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PROJECTION OF FIVE LINES IN PROJECTIVE SPACE

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ABSTRACT. The moduli space of five lines in $\mathbf{P}^2$ can be described by a quintic Del Pezzo surface in $\mathbf{P}^5$. Given five fixed lines in $\mathbf{P}^3$ and a fixed plane, we define a map from $\mathbf{P}^3$ to the quintic Del Pezzo surface by projecting the lines to the fixed plane, and taking the point on the Del Pezzo surface defined by the image lines as the image of the point of projection. We show that the fibers of this map are twisted cubic curves. Conversely, we show that the moduli space of curves in $\mathbf{P}^3$ with the five fixed lines as secants can be seen as isomorphic to the quintic Del Pezzo surface.

1. Invariants of five lines in $\mathbf{P}^2$

Consider a set of five lines in $\mathbf{P}^2$, where each line is described by the linear form $a_ix + b_iy + c_iz = 0$, $i = 1, \ldots, 5$. Using the coefficients $[a_i, b_i, c_i]$ of the linear forms which describe the lines, we can represent the lines by a 5×3 matrix X. We denote the 3×3 minor of X given by the ith, jth, and kth columns of X by $[ijk]$. For simplicity of notation, products of the 3×3 minors are written in *tableaux*, e.g.,

$$\begin{bmatrix} ijk \\ lmn \end{bmatrix} = [ijk][lmn].$$

Notice that, via duality, the same construction can be used to describe points as well as lines. Following [DO], we describe the moduli space of 5 lines in $\mathbf{P}^2$ by *homogeneous* invariants of the set of lines, which are products of the matrices $[ijk]$ in which each index occurs an equal number of times. The *degree* of a homogeneous invariant is defined to be the number of times each index occurs in the corresponding tableau. In [DO], it is shown that the following degree three invariants generate the moduli space P_2^5 of 5 lines in $\mathbf{P}^2$:

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$$t_0 = \begin{bmatrix} 124 \\ 134 \\ 135 \\ 235 \\ 245 \end{bmatrix}, \quad t_1 = \begin{bmatrix} 124 \\ 124 \\ 135 \\ 235 \\ 345 \end{bmatrix}, \quad t_2 = \begin{bmatrix} 123 \\ 134 \\ 135 \\ 245 \\ 245 \end{bmatrix},$$
$$t_3 = \begin{bmatrix} 123 \\ 124 \\ 135 \\ 245 \\ 345 \end{bmatrix}, \quad t_4 = \begin{bmatrix} 123 \\ 124 \\ 125 \\ 345 \\ 345 \end{bmatrix}, \quad t_5 = \begin{bmatrix} 123 \\ 123 \\ 145 \\ 245 \\ 345 \end{bmatrix}.$$

By applying straightening laws (see [Sf]) to the degree *six* invariants, we obtain the following relations between the functions t_i:

$$\begin{aligned} t_0t_5 &= t_3(t_2 - t_3 + t_5), \\ t_0t_3 &= t_1t_2, \\ t_1t_5 &= t_3(t_3 - t_4), \\ t_2t_4 &= t_3(t_3 - t_5), \\ t_0t_4 &= t_3(t_1 - t_3 + t_4). \end{aligned}$$

These equations define for us a quintic surface known as a *Del Pezzo* surface of degree five in $\mathbf{P}^5$. The quintic Del Pezzo surface can be obtained by blowing up four points in $\mathbf{P}^2$, as seen in [Ha, pp. 400–401].

In the above construction, only *semi-stable* configurations of lines in $\mathbf{P}^2$ will be represented as points in the moduli space (see [D]), where semi-stability is defined for five lines in $\mathbf{P}^2$ by the following criteria:

(1) No four lines may intersect in a single point.
(2) No double lines are allowed.

2. Projection of five lines in $\mathbf{P}^3$

We will now define a map from $\mathbf{P}^3$ to the quintic Del Pezzo surface, based on a set of five fixed lines in $\mathbf{P}^3$. In order to minimize the locus of points for which the map is undefined, we require that the fixed lines are in general position, as follows:

Definition 2.1. A set of five lines in $\mathbf{P}^3$ is in *general position* if the following conditions hold:

(1) No two of the lines intersect.
(2) There are exactly ten transversals of subsets of four of the five lines.

For a fixed set of lines $\{l_1, l_2, l_3, l_4, l_5\}$ in general position in $\mathbf{P}^3$, we define a map from $\mathbf{P}^3$ to $\mathbf{P}^5_2$ by projecting the lines to a fixed plane in $\mathbf{P}^3$, and taking the point on the Del Pezzo surface defined by the image lines as the image of the point of projection. Without loss of generality, we can choose the fixed plane to be the plane $d = 0$ in $\mathbf{P}^3$ with coordinates (a, b, c, d). Let l_i be given by points $v_i = (\alpha_i, \beta_i, \gamma_i, \delta_i), w_i = (\alpha'_i, \beta'_i, \gamma'_i, \delta'_i)$ for $i = 1, \ldots, 5$. Also without loss of generality, we can let $\delta_i = 0$, and we can assume that no line l_i is contained in the plane $d = 0$. Denote the point of projection by $p = (a, b, c, d)$. The coordinates of points on the image line $\mathrm{proj}_p(l_i)$ are then given as follows:

$$\begin{aligned}
\mathrm{proj}_p(v_i) &= (\alpha_i, \beta_i, \gamma_i), \\
\mathrm{proj}_p(w_i) &= (d\alpha'_i - a\delta'_i, d\beta'_i - b\delta'_i, d\gamma'_i - c\delta'_i).
\end{aligned}$$

Using Plücker coordinates to describe the image line itself, we get

$$\mathrm{proj}_p(l_i) = (x_i(p), y_i(p), z_i(p)),$$

where

$$\begin{aligned}
x_i &= (\alpha_i\beta'_i - \alpha'_i\beta_i)d - \alpha_i\delta'_i b + \beta_i\delta'_i a, \\
y_i &= (\alpha_i\gamma'_i - \alpha'_i\gamma_i)d - \alpha_i\delta'_i c + \gamma_i\delta'_i a, \\
z_i &= (\beta_i\gamma'_i - \beta'_i\gamma_i)d - \beta_i\delta'_i c + \gamma_i\delta'_i b.
\end{aligned}$$

We now have expressed the coordinates of the image lines as linear functions of the point of projection. Consider the 5×3 matrix given by the projected lines:

$$\begin{bmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2 \\
x_3 & y_3 & z_3 \\
x_4 & y_4 & z_4 \\
x_5 & y_5 & z_5
\end{bmatrix}.$$

We can use the 3×3 minors of this matrix to determine points on the quintic Del Pezzo surface. Consider the minor $[ijk]$. At first glance, the minors are cubic polynomials in $\mathbf{P}^3$. However, each minor factors as $d \cdot q_{ijk}$, where q_{ijk} can be viewed as the quadric surface which is determined by the three skew lines l_i, l_j, l_k. We can cancel the common factor d from all minors, and obtain the following map ϕ from $\mathbf{P}^3$ to the quintic Del Pezzo surface:

$$\begin{aligned}
f_0 &= q_{124}q_{134}q_{135}q_{235}q_{245}, \\
f_1 &= q_{124}^2 q_{135}q_{235}q_{345},
\end{aligned}$$

$$\begin{aligned} f_2 &= q_{123}q_{134}q_{135}q_{245}^2, \\ f_3 &= q_{123}q_{124}q_{135}q_{245}q_{345}, \\ f_4 &= q_{123}q_{124}q_{125}q_{345}^2, \\ f_5 &= q_{123}^2q_{145}q_{245}q_{345}. \end{aligned}$$

In this way, we can see that cancellation of the factor d simply extends the map ϕ to the plane $d = 0$. Note, however, that the map is not defined when p lies on any line l_i. Also, this map is defined only for points which project to stable sets of lines in $\mathbf{P}^2$. Therefore, when p lies on any transversal t_{ijkl} of four lines, the map ϕ is also undefined. The locus of indeterminacy of ϕ then consists of the following:

(1) the five lines l_i;
(2) the ten transversals t_{ijkl}, t'_{ijkl}.

Our assumption of general position precludes the possibility that any of the transversals intersect. Local computations show that the locus of indeterminacy of ϕ can be resolved by blowing up first the lines and then the transversals each once. We denote the resulting morphism by $\widehat{\phi}$.

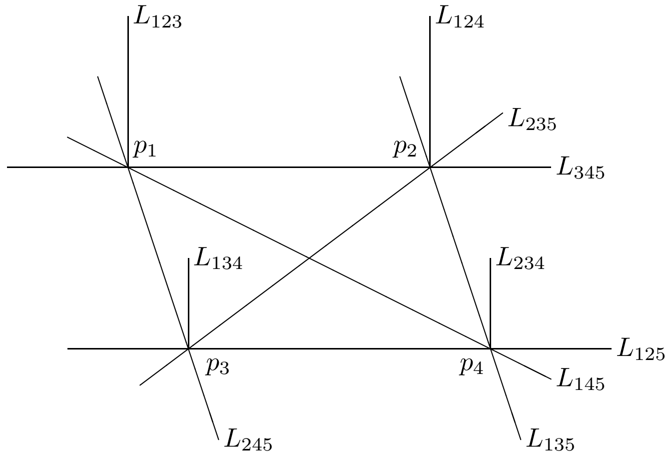
3. Description of the fibers

We can now prove the following proposition:

PROPOSITION 3.1. *A general fiber of the map $\widehat{\phi}$ is a twisted cubic curve which meets each of the five lines l_i in two points.*

We will use L_{ijk} to denote the line on DP_5 corresponding to the configurations in $\mathbf{P}_2^5$ in which the three lines l_i, l_j, l_k intersect in a common point. Recall that the Del Pezzo surface DP_5 can be obtained by blowing up $\mathbf{P}^2$ in four points p_1, p_2, p_3, p_4. Without loss of generality, we can match up the lines $\overline{p_ip_j}$ joining p_i to p_j and the exceptional curves $e_i = \widehat{p_i}$ with the lines L_{ijk} as follows:

$$\begin{aligned} L_{123} &= e_1 \\ L_{124} &= e_2 \\ L_{125} &= \overline{p_3p_4} \\ L_{134} &= e_3 \\ L_{135} &= \overline{p_2p_4} \\ L_{145} &= \overline{p_1p_4} \\ L_{234} &= e_4 \\ L_{235} &= \overline{p_2p_3} \\ L_{245} &= \overline{p_1p_3} \\ L_{345} &= \overline{p_1p_2} \end{aligned}$$



There are five pencils of conics on DP_5. These can be viewed as the inverse images under the map $\pi_1 : DP_5 \to \mathbf{P}^2$ of the following pencils on $\mathbf{P}^3$:

$$\begin{aligned}
\Theta_1 &= \text{lines through } p_1, \\
\Theta_2 &= \text{lines through } p_2, \\
\Theta_3 &= \text{lines through } p_3, \\
\Theta_4 &= \text{lines through } p_4, \\
\Theta_5 &= \text{conics through } p_1, p_2, p_3, p_4.
\end{aligned}$$

We will show that these conics are the images under ϕ of pencils of quartics in $\mathbf{P}^3$, as follows:

$$\begin{aligned}
C \in \Theta_1 &= \phi(q_{134}q_{245} - \alpha q_{124}q_{345}) \text{ for some } \alpha \neq \{0,1\}, \\
C \in \Theta_2 &= \phi(q_{123}q_{345} - \alpha q_{134}q_{235}) \text{ for some } \alpha \neq \{0,1\}, \\
C \in \Theta_3 &= \phi(q_{123}q_{245} - \alpha q_{124}q_{235}) \text{ for some } \alpha \neq \{0,1\}, \\
C \in \Theta_4 &= \phi(q_{123}q_{145} - \alpha q_{124}q_{135}) \text{ for some } \alpha \neq \{0,1\}, \\
C \in \Theta_5 &= \phi(q_{125}q_{345} - \alpha q_{135}q_{245}) \text{ for some } \alpha \neq \{0,1\}.
\end{aligned}$$

These are the five pencils of quartics in $\mathbf{P}^3$ which pass through $l_1, \dots, l_5$ and contain some l_i as a double line.

We will demonstrate this explicitly only for Θ_1, as the other cases are similar. Consider a hyperplane section H of DP_5 given by $f_0 = \alpha f_1$. This will be represented as the inverse image under π_1 of a cubic in $\mathbf{P}^2$ through p_1, p_2, p_3, p_4. In the coordinates f_i we have

$$q_{124}q_{134}q_{135}q_{235}q_{245} = \alpha q_{124}^2 q_{135}q_{235}q_{345}.$$

Note that points on the quadrics q_{ijk} will map under ϕ to points on the lines L_{ijk}. The hyperplane section H clearly vanishes on the lines $L_{124}, L_{135}, L_{235}$. Therefore, the cubic in $\mathbf{P}^2$ must include the lines $\overline{p_2p_3}$ and $\overline{p_2p_4}$. It must then also include a line through p_1 in order to contain all four points, and this line must not coincide with the lines $\overline{p_1p_2}$, $\overline{p_1p_3}$, or $\overline{p_1p_4}$ (by the assumption $\alpha \neq \{0,1\}$).

We can now see that the hyperplane section H is given by the above degree ten equation in $\mathbf{P}^3$, and maps via $\pi_1 \circ \phi$ to the cubic in $\mathbf{P}^2$ given by $\overline{p_2p_3}$, $\overline{p_2p_4}$, and some line $l \in \Theta_1$. As the quadrics $q_{124}, q_{135}, q_{235}$ map to the lines $L_{124}, L_{135}, L_{235}$ and not to l, we can cancel them from the above equation to see that $(\pi_1 \circ \phi)^{-1}$ (l) is given by $q_{134}q_{245} = \alpha q_{124}q_{345}$, as stated. Thus, we can state following lemma:

Lemma 3.2. *For some $\alpha, \beta \neq 0, 1$, the fiber F over p can be given by the following equations:*

$$\begin{array}{rcccl} Q_1(\alpha) & = & q_{134}q_{245} - \alpha q_{124}q_{345} & = & 0, \\ Q_2(\beta) & = & q_{123}q_{145} - \beta q_{124}q_{135} & = & 0. \end{array}$$

Consider DP_5 as the blowup of $\mathbf{P}^2$ in four points. We see that any point p which does not lie on any line L_{ijk} can be described as the intersection of two lines l_i, l_j from different pencils Θ_i, Θ_j. The elements of the pencils Θ_i, Θ_j thus determine the values of α, β above. We must, however, discard the locus of indeterminacy of ϕ and then determine the closure of F in $\mathbf{P}^3$. We see that, as the intersection of two quartics in $\mathbf{P}^3$, the fiber F initially has degree 16. However, notice that the intersection $Q_1 \cap Q_2$ contains the lines $l_1, \ldots, l_5$, with l_1 and l_4 contained as double lines. Also, $Q_1 \cap Q_2$ contains the six transversals $t_{1234}, t'_{1234}, t_{1345}, t'_{1345}, t_{1245}, t'_{1245}$. The remaining curve then has degree $16 - 7 - 6 = 3$. The symmetry of the configuration demonstrates that the fiber F must be irreducible in general, as each component must intersect each of the lines in the same number of points, and this is not possible in general if the fiber degenerates. It remains to verify that F will intersect each of the five lines at two points.

We turn to intersection theory on the quartic surfaces to complete the proof. Let

$$S = \lambda Q_1 + \mu Q_2 = 0$$

be a general member of the pencil of quartics through $l_1, \ldots, l_5$ determined by $Q_1(\alpha), Q_2(\beta)$. Consider Q_2 as a divisor on S. Moving to divisor notation, we have

$$\begin{aligned} 4H \sim Q_2 \sim{} & 2l_1 + l_2 + l_3 + 2l_4 + l_5 \\ & + t_{1234} + t'_{1234} + t_{1345} + t'_{1345} + t_{1245} + t'_{1245} + F, \end{aligned}$$

where H is a general hyperplane section.

We will compute the intersection with the fiber F. It is clear that a general fiber must intersect each line l_i the same number of times, and we have shown that a general fiber does not meet any transversal. This yields

$$4H \cdot F \sim 7F \cdot l_i + F^2.$$

Since F is cubic, we get

$$12 = 7F \cdot l_i - 2.$$

Therefore, F must meet each line twice, and the proof is complete.

In the case of five points, we showed that after blowing up $\mathbf{P}^3$, the ten lines l_{ij} connecting the points P_i, P_j mapped to the ten lines on the Del Pezzo surface. However, it is not true here that the exceptional divisors corresponding to the ten transversals t_{ijkl}, t'_{ijkl} map under $\hat{\phi}$ to the ten lines on the Del Pezzo. Instead, the corresponding divisors map to ten special conics C_{ijkl}, C'_{ijkl} on the Del Pezzo. We will show this only for the transversal t_{1235}, as proofs for the other transversals are similar.

Recall the family Q_1 of quartic curves in Lemma 3.2. By Bezout's Theorem, we can find a constant α such that $Q_1(\alpha) = 0$ on t_{1235}. The image of the exceptional divisors of t_{1235} under $\hat{\phi}$ must therefore lie in the hyperplanes $f_2 = \alpha f_3$ and $f_0 = \alpha f_1$.

Consider the hyperplane $f_2 = \alpha f_3$. This hyperplane cuts out a cubic curve on DP_5 which reduces as $L_{135} \cup C_{1235}$, where C_{1235} is a conic which corresponds to a line through p_1 under the map π_1. Local computations verify that the image of the proper transform of t_{1235} does not lie in the line L_{135}, but rather that fibers which intersect the transversal map to points on the conic C_{1235}. This can also be verified by the S_5 symmetry of the configuration.

The following theorem will complete our description of the fibers of $\hat{\phi}$:

Theorem 3.3. *All fibers of $\hat{\phi}$ have one of the following forms:*

(1) *The union of two lines, one in the rulings of each of the exceptional divisors corresponding to two transversals t_{ijkl}, t_{ijkm}, and the line given by the intersection of the planes determined by l_l, t_{ijkl} and l_m, t_{ijkm}, respectively. These are the fibers over the points of intersection of two special conics C_{ijkl}, C_{ijkm}.*

(2) *$L \cup X_{ijkl}$, where L is a fiber in the ruling of the proper transform of some transversal t_{ijkl} and X_{ijkl} is a conic which meets each line l_i, l_j, l_k, l_l once, the line l_m twice, and the transversal t_{ijkl} once. These are the fibers over the special conics C_{ijkl} on the Del Pezzo surface, and all fibers which do not lie in the intersection of two such conics have this form.*

(3) *A twisted cubic curve meeting each line l_i in two points. All fibers which do not map to points on the special conics C_{ijkl} are irreducible cubics of this form.*

Notice from the above argument that there are ten special conics on the Del Pezzo surface, corresponding to the ten transversals in $\mathbf{P}^3$. There are therefore ten special lines and ten special conics on DP_5 over which the fibers may degenerate. We will consider 5 cases:

1. General fibers over C_{ijkl}.
2. General fibers over L_{ijk}.
3. Fibers over $L_{ijk} \cap C_{ijkl}$.
4. Fibers over $L_{ijk} \cap L_{ilm}$.
5. Fibers over $C_{ijkl} \cap C_{ijkm}$.

Case 1. As before, we will show this for a specific conic C_{ijkl}, as proofs for the others can be obtained by permuting the indices. Consider the conic C_{1235}. Let α be such that

$$Q_1(\alpha) = q_{134}q_{245} - \alpha q_{124}q_{345} = 0$$

on t_{1235}. For a general β, let S be a general member of the pencil of quartics determined by $Q_1(\alpha), Q_2(\beta)$ as before. We consider $Q_2(\beta)$ as a divisor on S. We have

$$\begin{aligned} Q_2(\beta) \sim 4H \sim 2l_1 + l_2 + l_3 + 2l_4 + l_5 \\ + t_{1235} + t_{1234} + t'_{1234} + t_{1345} + t'_{1345} + t_{1245} + t'_{1245} + X, \end{aligned}$$

where X is a conic. We want to show that F meets each line l_1, l_2, l_3, l_5 once, the line l_4 twice, and the transversal t_{1235} once.

We will compute the intersection first of l_1 with the above divisor. By symmetry, this will also give us the intersection multiplicity of each line l_2, l_3, l_5 with X. We have

$$4 = 2(-2) + 7 + l_1 \cdot X,$$

verifying that X intersects the line l_i once.

For the line l_4, we have

$$4 = 2(-2) + 6 + l_4 \cdot X,$$

showing that l_4 must meet F twice. Finally, we have

$$4 = 5 - 2 + t_{1235} \cdot X,$$

verifying that F meets the transversal t_{1235} once and completing Case 1.

Case 2. Again, we will show this for the specific line L_{234}. Here, straightening laws show that $\alpha = 1$. Choosing a general β, we have $Q_2(\beta)$ as a divisor on $S = \lambda Q_1(\alpha) + \mu Q_2(\beta)$ as before:

$$\begin{aligned} Q_2 \sim 4H \sim 2l_1 + l_2 + l_3 + 2l_4 + l_5 \\ + t_{1234} + t'_{1234} + t_{1345} + t'_{1345} + t_{1245} + t'_{1245} + F. \end{aligned}$$

The rest of the proof follows as in Proposition 3.1.

Case 3. Consider the intersection of the conic C_{1235} with the line L_{125} on DP_5. Choose α as in Case 1 so that $Q_1(\alpha)$ vanishes on t_{1235}. Straightening laws show that the condition $\beta = 1$ causes the quartic $Q_2(\beta)$ to vanish on Q_{125}, and therefore the image to lie in the line L_{125}.

Computing intersections on a general member of the pencil S again,we have

$$\begin{aligned} Q_2(\beta) \sim 4H \sim 2l_1 + l_2 + l_3 + 2l_4 + l_5 \\ + t_{1235} + t_{1234} + t'_{1234} + t_{1345} + t'_{1345} + t_{1245} + t'_{1245} + F, \end{aligned}$$

where F must have degree two. The rest of the proof here follows as in Case 1, and shows that F must meet l_1, l_2, l_3, l_4 each once, l_4 twice, and t_{1235} once.

Case 4. Here, we will consider the specific example $L_{123} \cap L_{345}$. We know that the preimages of L_{123}, L_{345} lie in q_{123}, q_{345}, respectively. Therefore, the fiber will be the residual cubic curve which is the intersection of q_{123}, q_{345} after discarding the line l_3. It is easy to check that this line meets each line l_i twice, using intersection theory of the quadric surfaces q_{123}, q_{345}.

Case 5. This is the case which yields fibers of the second type discussed in the theorem. We will show what happens in the case $C_{1235} \cap C_{2345}$. Choose α, β such that $Q_1(\alpha)$ vanishes on t_{1235} and $Q_2(\beta)$ vanishes on t_{2345}, and consider $Q_2(\beta)$ as a divisor on S as before. We have

$$\begin{aligned} Q_2 \sim 4H \sim 2l_1 + l_2 + l_3 + 2l_4 + l_5 + t_{1235} + t_{2345} \\ + t_{1234} + t'_{1234} + t_{1345} + t'_{1345} + t_{1245} + t'_{1245} + L, \end{aligned}$$

where L must be a line in this case. Using arguments as above, we can determine that L must intersect both l_1 and l_4 each once, and the transversals t_{1235} and t_{2345} each once.

Consider the line L' given by the intersection of the planes determined by l_1, t_{1235}, and l_4, t_{2345}, respectively. Notice that this line meets t_{1235}, t_{2345}, l_1, and the double line l_4 of the quadric $Q_1(\alpha)$, all in different points. Thus, $Q_1(\alpha)$ must vanish on L'. A similar argument shows that $Q_2(\beta)$ vanishes on L', and we conclude that $L' = L$.

4. Twisted cubics with five fixed secants

In the above computations, we have shown that there is a two-dimensional family of twisted cubic curves, each of which has the five lines l_i as a secant. Comparison of this family to the moduli space of twisted cubics in $\mathbf{P}^3$ leads us to the following theorem:

THEOREM 4.1. *The subspace of the Hilbert scheme of curves in $\mathbf{P}^3$ consisting of twisted cubic curves which have the five lines l_i as secants is isomorphic to the quintic Del Pezzo surface.*

In this case, we take "twisted cubic curves" to mean stable cubic curves of genus zero. These can be irreducible rational cubics as before, or the union of a conic and a line intersecting it at one point, or the union of three lines which form a chain. To have the five fixed lines as secants, a reducible cubic must consist of either the union of a transversal to four of the lines and a conic in a plane π through the fifth line which passes through the points of intersections of π with the four lines and the transversal, or the union of two transversals t_{ijkl}, t_{ijkm} and the line given by the point of intersection of the planes determined by l_l and t_{ijkl}, and l_m and t_{ijkm}, respectively.

The proof of the theorem follows from a transformation of $\mathbf{P}^3$ which is known as *Wakeford's transformation* (see [SR, p. 186]). Given four skew lines in $\mathbf{P}^3$, Wakeford's transformation T maps $\mathbf{P}^3$ to $\mathbf{P}^3$ via cubic surfaces which pass through the four lines l_1, l_2, l_3, l_4 and their two transversals. It can be seen that the images of lines in $\mathbf{P}^3$ under T are twisted cubics in $\mathbf{P}^3$ which have four fixed lines l'_1, l'_2, l'_3, l'_4 as secants, and the images of twisted cubics in $\mathbf{P}^3$ which have as secants the four lines l_1, l_2, l_3, l_4 are lines in $\mathbf{P}^3$. The four lines l'_1, l'_2, l'_3, l'_4 are the images under T of the four quadrics Q_{ijk} defined by sets of three of the four lines l_1, l_2, l_3, l_4.

Given a fifth line l_5 in $\mathbf{P}^3$, its image under T will be a twisted cubic curve C. Then, the variety $\mathrm{Sec}(C)$ of lines in $\mathbf{P}^3$ which are secants to the curve C is birational to the variety of twisted cubics in $\mathbf{P}^3$ which have the five lines l_1, l_2, l_3, l_4, l_5 as secants.

We know that the secant variety of a twisted cubic is isomorphic to $\mathbf{P}^2$. However, we must blow up the points on $\mathrm{Sec}(C)$ which correspond to the four lines l'_1, l'_2, l'_3, l'_4, as these correspond to the four one-dimensional families of twisted cubics which have the five lines l_1, l_2, l_3, l_4, l_5 as secants and which lie on the quadrics Q_{ijk}. Thus, we see that the variety corresponds to the blowup of $\mathbf{P}^2$ in four points. This is the quintic Del Pezzo surface, as before.

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