \subset Q_m \times Q_m$ with $\nu \times \nu^n(G) > 1 - \epsilon$ and such that for all $(\omega, \omega') \in G$ we have $\bar{f}(\omega, \omega') < \epsilon$. For $(\omega, \omega') \in G$ there is a permutation π of B_m achieving this \bar{f} distance, and we set

$$\rho((\omega, \eta), (\omega', \eta')) = \begin{cases} \nu \times \nu^n(\omega, \omega')\mu(\eta) & \text{if } \eta' = \eta \circ \pi, \\ 0 & \text{otherwise.} \end{cases}$$

For $(\omega, \omega') \notin G$ we set

$$\rho((\omega, \eta), (\omega', \eta')) = \begin{cases} \nu \times \nu^n(\omega, \omega')\mu(\eta) & \text{if } \eta' = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

To see that ρ is a joining, it suffices to show that for a fixed $(\omega, \eta) \in Q_m \times P_m$

$$(9) \quad \sum_{(\omega', \eta') \in Q_m \times P_m} \rho((\omega, \eta), (\omega', \eta')) = \nu \times \mu(\omega, \eta)$$

and for a fixed $(\omega', \eta') \in Q_m \times P_m$

$$(10) \quad \sum_{(\omega, \eta) \in Q_m \times P_m} \rho((\omega, \eta), (\omega', \eta')) = \nu^n \times \mu(\omega', \eta').$$

To see that condition (9) holds we note that the sum in that expression can be rewritten as

$$\begin{aligned} & \sum_{\{(\omega', \eta') : (\omega, \omega') \in G\}} \rho((\omega, \eta), (\omega', \eta')) + \sum_{\{(\omega', \eta') : (\omega, \omega') \notin G\}} \rho((\omega, \eta), (\omega', \eta')) \\ &= \sum_{\{(\omega, \omega') \in G\}} \rho((\omega, \eta), (\omega', \eta \circ \pi)) + \sum_{\{(\omega, \omega') \notin G\}} \rho((\omega, \eta), (\omega', \eta)), \end{aligned}$$

where π is the appropriate permutation of B_m given by the definition of ρ . This is then equal to

$$\begin{aligned} & \sum_{\{(\omega, \omega') \in G\}} \nu \times \nu^n(\omega, \omega')\mu(\eta) + \sum_{\{(\omega, \omega') \notin G\}} \nu \times \nu^n(\omega, \omega')\mu(\eta) \\ &= \sum_{\{\omega' \in Q_m\}} \nu(\omega)\nu^n(\omega')\mu(\eta) = \nu \times \mu(\omega, \eta). \end{aligned}$$

A similar argument shows that condition (10) holds.

Let

$$\bar{G} = \left\{ ((\omega, \eta), (\omega', \eta')) : \bar{f}((\omega, \eta), (\omega', \eta')) < \epsilon \right\}.$$

It remains to show that $\rho(\bar{G}) > 1 - \epsilon$.

Note that \overline{G} contains all points of the form $((\omega, \eta), (\omega', \eta \circ \pi))$, where $(\omega, \omega') \in G$ and π is the permutation achieving the \overline{f} distance between ω and ω' . So we have

$$\begin{aligned} \rho(\overline{G}) &\geq \rho\left(\bigcup_{(\omega, \omega') \in G} \bigcup_{\eta} ((\omega, \eta), (\omega', \eta \circ \pi))\right) \\ &= \sum_{(\omega, \omega') \in G} \sum_{\eta} \nu \times \nu^n(\omega, \omega') \mu(\eta) > 1 - \epsilon. \end{aligned}$$

Thus Definition 4.1 is satisfied and we are done. □

The final piece left is to determine if the ergodic, isometric extensions of positive entropy LBI actions are themselves LBI. For the zero entropy case this was shown by the authors in [7]. In the positive entropy category this result is highly non-trivial, even in the one dimensional case. The proof depends on the characterization of isometric extensions of Bernoulli transformations due to Rudolph [17] and the fact that a positive entropy LB transformation induces a Bernoulli shift.

While the results in [17] have been extended to higher dimensions by Kammerer [9], the defining property of LBI \mathbb{Z}^d actions is slightly more complex: the notion of inducing is not available. Instead, we have to use even Kakutani equivalence.

We first give a brief sketch of the argument. Given an action (T, X, μ) , which is positive entropy and LBI, consider an ergodic, isometric extension T^h . By Theorem 1.1 any Bernoulli action S of equal entropy is even Kakutani equivalent to T . The idea is to use the even Kakutani equivalence between T and S to find an ergodic, isometric extension of S , S^g , which is even Kakutani equivalent to T^h .

We now establish some notation and the necessary definitions for the proof of the result. Let (C, ρ) be a compact, homogeneous metric space and let G be the group of all isometries of C . Note that G is a compact group [8]. Let m be the G -invariant measure on C , and (X, μ, T) a free, measure preserving, ergodic, zero entropy \mathbb{Z}^d action. Suppose $h : X \times \mathbb{Z}^d \rightarrow G$ is a measurable T cocycle. That is, for all $\vec{n}, \vec{m} \in \mathbb{Z}^d$ we have

$$(11) \quad h(x, \vec{n} + \vec{m}) = h(x, \vec{n}) \circ h(T_{\vec{n}}x, \vec{m}).$$

For $\vec{n} \in \mathbb{Z}^d$ we define $T^h : X \times C \rightarrow X \times C$ by

$$T_{\vec{n}}^h(x, c) = (T_{\vec{n}}x, h(x, \vec{n})(c)).$$

Then, by (11), T^h will be a measure preserving \mathbb{Z}^d action on $(X \times C, \mu \times m)$, and T^h will have the same entropy as T [8]. We will refer to T^h as an isometric group extension of T .

Our main result is as follows.

THEOREM 6.1. *Let (X, μ, T) be a free, ergodic, measure preserving, and LBI \mathbb{Z}^d action, G the group of isometries of a compact metric space C , and $h : X \times \mathbb{Z}^d \rightarrow G$ a T cocycle. If T^h is ergodic then T^h is LBI.*

Proof. Let T and T^h be as given in the theorem. We will assume that the entropy of T is positive, as the zero entropy case is already known (see [7]).

Let S be a Bernoulli action on a Lebesgue space (Y, \mathcal{G}, ν) with the same entropy as T . By Theorem 1.1, S and T are even Kakutani equivalent. Let ϕ denote the orbit equivalence given by Definition 2.1.

We will construct an isometric extension of S which is even Kakutani equivalent to T^h using the orbit equivalence ϕ . Define $g : Y \times \mathbb{Z}^d \rightarrow G$ by

$$g(y, \vec{n}) = h(\phi^{-1}y, \vec{T}(\phi^{-1}y, \phi^{-1}(S_{\vec{n}}y))).$$

This map is obviously G -valued and measurable, and it is a S -cocycle because

$$\begin{aligned} g(y, \vec{n} + \vec{m}) &= h(\phi^{-1}y, \vec{T}(\phi^{-1}y, \phi^{-1}(S_{\vec{n}+\vec{m}}y))) \\ &= h(\phi^{-1}y, \vec{T}(\phi^{-1}y, \phi^{-1}(S_{\vec{n}}y))) + \vec{T}(\phi^{-1}(S_{\vec{m}}y), \phi^{-1}(S_{\vec{n}+\vec{m}}y)) \\ &= h(\phi^{-1}y, \vec{T}(\phi^{-1}y, \phi^{-1}(S_{\vec{n}}y))) \circ h(\phi^{-1}(S_{\vec{m}}y), \vec{T}(\phi^{-1}(S_{\vec{n}}y), \phi^{-1}(S_{\vec{n}+\vec{m}}y))) \\ &= g(y, \vec{n}) \circ g(S_{\vec{m}}y, \vec{m}), \end{aligned}$$

as needed.

To show that T^h and S^g are even Kakutani equivalent, we show that $\phi^h = \phi \times \text{id} : X \times C \rightarrow Y \times C$ is an orbit equivalence between T^h and S^g satisfying Definition 2.1.

First we show that ϕ^h is an orbit equivalence. Note that

$$\phi^h(T_{\vec{n}}^h(x, c)) = \phi^h(T_{\vec{n}}x, h(x, \vec{n})(c)) = (\phi(T_{\vec{n}}x), h(x, \vec{n})(c)).$$

Since ϕ is an orbit equivalence, there exists \vec{m} with $S_{\vec{m}}(\phi x) = \phi(T_{\vec{n}}x)$, and thus $h(x, \vec{n})(c) = g(\phi x, \vec{m})(c)$. We then have

$$\phi^h(T_{\vec{n}}^h(x, c)) = (S_{\vec{m}}(\phi x), g(\phi x, \vec{m})(c)) = S_{\vec{m}}^g(\phi^h(x, c)),$$

i.e., ϕ^h maps orbits of T^h onto orbits of S^g .

Finally, to show that ϕ^h satisfies Definition 2.1, we first note that the T^h orbit of a point $(x, c) \in X \times C$ contains all (y, k) such that there exists an \vec{m} with $T_{\vec{m}}^h(x, c) = (y, k)$. Since $T_{\vec{m}}^h(x, c) = (T_{\vec{m}}x, h(x, \vec{m})(c))$, we must have $T_{\vec{m}}x = y$ and $h(x, \vec{m})(c) = k$. Thus $\vec{T}^h((x, c), (y, k)) = \vec{T}(x, y)$.

So for a fixed $\epsilon > 0$, if $A \subset X$ and $N \in \mathbb{N}$ are the set and constant, respectively, given by Definition 2.1 for ϕ , we claim that $A \times C$ and N are the corresponding set and constant for ϕ^h . Indeed,

$$\|\vec{T}^h(x, c), (x', c')\| - \vec{S}^g(\phi^h(x, c), \phi^h(x', c'))\| = \|\vec{T}(x, x') - \vec{S}(\phi x, \phi x')\|,$$

and if $x, x' \in A$ with $\|\vec{T}(x, x')\| > N$, then ϕ^h satisfies Definition 2.1 exactly because ϕ does.

By the results in [8], S^g is either a Bernoulli action itself, or it is isomorphic to the direct product of a Bernoulli action and a compact group rotation. In the first case we are done, by Theorem 1.1. In the second case, by the results in [5], S^g is then the direct product of a Bernoulli action and a zero entropy LB action. Thus, by Theorem 1.2, it is a LBI action. By Corollary 5.2 we are done.

□

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