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ABSOLUTE CONTINUITY OF PERIODIC SCHRÖDINGER OPERATORS WITH POTENTIALS IN THE KATO CLASS

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Abstract. We consider the Schrödinger operator $-\Delta + V$ in $\mathbb{R}^d$ with periodic potential V in the Kato class. We show that, if $d = 2$ or $d = 3$, the spectrum of $-\Delta + V$ is purely absolutely continuous.

1. Introduction

Let V be a real valued measurable function on $\mathbb{R}^d$, $d \geq 2$. V is said to belong to the Kato class K_d if

$$\lim_{r\to 0} \sup_{\mathbf{x}\in\mathbb{R}^d} \int_{|\mathbf{y}-\mathbf{x}|\leq r} \frac{|V(\mathbf{y})|d\,\mathbf{y}}{|\mathbf{y}-\mathbf{x}|^{d-2}} = 0, \quad \text{for } d \geq 3, \tag{1.1}$$

$$\lim_{r\to 0} \sup_{\mathbf{x}\in\mathbb{R}^d} \int_{|\mathbf{y}-\mathbf{x}|\leq r} |V(\mathbf{y})| \ln\left\{|\mathbf{y}-\mathbf{x}|^{-1}\right\} d\mathbf{y} = 0, \quad \text{for } d = 2. \tag{1.2}$$

It is well known that, if $V \in K_d$, then the quadratic form associated with $-\Delta+V$ defines a unique self-adjoint operator which we also denote by $-\Delta+V$ [7]. We refer the reader to [18] for the naturalness of the Kato class in the study of L^p properties of the semigroup $e^{-t(-\Delta+V)}$. The purpose of this paper is to show that, if $d = 2$ or $d = 3$ and $V \in K_d$ is a real periodic function on $\mathbb{R}^d$, then the spectrum of $-\Delta + V$ is purely absolutely continuous.

Main Theorem. *Let $A = (a_{jk})_{d\times d}$ be a symmetric, positive-definite matrix with real constant entries. Let $V \in K_d$ be a real periodic function on $\mathbb{R}^d$. If $d = 2$ or $d = 3$, then the spectrum of operator $DAD^T + V$ is purely absolutely continuous where $D = -i\nabla$ and $DAD^T = \sum_{j,k} D_j a_{jk} D_k$.*

A few remarks are in order.

Remark 1.3. For a Schrödinger operator $-\Delta + V$ with periodic potential V, the absolute continuity of the spectrum was first established by

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L. Thomas [21] in $\mathbb{R}^3$ under the assumption $V \in L^2_{\text{loc}}(\mathbb{R}^3)$. Thomas' result was subsequently extended to $\mathbb{R}^d$ by M. Reed and B. Simon [13] under the assumption $V \in L^r_{\text{loc}}(\mathbb{R}^d)$, where $r > d-1$ if $d \geq 4$ and $r = 2$ if $d = 2$ or $d = 3$. In [4] L. Danilov applied the approach of Thomas to the Dirac operator with a periodic potential. Recently, the absolute continuity of the magnetic Schrödinger operator $(-i\nabla - \mathbf{A}(\mathbf{x}))^2 + V(\mathbf{x})$ with periodic potentials $\mathbf{A}$ and V was investigated by R. Hempel and I. Herbst [5], [6], M. Birman and T. Suslina [1], [2], [3], A. Morame [12], and A. Sobolev [19]. In particular, the results in [2] and [3], pertaining to the case $-\Delta + V$, give the absolute continuity for $V \in L^p_{\text{loc}}(\mathbb{R}^d)$, where $p > 1$ if $d = 2$, $p = d/2$ if $d = 3$ or $d = 4$, and $p = d-2$ if $d \geq 5$. In [16], the author established the absolute continuity of $-\Delta + V$ under the condition $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$, $d \geq 3$. This is best possible in the context of the L^p spaces, in the sense that, under the periodicity condition, $L^{d/2}_{\text{loc}}$ is the largest space for which the self-adjoint operator $-\Delta + V$ may be defined by a quadratic form. The case $V \in$ weak-$L^{d/2}$ was also studied in [16].

Remark 1.4. In [17] the author investigated the periodic Schrödinger operator $-\Delta + V$ with potential V in the Morrey–Campanato class. The author showed that, if $d \geq 3$, $p \in ((d-1)/2, d/2]$, and

$$\limsup_{r\to 0} \sup_{\mathbf{x}\in\Omega} r^2 \left\{ \frac{1}{r^d} \int_{|\mathbf{y}-\mathbf{x}|\leq r} |V(\mathbf{y})|^p \, d\mathbf{y} \right\}^{1/p} < \varepsilon(p, d, \Omega), \tag{1.5}$$

where $\varepsilon(p, d, \Omega) > 0$ and Ω is a periodic cell for V, then $-\Delta + V$ has purely absolutely continuous spectrum. This improves the $L^{d/2}$ and weak-$L^{d/2}$ results in [16]. We point out that the Kato class considered in this paper is not comparable with the Morrey–Campanato class for $d \geq 3$ and $p > 1$. Indeed, one can construct a periodic potential V in $\mathbb{R}^3$ such that

$$|V(x)| \sim \frac{1}{|\mathbf{x}'|^2 \, |\ln(|\mathbf{x}'|)|^\delta} \quad \text{as} \quad |\mathbf{x}| \to 0, \tag{1.6}$$

where $\mathbf{x}' = (x_2, x_3)$. Then $V \in K_3$ if $\delta > 2$, but V does not satisfy (1.5) since $V \notin L^p_{\text{loc}}(\mathbb{R}^3)$ for any $p > 1$. On the other hand, if

$$|V(\mathbf{x})| \sim \frac{1}{|\mathbf{x}|^2 |\ln(|\mathbf{x}|)|^\delta} \quad \text{as} \quad |\mathbf{x}| \to 0, \tag{1.7}$$

then V satisfies (1.5) for $1 < p < 3/2$, if $\delta > 0$. However, $V \in K_3$ if and only if $\delta > 1$. Clearly, in the two-dimensional case, our result improves the L^p $(p > 1)$ result in [2].

Remark 1.8. By a change of coordinates, we may assume that V is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$.

Our main theorem is proved by using the approach of L. Thomas [21] and a new pointwise estimate on the kernel function of a certain integral operator.

To be more precise, let $\Omega = [0, 2\pi)^d \approx \mathbb{R}^d/(2\pi\mathbb{Z})^d = \mathbb{T}^d$. We consider a family of operators

$$\mathbb{H}_V(z\mathbf{a}+\mathbf{b}) = (\mathbf{D}+z\mathbf{a}+\mathbf{b})A(\mathbf{D}+z\mathbf{a}+\mathbf{b})^T + V, \qquad z \in \mathbb{C}, \tag{1.9}$$

defined on $L^2(\mathbb{T}^d)$ with $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$ fixed. Using the Floquet decomposition and Thomas' argument, we may reduce the main theorem to the problem of showing that the family of operators $\{\mathbb{H}_V(z\mathbf{a}+\mathbf{b})\colon z \in \mathbb{C}\}$ has no common eigenvalues. To this end, we will show that, for some appropriately chosen $\mathbf{a} \in \mathbb{R}^d$,

$$\left\| \{\mathbb{H}_V(z\mathbf{a}+\mathbf{b})\}^{-1} \right\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)} \to 0 \qquad \text{as} \qquad \rho \to \infty, \tag{1.10}$$

where $\langle \mathbf{b}, \mathbf{a}\rangle = 0$, $z = \delta + i\rho$, and δ is some fixed number depending on $\mathbf{a}$ and $\mathbf{b}$.

To prove (1.10), the key step is to show that

$$\begin{aligned} &\left\| V\left\{\mathbb{H}_0(z\mathbf{a}+\mathbf{b})\right\}^{-1} \right\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)} \\ &\qquad \le \begin{cases} C \sup\limits_{\mathbf{x}\in\Omega} \displaystyle\int_\Omega \frac{|V(\mathbf{y})|\, d\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}, & \text{if } d = 3, \\ C \sup\limits_{\mathbf{x}\in\Omega} \displaystyle\int_\Omega \left\{1 + \big|\ln|\mathbf{x}-\mathbf{y}|\big|\right\} |V(\mathbf{y})|\, d\mathbf{y}, & \text{if } d = 2, \end{cases} \end{aligned} \tag{1.11}$$

where $\mathbb{H}_0(z\mathbf{a}+\mathbf{b}) = (\mathbf{D}+z\mathbf{a}+\mathbf{b})A(\mathbf{D}+z\mathbf{a}+\mathbf{b})^T$. This will be done by establishing the following pointwise estimate on the kernel function $G_\rho(\mathbf{x}, \mathbf{y})$ of the operator $\{\mathbb{H}_0(z\mathbf{a}+\mathbf{b})\}^{-1}$:

$$|G_\rho(\mathbf{x}, \mathbf{y})| \le \begin{cases} \dfrac{C}{|\mathbf{x}-\mathbf{y}|}, & \text{if } d = 3, \\ C\left\{1 + \big|\ln|\mathbf{x}-\mathbf{y}|\big|\right\}, & \text{if } d = 2. \end{cases} \tag{1.12}$$

This paper is organized as follows. In Sections 2 and 3 we prove the kernel function estimate (1.12), and in Section 4 we prove the Main Theorem.

Throughout the rest of this paper we assume that $d = 2$ or $d = 3$, and that $V \in K_d$ is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. We use $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{T}^d)$. Finally we use C and c to denote positive constants, which may depend on the matrix A, and which are not necessarily the same at each occurrence.

2. Some preliminaries

We begin by choosing $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$ such that

$$|\mathbf{a}| = 1, \quad \mathbf{a}A = (s_0, 0, \cdots, 0), \quad s_0 > 0. \tag{2.1}$$

For $\mathbf{b} = (b_1, \cdots, b_d) \in \mathbb{R}^d$ with $\langle \mathbf{b}, \mathbf{a} \rangle = 0$ and $|\mathbf{b}| \le \sqrt{d}$, let

$$\delta = \frac{1}{a_1}\left(\frac{1}{2} - b_1\right). \tag{2.2}$$

Note that $a_1 > 0$ since $\mathbf{a}A\mathbf{a}^T = s_0 a_1 > 0$. We consider the operator

$$\mathbb{H}_0(\mathbf{k}) = (\mathbf{D} + \mathbf{k})A(\mathbf{D} + \mathbf{k})^T \tag{2.3}$$

defined on $L^2(\mathbb{T}^d)$, where

$$\mathbf{k} = (\delta + i\rho)\mathbf{a} + \mathbf{b} \quad \text{and} \quad \rho \ge 1. \tag{2.4}$$

For $\psi \in L^1(\Omega)$, let

$$\hat{\psi}(\mathbf{n}) = \frac{1}{(2\pi)^d}\int_\Omega e^{-i\langle \mathbf{n}, \mathbf{y}\rangle}\psi(\mathbf{y})\, d\mathbf{y}. \tag{2.5}$$

We may write

$$\{\mathbb{H}_0(\mathbf{k})\}^{-1}\psi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{\hat{\psi}(\mathbf{n})e^{i\langle \mathbf{n}, \mathbf{x}\rangle}}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} \tag{2.6}$$

for $\psi \in C^\infty(\mathbb{T}^d)$. Using (2.4) and (2.1), it is easy to see that

$$\begin{aligned}(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T &= (\mathbf{n}+\delta\mathbf{a}+\mathbf{b})A(\mathbf{n}+\delta\mathbf{a}+\mathbf{b})^T \\ &\quad - \rho^2 s_0 a_1 + 2i\rho s_0(n_1 + \delta a_1 + b_1).\end{aligned} \tag{2.7}$$

By (2.2) we have

$$\begin{aligned}\left|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right| &\ge 2\rho s_0\,|n_1 + \delta a_1 + b_1| \\ &= 2\rho s_0\left|n_1 + \frac{1}{2}\right| \ge \rho s_0,\end{aligned}$$

since n_1 is an integer.

We now choose $\eta \in C^\infty(\mathbb{R}_+)$ such that $\eta(r) = 1$ if $r \ge s_0^2$, and $\eta(r) = 0$ if $0 < r < s_0^2/2$. Then,

$$\eta\left(\frac{\left|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right|^2}{\rho^2}\right) = 1 \qquad \text{for any } \mathbf{n} \in \mathbb{Z}^d.$$

It follows that

$$\begin{aligned}\{\mathbb{H}_0(\mathbf{k})\}^{-1}\psi(\mathbf{x}) &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{e^{i\langle \mathbf{n}, \mathbf{x}\rangle}\hat{\psi}(\mathbf{n})\eta\left(\left|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right|^2/\rho^2\right)}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} \\ &= \int_\Omega G_\rho(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y},\end{aligned} \tag{2.8}$$

where

$$G_\rho(\mathbf{x}) = \frac{1}{(2\pi)^d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{e^{i\langle \mathbf{n},\mathbf{x}\rangle} \eta\left(\left|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right|^2/\rho^2\right)}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T}. \tag{2.9}$$

Note that, by the Plancherel theorem, we have $G_\rho \in L^2(\Omega)$ if $d = 2$ or $d = 3$.

Let

$$\varphi(\xi,\rho) = \xi A \xi^T - \rho^2 s_0 a_1 + 2i\rho s_0 \xi_1, \quad \text{where } \xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d. \tag{2.10}$$

Then,

$$h_\rho(\xi) = \frac{\eta\left(\left|\varphi(\xi,\rho)\right|^2/\rho^2\right)}{\varphi(\xi,\rho)} \in L^2(\mathbb{R}^d). \tag{2.11}$$

We denote its inverse Fourier transform by $F_\rho(\mathbf{x})$, i.e.,

$$F_\rho(\mathbf{x}) = (h_\rho)^\vee(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\langle \mathbf{x},\xi\rangle} h_\rho(\xi)\, d\xi. \tag{2.12}$$

Using the fact that $(-\mathbf{x})^\beta F_\rho(\mathbf{x})$ is the inverse Fourier transform of $\mathbf{D}^\beta h_\rho(\xi)$, one sees that

$$F_\rho(\mathbf{x}) = O\left(\frac{1}{|\mathbf{x}|^N}\right) \quad \text{as } |\mathbf{x}| \to \infty$$

for any $N \geq 1$. It follows that

$$\begin{aligned}
\frac{\eta\left(\left|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T\right|^2/\rho^2\right)}{(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T} &= \frac{\eta\left(|\varphi(\mathbf{n}+\delta\mathbf{a}+\mathbf{b},\rho)|^2/\rho^2\right)}{\varphi(\mathbf{n}+\delta\mathbf{a}+\mathbf{b},\rho)} = h_\rho(\mathbf{n}+\delta\mathbf{a}+\mathbf{b}) \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\langle \mathbf{n}+\delta\mathbf{a}+\mathbf{b},\mathbf{x}\rangle} F_\rho(\mathbf{x})\, d\mathbf{x} \\
&= \frac{1}{(2\pi)^d}\sum_{\mathbf{m}\in\mathbb{Z}^d}\int_{\Omega} e^{-i\langle \mathbf{n}+\delta\mathbf{a}+\mathbf{b},\mathbf{x}+2\pi\mathbf{m}\rangle} F_\rho(\mathbf{x}+2\pi\mathbf{m})\, d\mathbf{x} \\
&= \frac{1}{(2\pi)^d}\int_{\Omega} e^{-i\langle \mathbf{n},\mathbf{x}\rangle} \sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-i\langle \delta\mathbf{a}+\mathbf{b},\mathbf{x}+2\pi\mathbf{m}\rangle} F_\rho(\mathbf{x}+2\pi\mathbf{m})\, d\mathbf{x}.
\end{aligned}$$

In view of (2.9), this implies that

$$G_\rho(\mathbf{x}) = \frac{1}{(2\pi)^d}\sum_{\mathbf{m}\in\mathbb{Z}^d} e^{-i\langle \delta\mathbf{a}+\mathbf{b},\mathbf{x}+2\pi\mathbf{m}\rangle} F_\rho(\mathbf{x}+2\pi\mathbf{m}), \tag{2.13}$$

which is a form of Poisson summation formula [20]. In particular,

$$|G_\rho(\mathbf{x})| \leq \frac{1}{(2\pi)^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} \left|F_\rho(\mathbf{x}+2\pi\mathbf{n})\right|. \tag{2.14}$$

To estimate the function $F_\rho(\mathbf{x})$, we first note that

$$\varphi(\xi,\rho) = \rho^2\varphi\left(\xi/\rho, 1\right)$$

and

$$h_\rho(\xi)=\frac{\eta\left(\rho^2\left|\varphi\left(\xi/\rho,1\right)\right|^2\right)}{\rho^2\varphi\left(\xi/\rho,1\right)}.$$

It follows that

$$F_\rho(\mathbf{x})=(h_\rho)^\vee(\mathbf{x})=\rho^{d-2}\left(\frac{\eta\left(\rho^2\left|\varphi(\cdot,1)\right|^2\right)}{\varphi(\cdot,1)}\right)^\vee(\rho\mathbf{x}).$$

Let

$$f_\rho(\mathbf{x})=\left(\frac{\eta\left(\rho^2\left|\varphi(\cdot,1)\right|^2\right)}{\varphi(\cdot,1)}\right)^\vee(\mathbf{x}). \tag{2.15}$$

Then,

$$F_\rho(\mathbf{x})=\rho^{d-2}f_\rho(\rho\mathbf{x}). \tag{2.16}$$

Note that

$$\varphi(\xi,1)=\sum_{j,k}a_{jk}\xi_j\xi_k-s_0a_1+2is_0\xi_1, \tag{2.17}$$

$$|\varphi(\xi,1)|^2=\Big|\sum_{j,k}a_{jk}\xi_j\xi_k-s_0a_1\Big|^2+4s_0^2\xi_1^2.$$

A direct computation yields the estimates

$$\left|\frac{\partial^\ell}{\partial\xi_j^\ell}\left\{\frac{1}{\varphi(\xi,1)}\right\}\right|\le\frac{C_\ell(1+|\xi|)^\ell}{|\varphi(\xi,1)|^{\ell+1}}, \tag{2.18}$$

$$\left|\frac{\partial^\ell}{\partial\xi_j^\ell}\left\{\eta\left(\rho^2\left|\varphi(\xi,1)\right|^2\right)\right\}\right|\le C_\ell\rho^\ell \tag{2.19}$$

for $\ell\ge 0$ and $j=1,\dots,d$.

LEMMA 2.20. *Let $f_\rho(\mathbf{x})$ be defined by* (2.15). *Then, for any $\mathbf{x}\in\mathbb{R}^d$,*

$$|f_\rho(\mathbf{x})|\le\frac{C\rho^3}{|\mathbf{x}|^4},\qquad \textit{if } d=3, \tag{2.21}$$

$$|f_\rho(\mathbf{x})|\le\frac{C\rho^4}{|\mathbf{x}|^4},\qquad \textit{if } d=2. \tag{2.22}$$

Proof. Since $x_j^4f_\rho(\mathbf{x})$ is the inverse Fourier transform of

$$\frac{\partial^4}{\partial\xi_j^4}\left\{\frac{\eta\left(\rho^2|\varphi(\xi,1)|^2\right)}{\varphi(\xi,1)}\right\},$$

we have

$$
\begin{aligned}
\left|x_j^4 f_\rho(\mathbf{x})\right| &\le \int_{\mathbb{R}^d} \left| \frac{\partial^4}{\partial \xi_j^4} \left\{ \frac{\eta\left(\rho^2 |\varphi(\xi,1)|^2\right)}{\varphi(\xi,1)} \right\} \right| d\xi \\
&\le C \int_{\mathbb{R}^d} \sum_{\ell=0}^{4} \left| \frac{\partial^\ell}{\partial \xi_j^\ell} \left\{ \frac{1}{\varphi(\xi,1)} \right\} \right| \cdot \left| \frac{\partial^{4-\ell}}{\partial \xi_j^{4-\ell}} \left\{ \eta\left(\rho^2 |\varphi(\xi,1)|^2\right) \right\} \right| d\xi \\
&\le C \int_{|\varphi(\xi,1)| \ge c/\rho} \frac{(1+|\xi|)^4}{|\varphi(\xi,1)|^5}\, d\xi \\
&\qquad + C \int_{|\varphi(\xi,1)| \sim 1/\rho} \sum_{\ell=0}^{3} \frac{(1+|\xi|)^\ell}{|\varphi(\xi,1)|^{\ell+1}} \cdot \rho^{4-\ell}\, d\xi \\
&\le C \int_{|\varphi(\xi,1)| \ge c/\rho} \frac{(1+|\xi|)^4}{|\varphi(\xi,1)|^5}\, d\xi \\
&\le C \int_{\mathbb{R}^d} \frac{(1+|\xi|)^4}{\{|\varphi(\xi,1)| + 1/\rho\}}\, d\xi.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|\varphi(\xi,1)| &\sim |\xi_1| + \left|\xi A \xi^T - \mathbf{a} A \mathbf{a}^T\right| \\
&= |\xi_1| + \{|\xi B| + |\mathbf{a}B|\} \left||\xi B| - |\mathbf{a}B|\right|,
\end{aligned} \tag{2.23}
$$

where $B = \sqrt{A} \ge 0$. Using this it is not hard to see that

$$
I_1 := \int_{|\xi B| \ge 2|\mathbf{a}B|} \frac{(1+|\xi|)^4}{\{|\varphi(\xi,1)| + 1/\rho\}^5}\, d\xi \le C \int_{|\xi| \ge c} \frac{|\xi|^4}{|\xi|^{10}}\, d\xi \le C.
$$

Also,

$$
\begin{aligned}
I_2 &:= \int_{|\xi B| \le 2|\mathbf{a}B|} \frac{(1+|\xi|)^4}{\{|\varphi(\xi,1)| + 1/\rho\}^5}\, d\xi \\
&\le C \int_{|\xi B| \le 2|\mathbf{a}B|} \frac{d\xi}{\left\{|\xi_1| + \left||\xi B| - |\mathbf{a}B|\right| + 1/\rho\right\}^5} \\
&\le C \int_{|\xi| \le 2|\mathbf{a}B|} \frac{d\xi}{\left\{|(\xi B^{-1})_1| + \left||\xi| - |\mathbf{a}B|\right| + 1/\rho\right\}^5} \\
&\le C \int_{|\xi| \le 2|\mathbf{a}B|} \frac{d\xi}{\left\{|\xi_1| + \left||\xi| - |\mathbf{a}B|\right| + 1/\rho\right\}^5},
\end{aligned}
$$

where the last inequality follows by a rotation.

Now suppose $d = 3$. Then, using spherical coordinates with $\xi_1 = r\cos\theta$, we have

$$\begin{aligned} I_2 &\le C \int_0^{2|\mathbf{a}B|} r^2 \left\{ \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\left\{ r\cos\theta + \big| r - |\mathbf{a}B| \big| + 1/\rho \right\}^5} \right\} dr \\ &\le C \int_0^{2|\mathbf{a}B|} \frac{dr}{\left\{ \big| r - |\mathbf{a}B| \big| + 1/\rho \right\}^4} \\ &\le C \int_0^{|\mathbf{a}B|} \frac{dr}{\left\{ r + 1/\rho \right\}^4} \\ &\le C\rho^3. \end{aligned}$$

Similarly, if $d = 2$,

$$\begin{aligned} I_2 &\le C \int_0^{2|\mathbf{a}B|} r \left\{ \int_0^{\pi/2} \frac{d\theta}{\left\{ r\cos\theta + \big| r - |\mathbf{a}B| \big| + 1/\rho \right\}^5} \right\} dr \\ &\le C \int_0^{2|\mathbf{a}B|} \frac{dr}{\left\{ \big| r - |\mathbf{a}B| \big| + 1/\rho \right\}^5} \\ &\le C \int_0^{|\mathbf{a}B|} \frac{dr}{\left\{ r + 1/\rho \right\}^5} \\ &\le C\rho^4. \end{aligned}$$

Thus we have proved that, for $j = 1, \cdots, d$,

$$\left| x_j^4 f_\rho(\mathbf{x}) \right| \le C \left\{ I_1 + I_2 \right\} \le \begin{cases} C\rho^3, & \text{if } d = 3, \\ C\rho^4, & \text{if } d = 2. \end{cases}$$

The estimates (2.21) and (2.22) then follow. □

It follows from (2.16) and Lemma 2.20 that, for any $\mathbf{x} \in \mathbb{R}^d$,

$$|F_\rho(\mathbf{x})| \le \frac{C}{|\mathbf{x}|^4} \qquad \text{for } d = 2 \text{ or } 3. \tag{2.24}$$

This will be used to estimate the terms on the right hand side of (2.14), where $|\mathbf{x} + 2\pi\mathbf{n}| \ge 1/2$.

3. Pointwise estimate of the kernel function $G_\rho(\mathbf{x})$

In this section we will show that, if $|\mathbf{x}| \le 1/2$, then

$$|F_\rho(\mathbf{x})| \le \begin{cases} \dfrac{C}{|\mathbf{x}|}, & \text{if } d = 3, \\ C \ln \dfrac{1}{|\mathbf{x}|}, & \text{if } d = 2. \end{cases} \tag{3.1}$$

Together with (2.24) and (2.14), this implies that

$$|G_\rho(\mathbf{x})| \le \begin{cases} C\left\{1+\displaystyle\sum_{|\mathbf{x}+2\pi\mathbf{n}|\le 1/2}\frac{1}{|\mathbf{x}+2\pi\mathbf{n}|}\right\}, & \text{if } d=3,\\ C\left\{1+\displaystyle\sum_{|\mathbf{x}+2\pi\mathbf{n}|\le 1/2}\ln\frac{1}{|\mathbf{x}+2\pi\mathbf{n}|}\right\}, & \text{if } d=2.\end{cases} \tag{3.2}$$

To prove (3.1), we recall that $F_\rho(\mathbf{x}) = \rho^{d-2} f_\rho(\rho\mathbf{x})$ and write

$$f_\rho(\mathbf{x}) = \left\{\frac{1}{\varphi(\cdot,1)}\right\}^\vee(\mathbf{x}) + \left\{\frac{\eta\left(\rho^2|\varphi(\cdot,1)|^2\right)-1}{\varphi(\cdot,1)}\right\}^\vee(\mathbf{x}). \tag{3.3}$$

Lemma 3.4. *We have*

$$\int_{\mathbb{R}^d}\left|\frac{\eta\left(\rho^2|\varphi(\xi,1)|^2\right)-1}{\varphi(\xi,1)}\right| d\xi \le \frac{C}{\rho}.$$

Proof. Recall that $\eta(r) = 1$ for $r \ge s_0^2$. Thus, as in the proof of Lemma 2.20, we have

$$\begin{aligned}
\int_{\mathbb{R}^d}\left|\frac{\eta\left(\rho^2|\varphi(\xi,1)|^2\right)-1}{\varphi(\xi,1)}\right| d\xi &\le C\int_{|\varphi(\xi,1)|\le c/\rho}\frac{d\xi}{|\varphi(\xi,1)|}\\
&\le C\int_{|\xi_1|+\left||\xi|-|\mathbf{a}B|\right|\le c/\rho}\frac{d\xi}{|\xi_1|+\left||\xi|-|\mathbf{a}B|\right|}\\
&\le C\int_{\substack{|\xi_1|\le c/\rho\\ \left||\xi'|-|\mathbf{a}B|\right|\le c/\rho}}\frac{d\xi}{|\xi_1|+\left||\xi'|-|\mathbf{a}B|\right|}\quad\text{where } \xi=(\xi_1,\xi')\\
&\le C\int_{\left|r-|\mathbf{a}B|\right|\le c/\rho}\int_{|\xi_1|\le c/\rho}\frac{d\xi_1 dr}{|\xi_1|+\left|r-|\mathbf{a}B|\right|}\\
&\le \frac{C}{\rho}\int_{0<r<c}\int_{|\xi_1|\le c}\frac{d\xi_1 dr}{|\xi_1|+r}\\
&\le \frac{C}{\rho}.
\end{aligned}$$

□

It follows from Lemma 3.4 that

$$\left|\left\{\frac{\eta(\rho^2|\varphi(\cdot,1)|^2)-1}{\varphi(\cdot,1)}\right\}^\vee(\mathbf{x})\right| \le \frac{C}{\rho}. \tag{3.5}$$

To estimate the first term on the right hand side of (3.3), we first note that, by several changes of variables, we have

$$\left\{\frac{1}{\varphi(\cdot,1)}\right\}^\vee(\mathbf{x}) = \frac{|\mathbf{a}B|^{d-2}}{\det(B)}\left\{\frac{1}{|\xi|^2-1+2i\xi_1}\right\}^\vee\left(|\mathbf{a}B|\mathbf{x}B^{-1}O^{-1}\right), \tag{3.6}$$

where O is a $d\times d$ orthogonal matrix such that $\mathbf{a}BO^{-1}=(|\mathbf{a}B|,0,\cdots,0)$.

LEMMA 3.7. *Let $u(\mathbf{x})$ denote the inverse Fourier transform of $\left\{|\xi|^2-1+2i\xi_1\right\}^{-1}$ in $\mathbb{R}^d$, $d=2$ or $d=3$. Let $\mathbf{x}=(x_1,\mathbf{x}')\in\mathbb{R}^d$. Then,*

$$u(\mathbf{x})=2\pi\int_0^\infty J_0(|\mathbf{x}'|\,r)v(r,x_1)r\,dr,\quad \text{if } d=3, \tag{3.8}$$

$$u(\mathbf{x})=2\int_0^\infty \cos(|x_2|\,r)v(r,x_1)\,dr,\quad \text{if } d=2, \tag{3.9}$$

where

$$v(r,x_1)=\int_{\mathbb{R}}\frac{e^{ix_1\xi_1}}{r^2+\xi_1^2-1+2i\xi_1}\,d\xi_1, \tag{3.10}$$

and

$$J_0(t)=\frac{1}{2\pi}\int_0^{2\pi}e^{it\cos\omega}d\omega \tag{3.11}$$

is the Bessel function of the first kind of order 0.

Proof. One may verify that, for $R>0$,

$$\left\{|\xi|^2-1+2i\xi_1\right\}^{-1}\chi_{\{\xi\in\mathbb{R}^d:\,|\xi'|\le R\}}\in L^1\left(\mathbb{R}^d\right),$$

where $\xi=(\xi_1,\xi')$. Since $\left\{|\xi|^2-1+2i\xi_1\right\}^{-1}\in L^p\left(\mathbb{R}^d\right)$ for $3/2<p<2$, we have, by the Hausdorff–Young inequality [20],

$$u(\mathbf{x})=\lim_{R\to\infty}\int_{\substack{\xi\in\mathbb{R}^d\\|\xi'|\le R}}\frac{e^{i\langle\mathbf{x},\xi\rangle}}{|\xi|^2-1+2i\xi_1}\,d\xi,$$

where the limit is taken in the $L^{p'}$-space. From this, (3.8) and (3.9) follow by using Fubini's theorem and polar coordinates. We omit the details. □

LEMMA 3.12. *Let $v(r,x_1)$ be the function defined by* (3.10). *Then,*

$$v(r,x_1)=\begin{cases}\dfrac{\pi}{r}e^{x_1-r|x_1|}, & \text{if } r>1,\\[2mm] \dfrac{\pi}{r}\left\{e^{x_1-r|x_1|}-e^{(1-r)x_1}\right\}, & \text{if } 0<r<1.\end{cases}$$

Proof. First we write

$$v(r,x_1)=e^{x_1}\int_{\mathbb{R}}\frac{e^{ix_1(\xi_1+i)}}{r^2+(\xi_1+i)^2}\,d\xi_1.$$

Applying Cauchy's integral theorem to the function

$$w(z)=\frac{e^{ix_1z}}{r^2+z^2}=\frac{e^{ix_1z}}{(z+ri)(z-ri)},$$

we obtain

$$v(r,x_1) = \begin{cases} e^{x_1} \int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2+y^2}\, dy, & \text{if } r>1, \\ e^{x_1} \int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2+y^2}\, dy - \frac{\pi}{r} e^{(1-r)x_1}, & \text{if } 0<r<1. \end{cases} \tag{3.13}$$

By a routine application of the residue theorem, one may show that

$$\int_{\mathbb{R}} \frac{e^{ix_1 y}}{r^2+y^2}\, dy = \frac{\pi}{r} e^{-r|x_1|}; \tag{3.14}$$

see, e.g., [14, pp. 389–390]. This, together with (3.13), yields the lemma. □

LEMMA 3.15. *Let* $v(r,x_1)$ *be the function defined by* (3.10). *Then,*

$$|v(r,x_1)| \le \begin{cases} \frac{\pi}{r} e^{(1-r)\cdot|x_1|}, & \textit{if } r>1, \\ C\left\{ e^{(r-1)|x_1|} + |x_1|\, e^{-|x_1|/2} \right\}, & \textit{if } 0<r<1, \end{cases}$$

and

$$\left| \frac{\partial v}{\partial r}(r,x_1) \right| \le \begin{cases} \frac{C}{r^2}(1+r|x_1|) e^{(1-r)|x_1|}, & \textit{if } r>1, \\ C\left\{ |x_1|^2 e^{-|x_1|/2} + (1+|x_1|) e^{(r-1)|x_1|} \right\}, & \textit{if } 0<r<1. \end{cases}$$

Proof. We will only prove the second estimate, using Lemma 3.12. The proof of the first estimate is easier.

If $r>1$,

$$\frac{\partial v}{\partial r} = -\frac{\pi}{r^2} e^{x_1 - r|x_1|} + \frac{\pi}{r} e^{x_1 - r|x_1|}(-|x_1|).$$

Hence,

$$\left| \frac{\partial v}{\partial r} \right| = \frac{\pi(1+r|x_1|)}{r^2} e^{x_1 - r|x_1|} \le \frac{\pi(1+r|x_1|)}{r^2} e^{(1-r)|x_1|}.$$

Next suppose $0<r<1$. We may assume $x_1<0$ since $v(r,x_1)=0$ if $0<r<1$ and $x_1 \ge 0$. Note that, in this case, we have

$$\frac{\partial v}{\partial r} = -\frac{\pi}{r^2}\left\{ e^{(1+r)x_1} - e^{(1-r)x_1} \right\} + \frac{\pi x_1}{r}\left\{ e^{(1+r)x_1} + e^{(1-r)x_1} \right\}.$$

If $1/2 \le r < 1$, it is easy to see that

$$\begin{aligned} \left| \frac{\partial v}{\partial r} \right| &\le C\left| e^{(1+r)x_1} - e^{(1-r)x_1} \right| + C|x_1| \left\{ e^{(1+r)x_1} + e^{(1-r)x_1} \right\} \\ &\le C\{1+|x_1|\} e^{(r-1)|x_1|}. \end{aligned}$$

Also, if $0 < r < 1/2$ and $|rx_1| \geq 1$, then $|x_1| \geq 1/r$. It follows that

$$\begin{aligned}\left|\frac{\partial v}{\partial r}\right| &\leq C\,|x_1|^2\left\{e^{(1+r)x_1} + e^{(1-r)x_1}\right\}\\ &\leq C\,|x_1|^2\,e^{-|x_1|/2}.\end{aligned}$$

Finally, if $0 < r < 1/2$ and $|rx_1| < 1$, we use $e^t = 1 + t + O(t^2)$ for $|t| < 1$ to obtain

$$\begin{aligned}\frac{\partial v}{\partial r} &= -\frac{\pi}{r^2}e^{x_1}\left\{2rx_1 + O\big((rx_1)^2\big)\right\} + \frac{\pi x_1}{r}e^{x_1}\left\{2 + O\big((rx_1)^2\big)\right\}\\ &= \frac{\pi}{r^2}e^{x_1}O\big((rx_1)^2\big) + \frac{\pi x_1}{r}e^{x_1}O\big((rx_1)^2\big).\end{aligned}$$

It follows that

$$\left|\frac{\partial v}{\partial r}\right| \leq C\left\{|x_1|^2\,e^{x_1} + r\,|x_1|^3\,e^{x_1}\right\} \leq C\,|x_1|^2\,e^{-|x_1|}.$$

The proof is now complete. □

LEMMA 3.16. *Let $u(\mathbf{x})$ be the inverse Fourier transform of $\{|\xi|^2 - 1 + 2i\xi_1\}^{-1}$ in $\mathbb{R}^d$. Then, if $d = 3$,*

$$|u(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|},$$

and, if $d = 2$,

$$|u(\mathbf{x})| \leq \begin{cases} C\ln\dfrac{1}{|\mathbf{x}|}, & \text{if } |\mathbf{x}| \leq \dfrac{1}{2},\\ \dfrac{C}{|\mathbf{x}|}, & \text{if } |\mathbf{x}| > \dfrac{1}{2}.\end{cases}$$

Proof. We first consider the case $d = 3$. It follows from (3.8) that

$$\begin{aligned}u(\mathbf{x}) &= 2\pi\int_0^{1/|\mathbf{x}'|} J_0(|\mathbf{x}'|\,r)v(r,x_1)r\,dr + 2\pi\int_{1/|\mathbf{x}'|}^{\infty} J_0(|\mathbf{x}'|\,r)v(r,x_1)r\,dr\\ &= I_1 + I_2.\end{aligned}$$

By Lemma 3.15, $|v(r,x_1)|\,r \leq C$. This, together with the observation $|J_0(t)| \leq 1$, gives

$$|I_1| \leq 2\pi\int_0^{1/|\mathbf{x}'|}|v(r,x_1)|\,r\,dr \leq \frac{C}{|\mathbf{x}'|}.$$

To estimate I_2 we first assume that $|\mathbf{x}'| \leq 1$. Since

$$rJ_0(r) = \frac{d}{dr}\left\{rJ_1(r)\right\}, \tag{3.17}$$

where $J_\nu(r)$ denotes the Bessel function of the first kind of order ν (see [11]), we may use integration by parts to obtain

$$\begin{aligned} I_2 &= \frac{2\pi}{|\mathbf{x}'|^2}\int_1^\infty rJ_0(r)v\left(\frac{r}{|\mathbf{x}'|},x_1\right)dr \\ &= -\frac{2\pi}{|\mathbf{x}'|^2}J_1(1)v\left(\frac{1}{|\mathbf{x}'|},x_1\right) - \frac{2\pi}{|\mathbf{x}'|^3}\int_1^\infty rJ_1(r)\frac{\partial v}{\partial r}\left(\frac{r}{|\mathbf{x}'|},x_1\right)dr. \end{aligned}$$

It then follows from the estimate (see [11])

$$|J_1(r)| \le \frac{C}{r^{1/2}}, \quad \text{for } r \ge 1 \tag{3.18}$$

and Lemma 3.15 that

$$\begin{aligned} |I_2| &\le \frac{C}{|\mathbf{x}'|} + \frac{C}{|\mathbf{x}'|^3}\int_1^\infty r^{1/2}\left|\frac{\partial v}{\partial r}\left(\frac{r}{|\mathbf{x}'|},x_1\right)\right|dr \\ &\le \frac{C}{|\mathbf{x}'|} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_{1/|\mathbf{x}'|}^\infty r^{1/2}\left|\frac{\partial v}{\partial r}(r,x_1)\right|dr \\ &\le \frac{C}{|\mathbf{x}'|} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_{1/|\mathbf{x}'|}^\infty r^{1/2}\cdot\frac{1}{r^2}\cdot\{1+r|x_1|\}\,e^{(r-1)|x_1|}\,dr \\ &\le \frac{C}{|\mathbf{x}'|}. \end{aligned}$$

If $|\mathbf{x}'| \ge 1$, we write

$$\begin{aligned} I_2 &= \frac{2\pi}{|\mathbf{x}'|^2}\int_1^{|\mathbf{x}'|} rJ_0(r)v\left(\frac{r}{|\mathbf{x}'|},x_1\right)dr + \frac{2\pi}{|\mathbf{x}'|^2}\int_{|\mathbf{x}'|}^\infty rJ_0(r)v\left(\frac{r}{|\mathbf{x}'|},x_1\right)dr \\ &= I_{21} + I_{22}. \end{aligned}$$

Note that, using (3.17), integration by parts, and (3.18), we have

$$\begin{aligned} |I_{21}| &\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^3}\int_1^{|\mathbf{x}'|} r^{1/2}\left|\frac{\partial v}{\partial r}\left(\frac{r}{|\mathbf{x}'|},x_1\right)\right|dr \\ &\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_{1/|\mathbf{x}'|}^1 r^{1/2}\left|\frac{\partial v}{\partial r}(r,x_1)\right|dr \\ &\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^{3/2}}\int_0^1 |x_1|\,e^{(r-1)|x_1|}\,dr \\ &\le \frac{C}{|\mathbf{x}'|^{3/2}} \le \frac{C}{|\mathbf{x}'|}. \end{aligned}$$

Similarly,

$$
\begin{aligned}
|I_{22}| &= \frac{2\pi}{|\mathbf{x}'|^2} \left| \int_{|\mathbf{x}'|}^{\infty} \frac{d}{dr} \{ r J_1(r) \} \, v\left(\frac{r}{|\mathbf{x}'|}, x_1 \right) \, dr \right| \\
&\le \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^3} \int_{|\mathbf{x}'|}^{\infty} r^{1/2} \left| \frac{\partial v}{\partial r} \left(\frac{r}{|\mathbf{x}'|}, x_1 \right) \right| \, dr \\
&= \frac{C}{|\mathbf{x}'|^{3/2}} + \frac{C}{|\mathbf{x}'|^{3/2}} \int_1^{\infty} r^{1/2} \left| \frac{\partial v}{\partial r}(r, x_1) \right| \, dr \\
&\le \frac{C}{|\mathbf{x}'|^{3/2}} \le \frac{C}{|\mathbf{x}'|}.
\end{aligned}
$$

Thus we have proved that, for any $\mathbf{x} \in \mathbb{R}^3$,

$$
|u(\mathbf{x})| \le \frac{C}{|\mathbf{x}'|}. \tag{3.19}
$$

To finish the case $d = 3$, we still need to show that

$$
|u(\mathbf{x})| \le \frac{C}{|x_1|}, \quad \text{for any } \mathbf{x} \in \mathbb{R}^3. \tag{3.20}
$$

Clearly, (3.19) and (3.20) imply that $|u(\mathbf{x})| \le C/|\mathbf{x}|$ for any $\mathbf{x} \in \mathbb{R}^3$.

To see (3.20), we use Lemma 3.15 to obtain

$$
\begin{aligned}
|u(\mathbf{x})| &\le 2\pi \int_0^{\infty} |v(r, x_1)| \, r \, dr \\
&\le C \int_0^1 \left\{ e^{(r-1)|x_1|} + |x_1| \, e^{-|x_1|/2} \right\} dr + C \int_1^{\infty} e^{(1-r)|x_1|} \, dr \\
&\le \frac{C}{|x_1|}.
\end{aligned}
$$

We now consider the case $d = 2$. By Lemmas 3.7 and 3.5,

$$
\begin{aligned}
|u(\mathbf{x})| &= 2 \left| \int_0^{\infty} \cos(|x_2| \, r) v(r, x_1) \, dr \right| \\
&\le 2 \int_0^{\infty} |v(r, x_1)| \, dr \\
&\le C \int_0^1 \left\{ e^{(r-1)|x_1|} + |x_1| \, e^{-|x_1|/2} \right\} dr + C \int_1^{\infty} e^{(1-r)|x_1|} \frac{dr}{r} \\
&\le C \, |x_1| \, e^{-|x_1|/2} + C \int_0^1 e^{-r|x_1|} \, dr + C \int_0^{\infty} e^{-r|x_1|} \frac{dr}{r+1}.
\end{aligned}
$$

From this it is not hard to see that

$$|u(\mathbf{x})| \leq \begin{cases} C \ln \dfrac{1}{|x_1|}, & \text{if } |x_1| \leq \dfrac{1}{2}, \\ \dfrac{C}{|x_1|}, & \text{if } |x_1| > \dfrac{1}{2}. \end{cases} \tag{3.21}$$

Finally, we will show that

$$|u(\mathbf{x})| \leq \begin{cases} C \ln \dfrac{1}{|x_2|}, & \text{if } |x_2| \leq \dfrac{1}{2}, \\ \dfrac{C}{|x_2|}, & \text{if } |x_2| > \dfrac{1}{2}. \end{cases} \tag{3.22}$$

The desired estimate for $|u(\mathbf{x})|$ follows easily from (3.21) and (3.22).

To see (3.22) we write

$$\begin{aligned} u(\mathbf{x}) &= 2\int_0^{1/|x_2|} \cos(|x_2|\, r) v(r, x_1)\, dr + 2\int_{1/|x_2|}^{\infty} \cos(|x_2|\, r) v(r, x_1)\, dr \\ &= I_3 + I_4, \end{aligned}$$

as in the case of $d = 3$. If $|x_2| > 1/2$, by Lemma 3.15, we have

$$|I_3| \leq C \int_0^{1/|x_2|} |v(r, x_1)|\; dr \leq C \int_0^{1/|x_2|} dr \leq \frac{C}{|x_2|}.$$

Similarly, if $|x_2| \leq 1/2$,

$$|I_3| \leq 2\int_0^1 |v(r, x_1)|\; dr + 2\int_1^{1/|x_2|} |v(r, x_1)|\; dr \leq C + C\int_1^{1/|x_2|} \frac{dr}{r} \leq C \ln \frac{1}{|x_2|}.$$

To estimate I_4 we use integration by parts. Suppose $|x_2| \leq 1/2$. Then

$$\begin{aligned} |I_4| &= \frac{2}{|x_2|} \left| \int_{1/|x_2|}^{\infty} \frac{\partial}{\partial r} \left\{ \sin(|x_2|\, r) \right\} v(r, x_1)\, dr \right| \\ &\leq C + \frac{C}{|x_2|} \int_{1/|x_2|}^{\infty} \left| \frac{\partial v}{\partial r}(r, x_1) \right| \, dr \\ &\leq C + \frac{C}{|x_2|} \int_{1/|x_2|}^{\infty} \frac{1}{r^2} \left\{ 1 + r\, |x_1| \right\} e^{(1-r)|x_1|}\, dr \\ &\leq C \leq C \ln \frac{1}{|x_2|}. \end{aligned}$$

If $|x_2| > 1/2$, then

$$\begin{aligned}|I_4| &\le \frac{2}{|x_2|}\left|\int_{1/|x_2|}^{1} \frac{\partial}{\partial r}\{\sin(|x_2|\, r)\}\cdot v(r,x_1)\, dr\right| \\ &\quad + \frac{2}{|x_2|}\left|\int_{1}^{\infty} \frac{\partial}{\partial r}\{\sin(|x_2|\, r)\}\cdot v(r,x_1)\, dr\right| \\ &\le \frac{C}{|x_2|} + \frac{C}{|x_2|}\int_{1/|x_2|}^{1}\left|\frac{\partial v}{\partial r}(r,x_1)\right| dr + \frac{C}{|x_2|}\int_{1}^{\infty}\left|\frac{\partial v}{\partial r}(r,x_1)\right| dr \\ &\le \frac{C}{|x_2|}.\end{aligned}$$

This proves (3.22) and completes the proof of Lemma 3.16. □

It follows from Lemma 3.16 and (3.6) that

$$\left|\left\{\frac{1}{\varphi(\cdot,1)}\right\}^{\vee}(\mathbf{x})\right| \le \begin{cases} \dfrac{C}{|\mathbf{x}|}, & \text{if } d=3,\\ C\ln\left(1+\dfrac{1}{|\mathbf{x}|}\right), & \text{if } d=2. \end{cases} \tag{3.23}$$

This, together with (3.3) and (3.5), implies that

$$|f_\rho(\mathbf{x})| \le \begin{cases} C\left\{\dfrac{1}{\rho}+\dfrac{1}{|\mathbf{x}|}\right\}, & \text{if } d=3,\\ C\left\{\dfrac{1}{\rho}+\ln\left(1+\dfrac{1}{|\mathbf{x}|}\right)\right\}, & \text{if } d=2. \end{cases} \tag{3.24}$$

Thus, by (2.16), for any $\mathbf{x}\in\mathbb{R}^d$,

$$|F_\rho(\mathbf{x})| = \rho^{d-2}\,|f_\rho(\rho\mathbf{x})| \le \begin{cases} C\left\{1+\dfrac{1}{|\mathbf{x}|}\right\}, & \text{if } d=3,\\ C\left\{\dfrac{1}{\rho}+\ln\left(1+\dfrac{1}{\rho|\mathbf{x}|}\right)\right\}, & \text{if } d=2. \end{cases} \tag{3.25}$$

The estimate (3.1) now follows from (3.25), and the proof of (3.2) is complete.

4. Proof of the Main Theorem

Suppose $V \in K_d$. It is well known that, for any $\varepsilon > 0$, there exists a constant $C_{\varepsilon,V} > 0$ such that

$$\int_{\mathbb{R}^d} |g|^2|V|\, d\mathbf{x} \le \varepsilon \int_{\mathbb{R}^d} |\nabla g|^2\, d\mathbf{x} + C_{\varepsilon,V}\int_{\mathbb{R}^d} |g|^2\, d\mathbf{x} \tag{4.1}$$

for any $g \in H^1(\mathbb{R}^d)$; see [7], [18]. It follows from (4.1) that the quadratic form associated with $\mathbf{D}A\mathbf{D}^T + V$ generates a unique self-adjoint operator on $L^2(\mathbb{R}^d)$, which we also denote by $\mathbf{D}A\mathbf{D}^T + V$.

Let $\psi \in H^1(\mathbb{T}^d)$, where

$$H^1(\mathbb{T}^d) = \left\{ \phi \in L^2(\mathbb{T}^d) \colon \phi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}} e^{i\langle \mathbf{x},\mathbf{n}\rangle} \text{ and } \sum_{\mathbf{n}\in\mathbb{Z}^d} |\mathbf{n}|^2 \, |a_{\mathbf{n}}|^2 < \infty \right\}.$$

Extending ψ by periodicity to $\mathbb{R}^d$ and then applying (4.1) to $\psi\widetilde{\eta}$, where $\widetilde{\eta}$ is a C^∞ cut-off function such that $\widetilde{\eta} = 1$ on Ω, we obtain

$$\int_\Omega |\psi|^2 |V| \, d\mathbf{x} \le \varepsilon \int_\Omega |\nabla\psi|^2 \, d\mathbf{x} + \widetilde{C}_{\varepsilon,V} \int_\Omega |\psi|^2 \, d\mathbf{x} \tag{4.2}$$

for any $\varepsilon > 0$. This implies that, for any $\mathbf{k} \in \mathbb{C}^d$, the quadratic form associated with $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T + V$ on $\mathbb{T}^d$ defines a unique closed operator on $L^2(\mathbb{T}^d)$, which we denote by $\mathbb{H}_V(\mathbf{k})$. Moreover,

(4.3)

$$\text{Domain}\,(\mathbb{H}_V(\mathbf{k})) = \left\{ \psi \in H^1(\mathbb{T}^d) \colon \mathbb{H}_V(0)\psi = (\mathbf{D}A\mathbf{D}^T + V)\psi \in L^2(\mathbb{T}^d) \right\}.$$

Let $\mathbf{a} \in \mathbb{R}^d$ be a vector satisfying (2.1) and

$$L = \left\{ \mathbf{b} \in \mathbb{R}^d \colon \langle \mathbf{b}, \mathbf{a}\rangle = 0 \text{ and } |\mathbf{b}| \le \sqrt{d} \right\}. \tag{4.4}$$

Proposition 4.5. *If, for every $\mathbf{b} \in L$, the family of operators $\{\mathbb{H}_V(z\mathbf{a} + \mathbf{b}) \colon z \in \mathbb{C}\}$ has no common eigenvalue, then the spectrum of the operator $\mathbf{D}A\mathbf{D}^T + V$ on $L^2(\mathbb{R}^d)$ is purely absolutely continuous.*

Proof. See [13] and [16]. □

Fix $\mathbf{b} \in L$ and let

$$\delta = \frac{1}{a_1}\left(\frac{1}{2} - b_1\right),$$

as in (2.2). We will show that the family of operators $\{\mathbb{H}_V((\delta + i\rho)\mathbf{a} + \mathbf{b}) \colon \rho \ge 1\}$ has no common eigenvalue under the assumption of our main theorem.

We need the following estimate on the norm of $\{\mathbb{H}_0\big((\delta + i\rho)\mathbf{a} + \mathbf{b}\big)\}^{-1}$ on $L^1(\mathbb{T}^d)$.

Theorem 4.6. *There exists a constant $C > 0$ such that*

$$\left\| \left\{ \mathbb{H}_0\big((\delta + i\rho)\mathbf{a} + \mathbf{b}\big) \right\}^{-1} \right\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)} \le \begin{cases} \dfrac{C\ln(\rho+1)}{\rho^{1/2}}, & \textit{if } d = 3, \\ \dfrac{C}{\rho^{1/2}}, & \textit{if } d = 2. \end{cases}$$

Proof. In view of (2.8), it suffices to show that

$$\int_\Omega |G_\rho(\mathbf{x})| \, d\mathbf{x} \le \begin{cases} \dfrac{C\ln(\rho+1)}{\rho^{1/2}}, & \text{if } d = 3, \\ \dfrac{C}{\rho^{1/2}}, & \text{if } d = 2. \end{cases} \tag{4.7}$$

To this end, note that, by Hölder's inequality, (2.9), and the Plancherel theorem, we have

$$\begin{aligned}\int_\Omega |G_\rho(\mathbf{x})|\, d\mathbf{x} &\le |\Omega|^{1/2}\left\{\int_\Omega |G_\rho(\mathbf{x})|^2\, d\mathbf{x}\right\}^{1/2}\\ &= C\left\{\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{1}{|(\mathbf{n}+\mathbf{k})A(\mathbf{n}+\mathbf{k})^T|^2}\right\}^{1/2}\\ &\le C\left\{\sum_{\mathbf{n}\in\mathbb{Z}^d}\frac{1}{\left\{|(\mathbf{n}+\mathbf{b})A(\mathbf{n}+\mathbf{b})^T-\rho^2 a_1 s_0|+\rho\left|n_1+\frac{1}{2}\right|\right\}^2}\right\}^{1/2}.\end{aligned}$$

The desired estimate (4.7) follows from the proof of Lemma 3.2 in [16] (see the estimate (3.11) in [16]). We omit the details. □

The next theorem is a consequence of the pointwise estimate (3.2) of the kernel function G_ρ.

THEOREM 4.8. *There exists a constant* $C>0$ *such that*

$$\begin{aligned}&\left\|V\left\{\mathbb{H}_0\big((\delta+i\rho)\mathbf{a}+\mathbf{b}\big)\right\}^{-1}\right\|_{L^1(\mathbb{T}^d)\to L^1(\mathbb{T}^d)}\\ &\qquad\le\begin{cases}C\sup_{\mathbf{x}\in\Omega}\int_\Omega \frac{|V(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}, & \text{if } d=3,\\ C\sup_{\mathbf{x}\in\Omega}\int_\Omega |V(\mathbf{y})|\left\{1+|\ln|\mathbf{y}-\mathbf{x}||\right\}d\mathbf{y}, & \text{if } d=2.\end{cases}\end{aligned}$$

Proof. Recall that, if $\psi\in C^\infty(\mathbb{T}^d)$, then

$$\left\{\mathbb{H}_0\big((\delta+i\rho)\mathbf{a}+\mathbf{b}\big)\right\}^{-1}\psi(\mathbf{x})=\int_\Omega G_\rho(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y}.$$

It follows that

$$\begin{aligned}\left\|V\left\{\mathbb{H}_0\big((\delta+i\rho)\mathbf{a}+\mathbf{b}\big)\right\}^{-1}\psi\right\|_1 &\le \int_\Omega |V(\mathbf{x})|\left\{\int_\Omega |G_\rho(\mathbf{x}-\mathbf{y})|\,|\psi(\mathbf{y})|\,d\mathbf{y}\right\}d\mathbf{x}\\ &\le \sup_{\mathbf{y}\in\Omega}\int_\Omega |V(\mathbf{x})|\,|G_\rho(\mathbf{x}-\mathbf{y})|\,d\mathbf{x}\,\|\psi\|_1.\end{aligned}$$

The desired estimate now follows easily from (3.2). □

Proof of Main Theorem. We give the proof for the case $d=3$. The case $d=2$ can be handled in the same manner.

To show that $\{\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b})\colon \rho\ge 1\}$ has no common eigenvalue, we argue by contradiction. Suppose that there exists $E\in\mathbb{R}$ such that, for every $\rho\ge 1$, there exists $\psi_\rho\in\text{Domain}(\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b}))$ such that $\|\psi_\rho\|_2=1$ and

$$\mathbb{H}_V((\delta+i\rho)\mathbf{a}+\mathbf{b})\psi_\rho=E\psi_\rho.$$

Since $\psi_\rho \in H^1(\mathbb{T}^d)$, by the Cauchy inequality and (4.2), we have

$$\int_\Omega |\psi_\rho|\,|V|\,d\mathbf{x} \le \left\{\int_\Omega |V|\,d\mathbf{x}\right\}^{1/2}\left\{\int_\Omega |\psi_\rho|^2\,|V|\,d\mathbf{x}\right\}^{1/2} < \infty.$$

It follows that $(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho = E\psi_\rho - V\psi_\rho \in L^1(\mathbb{T}^d)$.

Let

$$V_N(\mathbf{x}) = \begin{cases} V(\mathbf{x}), & \text{if } |V(\mathbf{x})| > N,\\ 0, & \text{if } |V(\mathbf{x})| \le N.\end{cases}$$

Then,

$$\left\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\right\|_1 \le \{|E|+N\}\,\|\psi_\rho\|_1 + \|V_N\psi_\rho\|_1\,. \tag{4.9}$$

By Theorem 4.8,

$$\|V_N\psi_\rho\|_1 \le C\sup_{\mathbf{x}\in\Omega}\int_\Omega \frac{|V_N(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}\cdot\left\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\right\|_1\,. \tag{4.10}$$

Note that

$$\sup_{\mathbf{x}\in\Omega}\int_\Omega \frac{|V_N(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y} \le \sup_{\mathbf{x}\in\Omega}\int_{|\mathbf{y}-\mathbf{x}|\le r}\frac{|V(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y} + \frac{1}{r}\int_\Omega |V_N(\mathbf{y})|\,d\mathbf{y}.$$

It follows that

$$\lim_{N\to\infty}\sup_{\mathbf{x}\in\Omega}\int_\Omega \frac{|V_N(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y} \le \lim_{r\to 0}\sup_{\mathbf{x}\in\Omega}\int_{|\mathbf{y}-\mathbf{x}|\le r}\frac{|V(\mathbf{y})|}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y} = 0.$$

This implies that, if N is sufficiently large,

$$\|V_N\psi_\rho\|_1 \le \frac{1}{2}\left\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\right\|_1\,. \tag{4.11}$$

In view of (4.9) and (4.11), we obtain

$$\left\|(\mathbf{D}+\mathbf{k})A(\mathbf{D}+\mathbf{k})^T\psi_\rho\right\|_1 \le 2(|E|+N)\,\|\psi_\rho\|_1\,.$$

This, together with Theorem 4.6, gives

$$\frac{C\rho^{1/2}}{\ln(\rho+1)}\,\|\psi_\rho\|_1 \le 2(|E|+N)\,\|\psi_\rho\|_1$$

or

$$\frac{C\rho^{1/2}}{\ln(\rho+1)} \le 2(|E|+N),$$

for any $\rho \ge 1$. This is impossible if we let $\rho\to\infty$. □

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