Illinois Journal of Mathematics
Volume 47, Number 4, Winter 2003, Pages 1151–1166
S 0019-2082

ON THE ESTIMATION OF THE ORDER OF EULER-ZAGIER MULTIPLE ZETA-FUNCTIONS

HIDEAKI ISHIKAWA AND KOHJI MATSUMOTO

Abstract. We prove upper bound estimates for Euler-Zagier multiple zeta-functions. First, by shifting the paths of the relevant Mellin-Barnes type integrals to the right, we prove an estimate for general r-fold zeta-functions. Then, in the cases $r = 2$ and $r = 3$, we give further improvements by shifting the path suitably to the left.

1. Introduction

Let r be a positive integer, and define

$$(1)\quad \zeta_r(s_1, \ldots, s_r) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_r=1}^{\infty} n_1^{-s_1}(n_1+n_2)^{-s_2}\ldots(n_1+n_2+\cdots+n_r)^{-s_r},$$

where $s_1, \ldots, s_r$ are complex variables. This series is called the Euler-Zagier r-fold sum, and its values at positive integer arguments have been studied extensively by many mathematicians.

The series (1) may be regarded as an analytic function of several complex variables. From this point of view, we should first consider the problem of analytic continuation. In the case $r = 2$, this problem had already been discussed by Atkinson [4]. However, the investigation of the problem of analytic continuation for $r \geq 3$ has begun only recently. First, Arakawa and Kaneko [3] proved the analytic continuation of (1) as a function of one variable s_r. The continuation to the whole space $\mathbb{C}^r$ as a function of r variables was established by Zhao [15] and, independently, by Akiyama, Egami and Tanigawa [1]. The methods of continuation given in these three papers are all different from each other.

Still another proof of the analytic continuation was given by Matsumoto [10]. His method was based on the Mellin-Barnes integral formula ((2) below), which had been used successfully by Katsurada [8][9], who discovered a new

Received October 31, 2002; received in final form September 5, 2003.
2000 *Mathematics Subject Classification.* 11M41.

elegant proof of the analytic continuation of the case $r = 2$ of (1). The analytic continuations of various generalizations of (1) have been obtained in the papers [2], [11], [12], [13], [14].

A natural next step is the estimation of the order of $|\zeta_r(s_1, \dots, s_r)|$. Some upper bounds with respect to $t_r = \Im s_r$ were given in [6], [10], [12]. It is desirable, however, to obtain upper bounds with respect to all variables $t_j = \Im s_j$, $1 \le j \le r$. In the present paper, we do so using the method of the Mellin-Barnes formula. After reviewing the argument of [10] briefly in Section 2, we will give in Section 3 (Theorem 1) an upper bound of $|\zeta_r(s_1, \dots, s_r)|$ for general r. This result is a direct consequence of the formula (4), which is established by a "right-shift" of the path of integration. The estimate of the theorem is by no means best-possible. In Section 4 we will prove a refinement (Theorem 2) in the case $r = 2$, obtained by a suitable "left-shift" of (4). The method presented in Section 4 can, in principle, be applied to more general values $r \ge 3$, but the arguments becomes much more complicated. In the final section, we illustrate the basic idea by discussing a typical example in the case $r = 3$.

The authors express their sincere gratitude to Professor Yoshio Tanigawa for valuable discussions.

2. A review of the proof of analytic continuation

In the following sections, we use for brevity the notation

$$\begin{aligned} \boldsymbol{n}(j,r) &= n_j + n_{j+1} + \cdots + n_r, \\ \boldsymbol{k}(j,r) &= k_j + k_{j+1} + \cdots + k_r, \\ \boldsymbol{s}(j,r) &= s_j + s_{j+1} + \cdots + s_r, \\ \boldsymbol{\sigma}(j,r) &= \sigma_j + \sigma_{j+1} + \cdots + \sigma_r, \\ \boldsymbol{t}(j,r) &= t_j + t_{j+1} + \cdots + t_r. \end{aligned}$$

In this section we sketch the argument given in [10] to prove the analytic continuation of (1) by using the Mellin-Barnes integral formula

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s+z)\Gamma(-z)\lambda^z dz, \tag{2}$$

where s and λ are complex numbers with $\Re s > 0$, $\lambda \ne 0$, $|\arg \lambda| < \pi$, and c is real with $-\Re s < c < 0$. The path of integration is the vertical line from $c - i\infty$ to $c + i\infty$.

Let $r \ge 2$, $\xi > 0$, and first assume that $\sigma_j = \Re s_j \ge 1 + \xi$ $(1 \le j \le r)$. Then (1) is absolutely convergent. Putting $\lambda = n_r/\boldsymbol{n}(1, r-1)$, $c = -1 - \xi/2$ and $s = s_r$ in (2), dividing the both sides by

$$\Gamma(s_r) n_1^{s_1} (n_1 + n_2)^{s_2} \dots (\boldsymbol{n}(1, r-2))^{s_{r-2}} (\boldsymbol{n}(1, r-1))^{s_{r-1}+s_r},$$

and then summing with respect to $n_1, \dots, n_r$, we obtain

$$
\begin{aligned}
(3)\quad & \zeta_r(s_1, s_2, \dots s_r) \\
&= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1}+s_r+z)\zeta(-z)dz \\
&= I_r(c; s_1, s_2, \dots, s_r),
\end{aligned}
$$

say. Here $\zeta(-z)$ is the Riemann zeta-function. In the case $r=2$, Katsurada [8][9] obtained this formula in a somewhat more general form. His aim was to shift the path of integration and deduce asymptotic expansion formulas for certain mean values of Dirichlet $L-$functions and Lerch zeta-functions, but his argument also gives a new proof of the analytic continuation of $\zeta_2(s_1, s_2)$ (see Section 4 of [11]). Then Matsumoto [10] extended Katsurada's shifting argument to the case of general r.

Shift the path of integration in (3) to $\Re z = c_{r-1}$, where c_{r-1} is an arbitrary positive number. (This is the "right-shift" argument.) Counting the residues of relevant poles, we obtain

$$
\begin{aligned}
(4)\quad & \zeta_r(s_1, \dots, s_r) \\
&= \sum_{k_{r-1}=-1}^{[c_{r-1}]} \frac{B_{k_{r-1}+1}}{(k_{r-1}+1)!}\langle s_r\rangle_{k_{r-1}} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1}+s_r+k_{r-1}) \\
&\qquad + I_r(c_{r-1}; s_1, \dots, s_r),
\end{aligned}
$$

where B_k is the kth Bernoulli number and

$$
\langle s\rangle_k = \begin{cases} s(s+1)\dots(s+k-1) & \text{if} \quad k \geq 1, \\ 1 & \text{if} \quad k=0, \\ (s-1)^{-1} & \text{if} \quad k=-1. \end{cases}
$$

The series (1) is absolutely convergent in the region

$$
A(r) = \{(s_1, \dots, s_r) \in \mathbb{C}^r \mid \boldsymbol{\sigma}(j,r) > r-j+1 \; (1 \leq j \leq r)\}
$$

(Theorem 3 of [11]). Applying this fact to the factor ζ_{r-1}, we find easily that $I_r(c_{r-1}; s_1, \dots s_r)$ is holomorphic in the region

$$
D_r(c_{r-1}) = \{(s_1, \dots, s_r) \in \mathbb{C}^r \mid \boldsymbol{\sigma}(j,r) > r-j-c_{r-1} \; (1 \leq j \leq r)\}.
$$

If we already know that ζ_{r-1} can be continued to the whole space $\mathbb{C}^{r-1}$, then (4) implies that ζ_r can be continued to $D_r(c_{r-1})$. Since c_{r-1} is arbitrary, by induction on r, we conclude that $\zeta_r(s_1, \dots, s_r)$ can be continued meromorphically to the whole space $\mathbb{C}^r$, and any possible singularities are located on one of the following hyperplanes:

$$
\begin{aligned}
s_r &= 1, \\
s_{r-1}+s_r &= 2, 1, 0, -2, -4, -6, \dots, \\
\boldsymbol{s}(r-j+1, r) &= j-n \;\; (3 \leq j \leq r, \;\; n = 0, 1, 2, 3, \dots).
\end{aligned}
$$

We denote the union of these hyperplanes by $S(r)$. It is known that these are indeed singularities (Theorem 1 of Akiyama, Egami and Tanigawa [1]).

3. A general estimate

We first quote the following lemma.

LEMMA 1 (Matsumoto-Tanigawa [14], Lemma 2). *Let u, v, p, q, r be real numbers. Then*

$$\int_{-\infty}^{\infty}(1+|u+y|)^p(1+|v+y|)^q(1+|y|)^r\exp\Big(-\frac{\pi}{2}|u+y|-\frac{\pi}{2}|y|\Big)dy$$
$$=O\Big((1+U^p)(1+U^q+V^q)(1+U^{r+1})\exp\Big(-\frac{\pi}{2}|u|\Big)\Big),$$

where $U=1+|u|$, $V=1+|v|$, and the implied constant depends only on p, q and r.

In this section we estimate the right-hand side of (4) to obtain an upper bound of $|\zeta_r(s_1,\ldots,s_r)|$.

Assume $(s_1,\ldots,s_r)\in D_r(c_{r-1})\backslash S(r)$. By the functional equation of the Riemann zeta-function, we have

$$I_r(c_{r-1};s_1,\ldots s_r)=\frac{1}{2\pi i}\int_{(c_{r-1})}\frac{\Gamma(s_r+z)}{\Gamma(s_r)}\frac{\zeta(1+z)}{2(2\pi)^z\cos(\pi z/2)}\times\zeta_{r-1}(s_1,s_2,\ldots,s_{r-2},s_{r-1}+s_r+z)dz. \tag{5}$$

If $(s_1,\ldots,s_r)\in D_r(c_{r-1})$ and $\Re z=c_{r-1}$, then $(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z)\in A(r-1)$, so

$$\zeta_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z)=O(1)$$

on the right-hand side of (5). Hence, using Stirling's formula, we obtain

$$I_r(c_{r-1};s_1,\ldots s_r)$$
$$\ll\frac{1}{|\Gamma(s_r)|}\int_{-\infty}^{\infty}(1+|t_r+y|)^{\sigma_r+c_{r-1}-1/2}\exp\Big(-\frac{\pi}{2}|t_r+y|-\frac{\pi}{2}|y|\Big)dy.$$

Applying Lemma 1, we deduce

$$I_r(c_{r-1};s_1,\ldots s_r)\ll(1+(1+|t_r|)^{\sigma_r+c_{r-1}-1/2})(1+|t_r|)^{3/2-\sigma_r},$$

and hence

$$I_r(c_{r-1};s_1,\ldots s_r)\ll(1+|t_r|)^{c_{r-1}+1} \tag{6}$$

if $c_{r-1}>-\sigma_r+1/2$.

Next, we apply the "right-shift" argument of Section 2 to the factor $\zeta_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1}+s_r+k_{r-1})$ on the right-hand side of (4) to obtain

$$\begin{aligned}(7)\quad &\zeta_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+k_{r-1})\\ &= \sum_{k_{r-2}=-1}^{[c_{r-2}]} \frac{B_{k_{r-2}+1}}{(k_{r-2}+1)!}\langle s_{r-1}+s_r+k_{r-1}\rangle_{k_{r-2}}\\ &\qquad \times \zeta_{r-2}(s_1,\ldots,s_{r-3},s_{r-2}+s_{r-1}+s_r+k_{r-1}+k_{r-2})\\ &\quad + I_{r-1}(c_{r-2};s_1,\ldots,s_{r-2},s_{r-1}+s_r+k_{r-1}),\end{aligned}$$

where c_{r-2} is an arbitrary positive number. We see that $I_{r-1}(c_{r-2}; s_1, \ldots, s_{r-2}, s_{r-1}+s_r+k_{r-1})$ is holomorphic in the region

$$(8)\quad \{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \boldsymbol{\sigma}(j,r)+k_{r-1} > (r-1)-j-c_{r-2} \quad (1\le j\le r-1)\}.$$

If we choose

$$(9)\qquad c_{r-2}=c_{r-2}(k_{r-1})\ge c_{r-1}-k_{r-1}-1,$$

then the region (8) contains $D_r(c_{r-1})$. Hence $I_{r-1}(c_{r-2}; s_1, \ldots, s_{r-2}, s_{r-1}+s_r+k_{r-1})$ is holomorphic for $(s_1,\ldots,s_r)\in D_r(c_{r-1})$, and similarly to (6) we obtain

$$(10)\qquad I_{r-1}(c_{r-2};s_1,\ldots,s_{r-2},s_{r-1}+s_r+k_{r-1}) \ll (1+|t_{r-1}+t_r|)^{c_{r-2}+1}$$

if $c_{r-2} > -(\sigma_{r-1}+\sigma_r+k_{r-1})+1/2$.

Repeating this process, we obtain

$$\begin{aligned}(11)\quad &\zeta_r(s_1,\ldots,s_r)\\ &= \sum_{k_{r-1}=-1}^{[c_{r-1}]} \frac{B_{k_{r-1}+1}}{(k_{r-1}+1)!}\langle s_r\rangle_{k_{r-1}}\\ &\quad \times \left\{ \sum_{k_{r-2}=-1}^{[c_{r-2}]} \frac{B_{k_{r-2}+1}}{(k_{r-2}+1)!}\langle s_{r-1}+s_r+k_{r-1}\rangle_{k_{r-2}} \left\{ \ldots \right.\right.\\ &\quad \ldots \left\{ \sum_{k_2=-1}^{[c_2]} \frac{B_{k_2+1}}{(k_2+1)!}\langle \boldsymbol{s}(3,r)+\boldsymbol{k}(3,r-1)\rangle_{k_2} \left\{ \sum_{k_1=-1}^{[c_1]} \frac{B_{k_1+1}}{(k_1+1)!} \right.\right.\\ &\quad \left. \times \langle \boldsymbol{s}(2,r)+\boldsymbol{k}(2,r-1)\rangle_{k_1} \zeta(\boldsymbol{s}(1,r)+\boldsymbol{k}(1,r-1)) + I_2(c_1) \right\}\\ &\quad \left.\left.\left. + I_3(c_2) \right\} \ldots \right\} + I_{r-1}(c_{r-2}) \right\} + I_r(c_{r-1}),\end{aligned}$$

where $c_1, \ldots, c_{r-1}$ are positive numbers and

$$I_m(c_{m-1}) = I_m(c_{m-1}; s_1,\ldots,s_{m-1},\boldsymbol{s}(m,r)+\boldsymbol{k}(m,r-1))$$

(for $2 \le m \le r$; the empty sum is interpreted as zero), which is holomorphic in the region

$$\Big\{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \boldsymbol{\sigma}(j,r)+\boldsymbol{k}(m,r-1)>m-j-c_{m-1}\ (1\le j\le m)\Big\}.$$

This region contains $D_r(c_{r-1})$ if

$$c_{m-1}=c_{m-1}(k_m,\ldots,k_{r-1})\ge c_{r-1}-\boldsymbol{k}(m,r-1)-(r-m). \tag{12}$$

Under this condition, we obtain

$$I_m(c_{m-1})\ll(1+|\boldsymbol{t}(m,r)|)^{c_{m-1}+1} \tag{13}$$

if

$$c_{m-1}>-(\boldsymbol{\sigma}(m,r)+\boldsymbol{k}(m,r-1))+1/2. \tag{14}$$

Let $\theta(\sigma)$ be the infimum of the numbers α satisfying

$$\zeta(\sigma+it)=O\Big((1+|t|)^{\alpha}\Big).$$

It is known that $\theta(\sigma)=\frac{1}{2}-\sigma$ when $\sigma\le 0$; for the best known bounds for $\theta(\sigma)$ for $0<\sigma<1$, see Huxley [5]. From (11) and (13) we obtain

$$\begin{aligned}\zeta_r(s_1,\ldots,s_r)\ll&\sum_{k_{r-1}=-1}^{[c_{r-1}]}(1+|t_r|)^{k_{r-1}}\\ &\times\Bigg\{\sum_{k_{r-2}=-1}^{[c_{r-2}]}(1+|t_{r-1}+t_r|)^{k_{r-2}}\Bigg\{\ldots\Bigg\{\sum_{k_2=-1}^{[c_2]}(1+|\boldsymbol{t}(3,r)|)^{k_2}\\ &\times\Bigg\{\sum_{k_1=-1}^{[c_1]}(1+|\boldsymbol{t}(2,r)|)^{k_1}(1+|\boldsymbol{t}(1,r)|)^{\theta(\boldsymbol{\sigma}(1,r)+\boldsymbol{k}(1,r-1))}\\ &+(1+|\boldsymbol{t}(2,r)|)^{c_1+1}\Bigg\}+(1+|\boldsymbol{t}(3,r)|)^{c_2+1}\Bigg\}\ldots\\ &\ldots\Bigg\}+(1+|t_{r-1}+t_r|)^{c_{r-2}+1}\Bigg\}+(1+|t_r|)^{c_{r-1}+1}\end{aligned} \tag{15}$$

for $(s_1,\ldots,s_r)\in D_r(c_{r-1})\backslash S(r)$, if conditions (12) and (14) are satisfied for $2\le m\le r$. Therefore we now arrive at the following result.

Theorem 1. *Let $r\ge 2$ and $c_{r-1}>0$. Choose positive numbers $c_{r-2}=c_{r-2}(k_{r-1})$, $c_{r-3}=c_{r-3}(k_{r-2},k_{r-1})$, $\ldots$, $c_1=c_1(k_2,\ldots,k_{r-1})$ satisfying (12), where $k_2,\ldots,k_{r-1}$ are integers with $-1\le k_m\le[c_m]$ $(2\le m\le r-1)$.*

Then we have

$$\begin{aligned}\zeta_r(s_1,s_2,\ldots,s_r) &\ll (1+|t_r|)^{c_{r-1}+1}\\ &+\sum_{j=2}^{r-1}\max_{\substack{-1\le k_{r-1}\le[c_{r-1}]\\ -1\le k_{r-2}\le[c_{r-2}]\\ \cdots\\ -1\le k_j\le[c_j]}}(1+|t_r|)^{k_{r-1}}(1+|t_{r-1}+t_r|)^{k_{r-2}}\times\cdots\\ &\qquad\cdots\times(1+|\boldsymbol{t}(j+1,r)|)^{k_j}(1+|\boldsymbol{t}(j,r)|)^{c_{j-1}+1}\\ &+\max_{\substack{-1\le k_{r-1}\le[c_{r-1}]\\ -1\le k_{r-2}\le[c_{r-2}]\\ \cdots\\ -1\le k_1\le[c_1]}}(1+|t_r|)^{k_{r-1}}(1+|t_{r-1}+t_r|)^{k_{r-2}}\times\cdots\\ &\qquad\cdots\times(1+|\boldsymbol{t}(2,r)|)^{k_1}(1+|\boldsymbol{t}(1,r)|)^{\theta(\boldsymbol{\sigma}(1,r)+\boldsymbol{k}(1,r-1))}\end{aligned}$$

for any $(s_1,\ldots,s_r)\in D_r(c_{r-1})\backslash S(r)$ *which further satisfies* (14) *for* $2\le m\le r$.

4. The case of the double zeta-function

In the case of the double zeta-function, Theorem 1 implies

$$\zeta_2(s_1,s_2)\ll(1+|t_2|)^{c_1+1}+\max_{-1\le k_1\le[c_1]}(1+|t_2|)^{k_1}(1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2+k_1)} \tag{16}$$

for $(s_1,s_2)\in D_2(c_1)\backslash S(2)$, under the additional condition $c_1>-\sigma_2+1/2$. But this estimate is by no means best-possible. For instance, consider the case $s_1=it$, $s_2=i\alpha t$, where $t>0$ and α is a real constant. Then $(s_1,s_2)\in D_2(c_1)$ if $c_1>1$. Taking $c_1=1+\epsilon$, we obtain from (16)

$$\zeta_2(it,i\alpha t)\ll(1+|t|)^{2+\epsilon}. \tag{17}$$

However, in the case $\alpha=1$, from the obvious relation

$$\zeta(it)\zeta(it)=\zeta(2it)+2\zeta_2(it,it), \tag{18}$$

we immediately obtain

$$\zeta_2(it,it)\ll(1+|t|)^{1+\epsilon}, \tag{19}$$

which is much better than (17). This is the consequence of the fortunate relation (18), but actually we can improve (17) without using this relation. The purpose of this section is to prove such an improvement (Theorem 2 below), which implies the following result.

Corollary. *For any fixed real* α $(\neq -1)$, *we have*

$$\zeta_2(it,i\alpha t)\ll(1+|t|)^{3/2+\epsilon}. \tag{20}$$

We begin by applying formula (4) with $r=2$, $0<c_1<1$. Then

$$\zeta_2(s_1,s_2)=\frac{\zeta(s_1+s_2-1)}{s_2-1}+\zeta(s_1+s_2)+I_2(c_1;s_1,s_2) \tag{21}$$

in the region $D_2(c_1)$. Let η be a small positive number satisfying $0 < \eta < c_1$, and let

$$D_2^*(\eta) = \left\{ (s_1, s_2) \in \mathbb{C}^2 \,\middle|\, \begin{array}{ccc} \sigma_2 & > & -\eta \\ \sigma_1 + \sigma_2 & < & 1 - \eta \end{array} \right\}.$$

Then $D_2(c_1) \cap D_2^*(\eta) \neq \emptyset$. Fix an element (s_1, s_2) of this intersection, and shift the path of integration of $I_2(c_1; s_1, s_2)$ to $\Re z = \eta$. (This is the "left-shift" argument.) The only relevant pole is $z = 1 - s_1 - s_2$. Hence we have

$$\begin{aligned}(22)\quad \zeta_2(s_1, s_2) = {} & \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} + \zeta(s_1 + s_2) \\ & + \frac{\Gamma(1 - s_1)}{\Gamma(s_2)} \Gamma(s_1 + s_2 - 1) \zeta(s_1 + s_2 - 1) + I_2(\eta; s_1, s_2),\end{aligned}$$

and $I_2(\eta; s_1, s_2)$ is holomorphic in $D_2^*(\eta)$. This gives the meromorphic continuation of $\zeta_2(s_1, s_2)$ to $D_2^*(\eta)$.

Next, we assume that (s_1, s_2) is an arbitrary element of $D_2^*(\eta)$, and estimate $\zeta_2(s_1, s_2)$ by using (22). By Stirling's formula we have

$$(23)\quad \frac{\Gamma(1 - s_1)}{\Gamma(s_2)} \Gamma(s_1 + s_2 - 1) \zeta(s_1 + s_2 - 1) \ll (1 + |t_1|)^{1/2 - \sigma_1} (1 + |t_2|)^{1/2 - \sigma_2}$$

and

$$\begin{aligned}(24)\qquad I_2(\eta; s_1, s_2) \ll {} & e^{\frac{\pi}{2}|t_2|} (1 + |t_2|)^{1/2 - \sigma_2} \int_{-\infty}^{\infty} e^{-\frac{\pi}{2}|t_2 + y| - \frac{\pi}{2}|y|} \\ & \times (1 + |t_2 + y|)^{\sigma_2 + \eta - 1/2} (1 + |t_1 + t_2 + y|)^{\theta(\sigma_1 + \sigma_2 + \eta)} dy.\end{aligned}$$

Therefore our problem is reduced to the evaluation of the integral

$$(25)\qquad J = \int_{-\infty}^{\infty} (1 + |y + u|)^p (1 + |y + v|)^q \exp\Big(-\frac{\pi}{2}|y + u| - \frac{\pi}{2}|y| \Big) dy,$$

where u, v, p, q are real numbers and $p > -1$, $q \geq 0$. We could estimate this integral by means of Lemma 1, but in the following lemma we give more refined estimates.

LEMMA 2. *Assume $p > -1$ and $q \geq 0$. Then the integral J can be estimated as follows.*

(i) *When $uv \leq 0$, then*

$$(26)\qquad J \ll e^{-\frac{\pi}{2}|u|} (1 + |u|)^{p+1} (1 + |u - v|)^q.$$

(ii) *When $uv > 0$ and $|u| < |v|$, then*

$$(27)\qquad J \ll e^{-\frac{\pi}{2}|u|} (1 + |u|)^{p+1} (1 + |v|)^q.$$

(iii) *When $uv > 0$ and $|u| \geq |v|$, then*

$$J \ll e^{-\frac{\pi}{2}|u|}\Big\{\max\big\{(1+|u|)^p, (1+|u-v|)^p\big\}(1+|v|)^{q+1} + (1+|u-v|)^{p+q+1}\Big\}. \tag{28}$$

Remark 1. The following proof can also be applied to the case $p \leq -1$. In this case, the conclusion is as follows. In (26) and (27), the factor $(1+|u|)^{p+1}$ is to be replaced by $\log(1+|u|)$ (if $p=-1$) or 1 (if $p<-1$). In (28), the factor $(1+|u-v|)^{p+q+1}$ is to be replaced by $(1+|u-v|)^q \log(1+|u-v|)$ (if $p=-1$) or $(1+|u-v|)^q$ (if $p<-1$).

To prove Lemma 2, we may assume $u \geq 0$ without loss of generality, because the results in the case $u < 0$ can be deduced from the case $u \geq 0$ by replacing, in (25), u, v, y by $-u$, $-v$, $-y$, respectively.

First consider the case (i), that is, $u \geq 0$ and $v \leq 0$. We divide the integral J into four parts,

$$J = \int_{-\infty}^{-u} + \int_{-u}^{0} + \int_{0}^{-v} + \int_{-v}^{\infty} = J_1 + J_2 + J_3 + J_4,$$

say. We put $-y-u=\tau$ in J_1 to obtain

$$\begin{aligned} J_1 &= e^{-\frac{\pi}{2}u}\int_0^\infty e^{-\pi\tau}(1+\tau)^p(1+\tau+u-v)^q d\tau \\ &\ll e^{-\frac{\pi}{2}u}\left\{\int_0^{u-v} e^{-\pi\tau}(1+\tau)^p(1+u-v)^q d\tau + \int_{u-v}^\infty e^{-\pi\tau}(1+\tau)^{p+q}d\tau\right\} \\ &\ll e^{-\frac{\pi}{2}u}(1+u-v)^q. \end{aligned}$$

As for J_2, we put $y+u=\tau$ to obtain

$$\begin{aligned} J_2 &= e^{-\frac{\pi}{2}u}\int_0^u (1+\tau)^p(1-\tau+u-v)^q d\tau \\ &\leq e^{-\frac{\pi}{2}u}(1+u-v)^q\int_0^u (1+\tau)^p d\tau \\ &\ll e^{-\frac{\pi}{2}u}(1+u)^{p+1}(1+u-v)^q. \end{aligned}$$

The integral J_3 can be treated similarly to J_2 and we get

$$J_3 = O(e^{-\frac{\pi}{2}u}(1+u)^p(1+u-v)^q).$$

As for J_4, we put $y+v=\tau$ and proceed similarly to the case of J_1 to obtain $J_4 = O(e^{-\frac{\pi}{2}u+\pi v}(1+u-v)^p)$, from which we deduce

$$J_4 \ll e^{-\frac{\pi}{2}u}(1+u)^p. \tag{29}$$

In fact, if $0 \leq -v < u$ we simply use $e^{\pi v} \leq 1$ and $(1+u-v)^p \ll (1+u)^p$. If $-v \geq u$, then

$$e^{-\frac{\pi}{2}u+\pi v} \leq e^{-\pi u-\frac{\pi}{4}(u-v)}$$

and $e^{-\frac{\pi}{4}(u-v)}(1+u-v)^p = O(1)$, and hence (29) follows. Collecting the above results, we obtain (26).

The proof of (27) in the case $v > u > 0$ is similar, by dividing the integral J into the parts

$$J = \int_{-\infty}^{-v} + \int_{-v}^{-u} + \int_{-u}^{0} + \int_{0}^{\infty} = J_1' + J_2' + J_3' + J_4',$$

say. We omit the details and only note that the integral J_4' is treated by splitting the integral further at $y = u$ and $y = v$ and estimating each part separately.

In the case $u \geq v > 0$, we divide J into

$$J = \int_{-\infty}^{-u} + \int_{-u}^{-v} + \int_{-v}^{0} + \int_{0}^{\infty} = J_1'' + J_2'' + J_3'' + J_4'',$$

say. The treatment of J_1'' is exactly the same as that of J_1. Next, we put $-y-v=\tau$ in J_2'' to obtain

$$\begin{aligned} J_2'' &= e^{-\frac{\pi}{2}u}\int_0^{u-v}(1-\tau+u-v)^p(1+\tau)^q d\tau \\ &\leq e^{-\frac{\pi}{2}u}\left\{\int_0^{(u-v)/2}(1+u-v)^p(1+\tau)^q d\tau \right. \\ &\qquad \left. + \int_{(u-v)/2}^{u-v}(1-\tau+u-v)^p(1+u-v)^q d\tau\right\} \\ &\ll e^{-\frac{\pi}{2}u}(1+u-v)^{p+q+1}. \end{aligned}$$

As for J_3'', we put $y+u=\tau$ to obtain

$$\begin{aligned} J_3'' &\ll e^{-\frac{\pi}{2}u}\int_{u-v}^{u}(1+\tau)^p(1+\tau-u+v)^q d\tau \\ &\ll e^{-\frac{\pi}{2}u}\max\Big\{(1+u)^p,\ (1+u-v)^p\Big\}(1+v)^{q+1}. \end{aligned}$$

The integral J_4'' can be treated similarly to J_4' and we obtain

$$J_4'' = O(e^{-\frac{\pi}{2}u}(1+u)^p(1+v)^q).$$

The estimate (28) now follows, and the proof of Lemma 2 is complete.

We estimate $I_2(\eta; s_1, s_2)$ by applying Lemma 2 to the right-hand side of (24). Then, combining with (22) and (23), we obtain the following result.

Theorem 2. *Let $0 < \eta < 1$. If $(s_1, s_2) \in D_2^*(\eta)$, then we have*

$$\begin{aligned} (30)\quad \zeta_2(s_1,s_2) &\\ &\ll (1+|t_2|)^{-1}(1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2-1)} + (1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2)} \\ &\qquad + (1+|t_1|)^{1/2-\sigma_1}(1+|t_2|)^{1/2-\sigma_2} + |I_2(\eta; s_1, s_2)|, \end{aligned}$$

and $I_2(\eta; s_1, s_2)$ *is bounded by*

$$\ll (1+|t_2|)^{\eta+1}(1+|t_1|)^{\theta(\sigma_1+\sigma_2+\eta)} \tag{31}$$

if $t_2(t_1+t_2) \le 0$, *by*

$$\ll (1+|t_2|)^{\eta+1}(1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2+\eta)} \tag{32}$$

if $t_2(t_1+t_2) > 0$ *and* $|t_2| < |t_1+t_2|$, *and by*

$$\begin{aligned} &\ll (1+|t_2|)^{1/2-\sigma_2}\Bigg(\max\Big\{(1+|t_1|)^{\sigma_2+\eta-1/2}, (1+|t_2|)^{\sigma_2+\eta-1/2}\Big\} \\ &\qquad \times (1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2+\eta)+1} + (1+|t_1|)^{\sigma_2+\eta+1/2+\theta(\sigma_1+\sigma_2+\eta)}\Bigg) \end{aligned} \tag{33}$$

if $t_2(t_1+t_2) > 0$ *and* $|t_2| \ge |t_1+t_2|$.

This theorem, applied with $s_1 = it$, $s_2 = i\alpha t$ and $\eta = \epsilon$, implies the corollary stated above, and therefore refines Theorem 1 in the case $r = 2$.

5. The case of the triple zeta-function

In this section we illustrate how to refine Theorem 1 in the case $r = 3$. Since the argument is much more complicated than in the case $r = 2$ presented in the preceding section, we restrict our consideration to a typical example, namely $(s_1, s_2, s_3) = (-it, it, it)$, where t is a non-zero real number.

If we put $c_2 = 2 + \epsilon$ and $c_1 = c_1(k_2) = 1 - k_2 + \epsilon$, then $(-it, it, it) \in D_3(c_2)\backslash S_3$ and we can apply Theorem 1. The result is

$$\zeta_3(-it, it, it) = O\Big((1+|t|)^{3+\epsilon}\Big). \tag{34}$$

The purpose of this section is to prove the following improvement of (34):

Theorem 3. *We have*

$$\zeta_3(-it, it, it) = O\Big((1+|t|)^{5/2+\epsilon}\Big)$$

for any $t \ne 0$.

In order to prove this theorem, we again use the "left-shift" argument, but this time we will need to shift the path to the left twice.

Our starting point is formula (4) with $r = 3$ and $0 < c_2 < 1$, that is,

$$\begin{aligned} &\zeta_3(s_1, s_2, s_3) \\ &\qquad = \frac{\zeta_2(s_1, s_2+s_3-1)}{s_3-1} - \frac{1}{2}\zeta_2(s_1, s_2+s_3) + I_3(c_2; s_1, s_2, s_3), \end{aligned}$$

which is valid in $D_3(c_2)$. Let $0<\mu<c_2$ and

$$D_3^*(\mu)=\left\{(s_1,s_2,s_3)\in\mathbb{C}^3 \;\middle|\; \begin{array}{ccccc} -\mu & < & \sigma_3 & & \\ & & \sigma_2+\sigma_3 & < & 1-\mu \\ 1-\mu & < & \sigma_1+\sigma_2+\sigma_3 & < & 2-\mu \end{array}\right\}.$$

We fix a point $(s_1,s_2,s_3)\in D_3(c_2)\cap D_3^*(\mu)$, and shift the path of $I_3(c_2;s_1,s_2,s_3)$ to $\Re z=\mu$. The definition of $S(2)$ implies that the poles of $\zeta_2(s_1,s_2+s_3+z)$ as a function in z are $z=1-s_2-s_3$ and $z=-s_1-s_2-s_3+n$ ($n=2,1,0,-2,-4,-6,\dots$). Two of them ($z=1-s_1-s_2-s_3$, $z=2-s_1-s_2-s_3$) are located in the strip $\mu<\Re z<c_2$. We may assume these two poles are not at the same point, because we may choose our fixed point with the condition $s_1\neq 1$. The residues of $\zeta_2(s_1,s_2+s_3+z)$ at $z=1-s_2-s_3$ and $z=2-s_1-s_2-s_3$ are $\zeta(s_1)$ and $(1-s_1)^{-1}$, respectively. (These can be calculated by using the expression (21).) These two poles are the only poles of the integrand of $I_3(c_2;s_1,s_2,s_3)$ whose residues we need to take into account, and we therefore obtain

$$\begin{aligned}(35)\quad &\zeta_3(s_1,s_2,s_3)\\ &=\frac{\zeta_2(s_1,s_2+s_3-1)}{s_3-1}-\frac{1}{2}\zeta_2(s_1,s_2+s_3)\\ &\quad+\frac{\Gamma(1-s_2)}{\Gamma(s_3)}\Gamma(s_2+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1)\\ &\quad+\frac{\Gamma(2-s_1-s_2)}{\Gamma(s_3)}\Gamma(s_1+s_2+s_3-2)\zeta(s_1+s_2+s_3-2)\frac{1}{1-s_1}\\ &\quad+I_3(\mu;s_1,s_2,s_3),\end{aligned}$$

which gives the continuation of $\zeta_3(s_1,s_2,s_3)$ to $D_3^*(\mu)$.

Next, let λ be a number satisfying $0<\lambda<\mu$, and define

$$D_3^{**}(\lambda)=\left\{(s_1,s_2,s_3)\in\mathbb{C}^3 \;\middle|\; \begin{array}{ccccc} -\lambda & < & \sigma_3 & & \\ & & \sigma_2+\sigma_3 & < & 1-\lambda \\ -\lambda & < & \sigma_1+\sigma_2+\sigma_3 & < & 1-\lambda \end{array}\right\}.$$

Now we fix a point $(s_1,s_2,s_3)\in D_3^*(\mu)\cap D_3^{**}(\lambda)$, and shift the path of $I_3(\mu;s_1,s_2,s_3)$ to $\Re z=\lambda$. This time the only relevant pole is $z=1-s_1-s_2-s_3$. Hence we obtain

$$\begin{aligned}(36)\quad &\zeta_3(s_1,s_2,s_3)\\ &=\frac{\zeta_2(s_1,s_2+s_3-1)}{s_3-1}-\frac{1}{2}\zeta_2(s_1,s_2+s_3)\\ &\quad+\frac{\Gamma(1-s_2)}{\Gamma(s_3)}\Gamma(s_2+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1)\\ &\quad+\frac{\Gamma(2-s_1-s_2)}{\Gamma(s_3)}\Gamma(s_1+s_2+s_3-2)\zeta(s_1+s_2+s_3-2)\frac{1}{1-s_1}\end{aligned}$$

$$
\begin{aligned}
&-\frac{\Gamma(1-s_1-s_2)}{2\Gamma(s_3)}\Gamma(s_1+s_2+s_3-1)\zeta(s_1+s_2+s_3-1)\\
&+I_3(\lambda;s_1,s_2,s_3)\\
=\ &A_1-A_2+A_3+A_4-A_5+I_3(\lambda;s_1,s_2,s_3),
\end{aligned}
$$

say. Since $I_3(\lambda;s_1,s_2,s_3)$ is holomorophic in $D_3^{**}(\lambda)$, the formula (36) gives the continuation of $\zeta_3(s_1,s_2,s_3)$ to $D_3^{**}(\lambda)$.

Since $(-it,it,it)\in D_3^{**}(\lambda)$, we can evaluate the order of $\zeta_3(-it,it,it)$ by using (36). We may assume $t>0$.

First we estimate A_j $(1\le j\le 5)$ at the point $(s_1,s_2,s_3)=(-it,it,it)$. The corollary in Section 4 implies $A_2=O((1+t)^{3/2+\epsilon})$. To estimate A_1, we use (4) with $r=2$, $c_1=2+\epsilon$. We have

$$\zeta_2(-it,2it-1)=\frac{\zeta(it-2)}{2it-2}-\frac{1}{2}\zeta(it-1)+\frac{1}{12}(2it-1)\zeta(it)+I_2(c_1;-it,2it-1),$$

because $B_3=0$. From Lemma 2 (iii) (or from (6)) we have

$$I_2(c_1;-it,2it-1)\ll(1+t)^{3+\epsilon}.$$

Hence we obtain $A_1=O((1+t)^{2+\epsilon})$. By using Stirling's formula it is easy to see that $A_4\ll(1+t)^{-1/2}$, $A_5\ll(1+t)^{1/2}$, and that A_3 is of exponential decay. Therefore we obtain

$$\zeta_3(-it,it,it)\ll(1+t)^{2+\epsilon}+|I_3(\lambda;-it,it,it)|. \tag{37}$$

Our remaining task is to estimate the integral $I_3(\lambda;-it,it,it)$. Again by Stirling's formula we have

$$
\begin{aligned}
I_3(\lambda;-it,it,it)\ll e^{\frac{1}{2}\pi t}(1+t)^{1/2}&\int_{-\infty}^{\infty}\exp\Big(-\frac{\pi}{2}|y|-\frac{\pi}{2}|y-t|\Big)\\
&\times(1+|y-t|)^{\lambda-1/2}|\zeta_2(-it,\lambda+i(2t-y))|dy.
\end{aligned} \tag{38}
$$

From (22) with $\eta=\epsilon$, using (23) and the fact that $\theta(\sigma)=1/2-(2/3)\sigma$ for $0\le\sigma\le 1/2$, we obtain

$$
\begin{aligned}
&\zeta_2(-it,\lambda+i(2t-y))\\
&\quad\ll(1+|t-y|)^{3/2-\lambda}(1+|2t-y|)^{-1}+(1+|t-y|)^{1/2-(2/3)\lambda}\\
&\qquad+(1+t)^{1/2}(1+|2t-y|)^{1/2-\lambda}+|I_2(\epsilon;-it,\lambda+i(2t-y))|\\
&\quad=h_1(t,y)+h_2(t,y)+h_3(t,y)+h_4(t,y),
\end{aligned}
$$

say. Substituting this estimate into the right-hand side of (38), we obtain

$$I_3(\lambda;-it,it,it)\ll\sum_{j=1}^{4}H_j, \tag{39}$$

where

$$H_j = e^{\frac{\pi}{2}t}(1+t)^{1/2}\int_{-\infty}^{\infty}\exp\Big(-\frac{\pi}{2}|y|-\frac{\pi}{2}|y-t|\Big) \times (1+|y-t|)^{\lambda-1/2}h_j(t,y)dy$$

$(1\le j\le 4)$. We apply Lemma 1 to H_1, H_2 and H_3 to obtain

$$H_1 \ll (1+t)^{5/2}, \quad H_2 \ll (1+t)^{3/2+(1/3)\lambda}, \quad H_3 \ll (1+t)^{5/2-\lambda}. \tag{40}$$

Concerning H_4, we first estimate $h_4(t,y)$ by (31)–(33) of Theorem 2. The results are

$$h_4(t,y) \ll (1+|2t-y|)^{1+\epsilon}(1+|t-y|)^{1/2-(2/3)(\lambda+\epsilon)}$$

if $y>2t$,

$$h_4(t,y) \ll (1+|2t-y|)^{1+\epsilon}(1+t)^{1/2-(2/3)(\lambda+\epsilon)}$$

if $t\le y\le 2t$, and

$$h_4(t,y) \ll (1+|2t-y|)^{1/2-\lambda}\Big\{\max\Big((1+t)^{\lambda+\epsilon-1/2}, \quad (1+|2t-y|)^{\lambda+\epsilon-1/2}\Big) \times (1+|t-y|)^{3/2-(2/3)(\lambda+\epsilon)} + (1+t)^{1+(1/3)(\lambda+\epsilon)}\Big\}$$

if $y<t$. Therefore

$$H_4 \ll e^{\frac{\pi}{2}t}(1+t)^{1/2}(J_1+J_2+J_3+J_4),$$

where

$$J_1 = \int_{2t}^{\infty}\exp\Big(-\frac{\pi}{2}|y|-\frac{\pi}{2}|y-t|\Big)(1+|y-t|)^{\lambda-(2/3)(\lambda+\epsilon)}(1+|y-2t|)^{1+\epsilon}dy,$$

$$J_2 = (1+t)^{1/2-(2/3)(\lambda+\epsilon)}\int_{t}^{2t}\exp\Big(-\frac{\pi}{2}|y|-\frac{\pi}{2}|y-t|\Big) \times (1+|y-t|)^{\lambda-1/2}(1+|y-2t|)^{1+\epsilon}dy,$$

$$J_3 = \int_{-\infty}^{t}\exp\Big(-\frac{\pi}{2}|y|-\frac{\pi}{2}|y-t|\Big)(1+|y-t|)^{\lambda+1-(2/3)(\lambda+\epsilon)} \times (1+|y-2t|)^{1/2-\lambda}\max\Big\{(1+t)^{\lambda+\epsilon-1/2}, \quad (1+|y-2t|)^{\lambda+\epsilon-1/2}\Big\}dy,$$

$$J_4 = (1+t)^{1+(1/3)(\lambda+\epsilon)}\int_{-\infty}^{t}\exp\Big(-\frac{\pi}{2}|y|-\frac{\pi}{2}|y-t|\Big) \times (1+|y-t|)^{\lambda-1/2}(1+|y-2t|)^{1/2-\lambda}dy.$$

We now choose $\lambda=2\epsilon$. Then $\lambda-(2/3)(\lambda+\epsilon)=0$, and applying Lemma 1 we obtain

$$J_1 \ll (1+t)^{2+\epsilon}e^{-\frac{\pi}{2}t}. \tag{41}$$

If $y < t$ then $t < |y - 2t|$, and hence

$$\max\Big\{(1+t)^{\lambda+\epsilon-1/2}, \quad (1+|y-2t|)^{\lambda+\epsilon-1/2}\Big\} = (1+t)^{\lambda+\epsilon-1/2}.$$

Hence, applying Lemma 1 again, we obtain

$$J_3 \ll (1+t)^{2+\epsilon} e^{-\frac{\pi}{2}t}. \tag{42}$$

If we apply Lemma 1 to J_2 and J_4, we only obtain the estimate $O((1+t)^{3+\epsilon})$, which does not improve on (34). Therefore we need to estimate J_2 and J_4 more carefully.

Putting $y - t = y'$, we have

$$\begin{aligned} J_2 &= (1+t)^{1/2-2\epsilon} e^{-\frac{\pi}{2}t} \int_0^t e^{-\pi y'} (1+y')^{\lambda-1/2} (1+t-y')^{1+\epsilon} dy' \\ &\le (1+t)^{1/2-2\epsilon} e^{-\frac{\pi}{2}t} \int_0^t e^{-\pi y'} (1+t)^{1+\epsilon} dy' \\ &\ll (1+t)^{3/2-\epsilon} e^{-\frac{\pi}{2}t}. \end{aligned} \tag{43}$$

Next, we divide J_4 into two parts,

$$J_4 = (1+t)^{1+\epsilon} \left(\int_{-\infty}^0 + \int_0^t \right) = (1+t)^{1+\epsilon} (J_{41} + J_{42}),$$

say. Then

$$\begin{aligned} J_{41} &= e^{-\frac{\pi}{2}t} \int_0^\infty e^{-\pi y} (1+t+y)^{\lambda-1/2} (1+2t+y)^{1/2-\lambda} dy \\ &\ll e^{-\frac{\pi}{2}t} \Big\{ \int_0^{2t} e^{-\pi y} (1+t)^{\lambda-1/2} (1+t)^{1/2-\lambda} dy \\ &\qquad + \int_{2t}^\infty e^{-\pi y} (1+y)^{\lambda-1/2} (1+y)^{1/2-\lambda} dy \Big\} \\ &\ll e^{-\frac{\pi}{2}t}. \end{aligned}$$

Also, putting $t - y = y'$, we have

$$\begin{aligned} J_{42} &= e^{-\frac{\pi}{2}t} \int_0^t (1+y')^{\lambda-1/2} (1+t+y')^{1/2-\lambda} dy' \\ &\ll e^{-\frac{\pi}{2}t} (1+t)^{1/2-\lambda} \int_0^t (1+y')^{\lambda-1/2} dy' \\ &\ll e^{-\frac{\pi}{2}t} (1+t). \end{aligned}$$

Hence we obtain

$$J_4 \ll (1+t)^{2+\epsilon} e^{-\frac{\pi}{2}t}. \tag{44}$$

Collecting (41)–(44) we obtain

$$H_4 \ll (1+t)^{5/2+\epsilon}.$$

Combining this with (39) and (40) we now arrive at the assertion of Theorem 3.

Remark 2. The estimate of Lemma 1 is somewhat crude, and we can improve some of the above estimates, which were obtained by using Lemma 1, by proceeding more carefully. However, the most crucial estimates are those of J_3 and J_{42}, and these cannot be improved.

References

[1] S. Akiyama, S. Egami, and Y. Tanigawa, *Analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith. **98** (2001), 107–116. MR **2002c:**11113

[2] S. Akiyama and H. Ishikawa, *On analytic continuation of multiple L-functions and related zeta-functions*, Analytic number theory (Beijing/Kyoto, 1999), Dev. Math., vol.6, Kluwer Acad. Publ., Dordrecht, 2002, pp. 1–16. MR **2003b:**11093

[3] T. Arakawa and M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions*, Nagoya Math. J. **153** (1999), 189–209. MR **2000e:**11113

[4] F. V. Atkinson, *The mean-value of the Riemann zeta function*, Acta Math. **81** (1949), 353–376. MR 11,234d

[5] M. N. Huxley, *Exponential sums and the Riemann zeta function. IV*, Proc. London Math. Soc. (3) **66** (1993), 1–40. MR **93j:**11056

[6] H. Ishikawa, *On analytic properties of a multiple L-function*, Analytic extension formulas and their applications (Fukuoka, 1999/Kyoto, 2000), Soc. Anal. Appl. Comput., vol. 9, Kluwer Acad. Publ., Dordrecht, 2001, pp. 105–122. MR **2002b:**11119

[7] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons Inc., New York, 1985. MR **87d:**11062

[8] M. Katsurada, *An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions*, Collect. Math. **48** (1997), 137–153. MR **98d:**11102

[9] ———, *An application of Mellin-Barnes type of integrals to the mean square of L-functions*, Liet. Mat. Rink. **38** (1998), 98–112. MR **99k:**11130

[10] K. Matsumoto, *Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series*, Nagoya Math. J. **172** (2003), 59–102.

[11] ———, *On the analytic continuation of various multiple zeta-functions*, Number theory for the millennium II (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 417–440. MR 1 956 262

[12] ———, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions. I*, J. Number Theory **101** (2003), 223–243. MR 1 989 886

[13] ———, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions. II*, Analytic and probabilistic methods in number theory (Palanga, 2001), TEV, Vilnius, 2002, pp. 188–194. MR **2003m:**11148

[14] K. Matsumoto and Y. Tanigawa, *The analytic continuation and the order estimate of multiple Dirichlet series*, J. Théor. Nombres Bordeaux **15** (2003), 267–274.

[15] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Soc. **128** (2000), 1275–1283. MR **2000j:**11132

Hideaki Ishikawa, Graduate School of Science and Technology, Niigata University, Niigata, 950-2181, Japan

E-mail address: hidemika@isis.ocn.ne.jp

Kohji Matsumoto, Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan

E-mail address: kohjimat@math.nagoya-u.ac.jp