Illinois Journal of Mathematics
Volume 45, Number 4, Winter 2001, Pages 1145–1160
S 0019-2082

UNIFORM $\mathbf{L_p(w)}$ SPACES

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ABSTRACT. $L_p(w)$ spaces ($0 < p < 1$) were developed by J. W. Roberts to serve as a special class of trivial-dual spaces which admit compact operators and to provide counterexamples to various interesting problems. Roberts showed that any separable, trivial-dual p-Banach space is a quotient of some *uniform* $L_p(w)$ *space*. Uniform $L_p(w)$ spaces are indexed by a sequence of finite dimensional spaces $\langle X_n \rangle$ in L_p and a sequence of constants $\langle c_n \rangle$ such that $1 \le c_0 \le c_1 \le c_2 \le \cdots$. If $\langle c_n \rangle$ is bounded, the resulting space is isomorphic to L_p. Hence these spaces can be thought of as generalized L_p spaces. We prove that if $c_n \uparrow \infty$, the corresponding $L_p(w)$ space admits compact operators and is thus not isomorphic to L_p. Further, we show that there is no non-zero continuous linear operator from L_p into any $L_p(w)$, where $c_n \uparrow \infty$. Using and sharpening a result of Roberts, we also demonstrate that for any separable, trivial-dual p-Banach space S there exists a uniform $L_p(w)$ space X_S with $\mathcal{L}(S, X_S) = \{0\}$.

1. Introduction

In this paper we investigate a generalization of the spaces L_p, $0 < p < 1$, the so-called "$L_p(w)$ spaces." (Throughout the paper, p will be in the range $0 < p < 1$.) This class of separable, trivial-dual p-Banach spaces was introduced in 1981 by Roberts [7] to serve as "domain-spaces" for compact operators, after Kalton and Shapiro [4] had proved the existence of trivial-dual spaces which admit compact operators.

A particularly nice sub-class of $L_p(w)$ spaces is the class of *uniform* $L_p(w)$-*spaces*. Although one could consider these spaces as weighted L_p-spaces, they are fundamentally different from the spaces L_p. For example, these spaces tend to admit compact operators, while the spaces L_p cannot be the domain of a compact operator, as was shown in 1976 by Kalton (see [1] and [2]). Furthermore, at least some of the uniform $L_p(w)$ spaces do not contain pathological compact convex sets (Rowe [9]), while L_p boasts an abundance of these (Roberts [5]).

Received December 21, 1999; received in final form August 23, 2000.
2000 *Mathematics Subject Classification.* Primary 46A16. Secondary 46B03, 46E30.

Another interesting result due to Roberts [5] is that uniform $L_p(w)$ spaces are *projective* among separable, trivial-dual p-Banach spaces; in other words, any separable trivial-dual p-Banach space is a quotient of some uniform $L_p(w)$ space. On the other hand, there is no *projective space* in the class of separable, trivial-dual p-Banach spaces (Roberts [8]). (A space X is a *trivial-dual space* if its dual consists of only the zero functional. X is a *projective space* in a class of spaces $\mathcal{C}$ if $X \in \mathcal{C}$ and every space in $\mathcal{C}$ is a quotient of X.)

In the present paper we investigate the possibility of an isomorphism between L_p and a uniform $L_p(w)$ space and characterize the spaces that admit compact operators. Furthermore, answering a question of Roberts, we demonstrate the "repellent nature" of uniform $L_p(w)$ spaces, i.e., the fact that for any separable, trivial-dual p-Banach space X there is a uniform $L_p(w)$ with $\mathcal{L}(X, L_p(w)) = \{0\}$.

Our notation is rather standard. A *p-norm* $\|\cdot\|$ on a real vector space X is a map $\|\cdot\|\colon X \to \mathbf{R}$ such that for all $x \in X$ we have

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|x + y\| \leq \|x\| + \|y\|$,
(iii) $\|\alpha x\| = |\alpha|^p \|x\|$ for all α and x.

If $\|\cdot\|$ is a p-norm, then $d(x, y) := \|x - y\|$ defines a *(translation)-invariant metric* on X. A complete p-normed space is called a *p-Banach space.*

The most prominent examples of p-Banach spaces are L_p, l_p, and H^p, $0 < p < 1$.

A closed ball of radius ϵ centered at the origin will be denoted by B_ϵ, and the *Hausdorff-distance* of two sets A, B is defined by

$$\mathcal{H}_{\|\cdot\|}(A, B) = \inf\{r : A \subset \tau_r(B) \text{ and } B \subset \tau_r(A)\},$$

where for any $r \geq 0$ and $S \subset X$, $\tau_r(S) = \{x : d(x, S) \leq r\}$.

A continuous linear operator between the F-spaces X and Y is called *compact* if there is a neighborhood of 0 in X whose image is compact in Y.

Finally, given p with $0 < p < 1$, we let $\mathcal{B}$ denote the class of separable, trivial-dual p-Banach spaces.

2. Construction of $L_p(w)$ spaces

For each $n \in N \cup \{0\}$, let Π_n denote a finite partition of $[0, 1]$ into intervals such that the sequence $\langle \Pi_n \rangle$ satisfies

(1) $\Pi_0 = \{[0, 1]\}$;
(2) Π_{n+1} refines Π_n;
(3) if $I \in \Pi_n$, then $I = \bigcup_{j=1}^k I_j$, where $k \geq 2$ and each of the intervals I_j has the same length.

Letting $\Pi = \bigcup_n \{I : I \in \Pi_n\}$, a function $w\colon \Pi \to (0, \infty)$ is a *weight-function* if it satisfies

(4) $w([0, 1]) = 1$;

(5) if $I \in \Pi_n$ and $I = \bigcup_{j=1}^{k} I_j$, where $I_1, \ldots, I_k \in \Pi_{n+1}$, then $w(I) \leq \sum_{j=1}^{k} w(I_j)$;
(6) $\lim_{n\to\infty} \max_{I\in\Pi_n} \frac{1}{|I|^p} w(I) = 0$.

Now suppose $w(\cdot)$ is a weight function on a collection Π. For each n, let

$$X_n := \operatorname{span}\{1_I : I \in \Pi_n\},$$

and define a p-norm $\|\cdot\|_n$ on X_n by

$$\left\| \sum_{I\in\Pi_n} \alpha_I 1_I \right\|_n = \sum_{I\in\Pi_n} |\alpha_I|^p w(I).$$

Clearly, each $(X_n, \|\cdot\|_n)$ is isometrically isomorphic to l_p^M, where $M = |\Pi_n|$. Furthermore, we see that $X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$, and $\|\cdot\|_n \leq \|\cdot\|_{n+1}$ on each of the X_n.

Next, let $X_\infty = \bigcup_{n=1}^{\infty} X_n$, and for each $x \in X_\infty$ define $|||x|||$ by

$$|||x||| = \inf\{\|x_0\|_0 + \cdots + \|x_n\|_n : x = x_0 + \cdots + x_n, x_k \in X_k\}.$$

Note that $|||\cdot|||$ is easily seen to be a p-norm on X_∞. We define the space $L_p(w)$ to be the completion of X_∞ with respect to $|||\cdot|||$.

Condition (6) above ensures that the space $L_p(w)$ has trivial dual (see [8]).

The $L_p(w)$ space X is said to be *uniform* if all intervals in each Π_n have the same length and are *equally weighted*; i.e., if $\Pi_n = \{I_1, I_2, \ldots, I_{M_n}\}$, then $w(I_1) = w(I_2) = \cdots = w(I_{M_n})$.

Observe that if $L_p(w)$ is uniform and $x_n \in X_n$, then

$$\|x_n\|_n = c_n \|x_n\|_p,$$

where $\|\cdot\|_p$ is the usual p-norm, and $c_n = w(I)/|I|$ for each $I \in \Pi_n$. So if $x \in X$, then

$$|||x||| = \inf\left\{ \sum_{i=1}^{n} c_i \|x_i\|_p : \sum_{i=1}^{n} x_i = x,\ x_i \in X_i \right\}.$$

Furthermore, because $\|\cdot\|_{n+1} \geq \|\cdot\|_n$ on each X_n, the sequence $\langle c_n \rangle$ is non-decreasing. In other words, we can think of X as being indexed by $\langle \Pi_n \rangle$ and a non-decreasing positive sequence $\langle c_n \rangle$. Clearly, if $\langle c_n \rangle$ is bounded, $|||\cdot|||$ is equivalent to $\|\cdot\|_p$, and $L_p(w)$ is just L_p. On the other hand, unbounded sequences tend to give rise to spaces quite different from L_p, as the following results of Rowe [9] and Sisson [10] illustrate.

THEOREM 2.1 (ROWE). *There is a uniform $L_p(w)$ space with the following property. If K is a compact convex subset of $L_p(w)$, then there is an affine homeomorphism mapping K into a compact convex subset of a locally convex topological vector space.*

Corollary. *There is a uniform $L_p(w)$ space with no pathological compact convex subsets.*

This is, of course, in sharp contrast with properties of L_p (see [5]).
Sisson [10] proved the following result (see also Kalton [1]).

Theorem 2.2 (Sisson). *Let X be a uniform $L_p(w)$ space such that its sequence $\langle c_n \rangle$ has the property that for all $n \in N$, $\lambda \le c_{n+1}/c_n$ for some fixed $\lambda > 1$. Then X admits compact operators.*

At this point, the following questions arise quite naturally.

(a) Is it possible that some uniform $L_p(w)$ space ($c_n \uparrow \infty$) is isomorphic to L_p? To be more specific, what growth conditions must we impose on $\langle c_n \rangle$ so that the resulting $L_p(w)$ space is not isomorphic to L_p?
(b) Is it possible that some uniform $L_p(w)$ space ($c_n \uparrow \infty$) fails to admit compact operators?

3. Uniform $L_p(w)$ spaces and L_p

In this section we provide a simple answer to the questions posed above. The results indicate that "all" uniform $L_p(w)$ spaces ($c_n \uparrow \infty$) are fundamentally different from L_p. Regarding a possible isomorphism, the second theorem will imply that it is not possible to construct even a non-trivial continuous linear operator from L_p into $L_p(w)$. Before stating and proving our results, we define the *identity operator* I to be the identity on X_∞. (Note that I is norm-decreasing since the weight of an interval is replaced by its length and $w(I) \ge |I|$.) We then extend I to $L_p(w)$, which is the completion of X_∞. Thus I is a norm-one linear operator mapping $L_p(w)$ into L_p.

Theorem 3.1. *For a uniform $L_p(w)$ space X, the following are equivalent:*

(1) *X is not isomorphic to L_p.*
(2) *$c_n \uparrow \infty$.*
(3) *$I\colon X \to L_p$ is compact.*
(4) *X admits compact operators.*

Proof. (1) $\Rightarrow$ (2) follows directly by contraposition.
(3) $\Rightarrow$ (4) is trivial.
(4) $\Rightarrow$ (1) follows easily from the fact that L_p does not admit compact operators.
In order to prove that (2) implies (3), we need a few lemmas.

Lemma 3.2. *Let $(X, \|\cdot\|)$ be an F-space, $\langle K_n \rangle$ a sequence of compact subsets of X with $K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$, and let $\langle \epsilon_n \rangle$ be a sequence of* *positive numbers such that* $\sum_n \epsilon_n < \infty$ *and*

$$\mathcal{H}_{\|\cdot\|}(K_n, K_{n+1}) < \epsilon_{n+1}.$$

Then $\bigcup K_n$ *is relatively compact.*

The proof of this result may be found on p. 203 of [3].

LEMMA 3.3. *If* X *is an* $L_p(w)$ *space and* $x \in X_N$*, then there exist* $x_0 \in X_0, \dots, x_N \in X_N$ *such that*

$$|||x||| = \sum_{k=0}^{N} \|x_k\|_k.$$

Moreover, $|||x_k||| = \|x_k\|_k$ *for all* k*. (The elements* x_k *are called norm-attaining.)*

Proof. Let N be fixed, and for $x \in X_N$ and a natural number $n \geq N$ define $|||\cdot|||_n$ by

$$|||x|||_n = \inf \left\{ \|x_0\|_0 + \cdots + \|x_n\|_n : \sum_{k=0}^{n} x_k = x,\ x_k \in X_k \right\}.$$

We first show that when computing $|||x|||$ for $x \in X_N$, only the infimum over the first N spaces must be considered. To this end, it is enough to see that $|||\cdot|||_n = |||\cdot|||_N$ whenever $n \geq N$.

Clearly, $|||\cdot|||_{N+1} \leq |||\cdot|||_N$ on X_N. For the reverse inequality, let us assume that x is the sum of elements from X_{N+1}:

$$x = x_1 + \cdots + x_N + x_{N+1}.$$

Then

$$x_{N+1} = x - x_1 - \cdots - x_N,$$

and since the right hand side is in X_N, so is x_{N+1}. Since $\|\cdot\|_N \leq \|\cdot\|_{N+1}$ on X_N, we obtain

$$\begin{aligned} \|x_1\|_1 + \cdots + \|x_{N+1}\|_{N+1} &\geq \|x_1\|_1 + \cdots + \|x_N\|_N + \|x_{N+1}\|_N \\ &\geq \|x_1\|_1 + \cdots + \|x_N + x_{N+1}\|_N, \end{aligned}$$

which shows that $|||\cdot|||_N \leq |||\cdot|||_{N+1}$ on X_N. The above claim now follows by induction.

To see that the infimum in the definition of $|||x|||$ is attained, let us define the map

$$\Phi(x_1, \dots, x_n) = \|x_1\|_1 + \cdots + \|x_n\|_n$$

on the set

$$K_x = \left\{ (x_1, \dots, x_n) : \sum_{k=1}^{N} x_k = x,\ \sum_{k=1}^{N} \|x_k\|_k \leq 2\,|||x|||\,, x_k \in X_k \right\}.$$

The set K_x is certainly compact by finite-dimensionality, and Φ is continuous and therefore assumes its infimum, $|||x|||$, on K_x.

Note that

$$|||x||| \leq |||x_1||| + \cdots + |||x_N||| \leq \|x_1\|_1 + \cdots + \|x_N\|_N = |||x||| .$$

Hence both inequalities are actually equalities, showing that the elements x_k are indeed norm-attaining; i.e., we have $|||x_k||| = \|x_k\|_k$ for all k, $1 \leq k \leq N$. □

Remark. Lemma 3.3 can be generalized as follows (see [8]):

If X is an $L_p(w)$ space with $c_n \uparrow \infty$ and $x \in X$, then for each k there exist $x_k \in X_k$ such that $x = \sum_{k=0}^{\infty} x_k$ and $|||x||| = \sum_{k=0}^{\infty} \|x_k\|_k$.

Since the proof of this result is quite involved, we have tried to avoid using the result as much as possible. Indeed, we will not use the result until the proof of the main theorem in Section 4. We could have avoided it even there, but we feel that referring to it makes the proof more readable.

Returning to the proof of the implication (2) ⇒ (3) of Theorem 3.1, consider a uniform $L_p(w)$ space X, with associated constants $c_n \uparrow \infty$. Choose $\langle c_{n_k} \rangle$, a subsequence of $\langle c_n \rangle$, such that

$$\sum_{k=1}^{\infty} \frac{1}{c_{n_k}} < \infty.$$

To see that the identity operator I is compact, we let

$$B_n := B \cap X_n$$

and

$$C_k := I(B_{n_k}).$$

We claim that for each k, $\mathcal{H}_{\|\cdot\|_p}(C_k, C_{k+1}) < 1/c_{n_k}$. This will allow us to apply Lemma 3.2, since by finite-dimensionality each C_k is compact, and clearly $C_0 \subset C_1 \subset \cdots \subset C_k \subset \cdots$. To establish the above claim, let $x \in B_{n_{k+1}}$ be arbitrary. By Proposition 2.1, for each j, $0 \leq j \leq n_{k+1}$, there exists $x_j \in X_j$ such that

$$x = x_0 + \cdots + x_{n_k} + \cdots + x_{n_{k+1}}$$

and

$$\begin{aligned} |||x||| &= \|x_0\|_0 + \cdots + \|x_{n_k}\|_{n_k} + \cdots + \|x_{n_{k+1}}\|_{n_{k+1}} \\ &= c_0\|x_0\|_p + \cdots + c_{n_k}\|x_{n_k}\|_p + \cdots + c_{n_{k+1}}\|x_{n_{k+1}}\|_p < 1, \end{aligned}$$

where the last inequality is true by assumption. Now define

$$y := x_0 + \cdots + x_{n_k}.$$

Clearly $y \in B_{n_k}$ (since $|||\, y \,||| \le \|x_0\|_0 + \cdots + \|x_{n_k}\|_{n_k} < 1$), and furthermore,

$$\begin{aligned} c_{n_k}\|x-y\|_p &= c_{n_k}\|x_{n_k+1} + \cdots + x_{n_{k+1}}\|_p \\ &\le c_{n_k}\|x_{n_k+1}\|_p + c_{n_k}\|x_{n_k+2}\|_p + \cdots + c_{n_k}\|x_{n_{k+1}}\|_p \\ &\le c_{n_k+1}\|x_{n_k+1}\|_p + c_{n_k+2}\|x_{n_k+2}\|_p + \cdots + c_{n_{k+1}}\|x_{n_{k+1}}\|_p < 1. \end{aligned}$$

Therefore we obtain

$$\|x-y\|_p < \frac{1}{c_{n_k}},$$

which proves our claim.

By Lemma 3.2, it follows that $I\left(B \cap \bigcup_k X_{n_k}\right) = I\left(\bigcup_k B_{n_k}\right)$ is relatively compact in L_p. Hence $I\colon X \to L_p$ is a compact operator. □

Our next result significantly sharpens the non-isomorphism statement of Theorem 3.1.

THEOREM 3.4. *The identity operator $I\colon L_p(w) \to L_p$ is one-to one.*

COROLLARY. *If X is a space with no non-trivial compact operators and $L_p(w)$ is uniform with $c_n \uparrow \infty$, then $\mathcal{L}(X, L_p(w)) = \{0\}$. In particular, there are no non-trivial continuous linear operators between L_p and $L_p(w)$.*

Proof. Indeed, if $T\colon X \to L_p(w)$ is non-trivial and continuous, then the composition operator

$$I \circ T\colon X \to L_p$$

is also non-trivial and compact, contradicting our hypotheses on X. □

To prove the theorem, we need the following lemma.

LEMMA 3.5. *Let $x \in L_p(w)$ and $\epsilon > 0$ be fixed. Then there exist $x_0 \in X_0$, $x_1 \in X_1, \ldots, x_n \in X_n, \ldots$ such that*

$$x = \sum_{n=0}^{\infty} x_n \quad \textit{and} \quad |||\, x \,||| + \epsilon \ge \sum_{n=0}^{\infty} \|x_n\|_n.$$

Proof. Let $x \in L_p(w)$ and $\epsilon > 0$ be given. Pick a sequence $\epsilon_n \downarrow 0$, with $\sum_n \epsilon_n < \epsilon/2$. Since X_∞ is dense in X, there is a sequence of simple functions $\langle y_n \rangle \in X_\infty$ such that

$$|||\, x - (y_1 + y_2 + \cdots + y_n) \,||| < \epsilon_n.$$

Now, by Lemma 3.3, for each n there exist $y_{nk} \in X_k$ such that

$$y_n = \sum_{k=0}^{N_n} y_{nk} \quad \text{and} \quad |||\, y_n \,||| = \sum_{k=0}^{N_n} \|y_{nk}\|_k.$$

Further, observe that

$$|||\, y_n \,||| \le |||\, x - (y_1 + \cdots + y_{n-1}) \,||| + |||\, x - (y_1 + \cdots + y_n) \,||| < \epsilon_{n-1} + \epsilon_n$$

for all $n \geq 2$; i.e., we have

$$\|y_{n1}\|_1 + \|y_{n2}\|_2 + \cdots + \|y_{nN_n}\|_{N_n} < \epsilon_n + \epsilon_{n-1}$$

for $n \geq 2$. Similarly,

$$|||\, y_1 \,||| \leq |||\, x \,||| + |||\, x - y_1 \,||| < |||\, x \,||| + \epsilon_1,$$

which implies

$$\|y_{11}\|_1 + \|y_{12}\|_2 + \cdots + \|y_{1N_1}\|_{N_1} < |||\, x \,||| + \epsilon_1.$$

Hence

$$\sum_{n,k} \|y_{nk}\|_k < |||\, x \,||| + 2\sum_n \epsilon_n < |||\, x \,||| + \epsilon,$$

and if we define

$$\begin{aligned} x_0 &:= y_{10} + y_{20} + y_{30} + \cdots \in X_0, \\ x_1 &:= y_{11} + y_{21} + y_{31} + \cdots \in X_1, \\ &\vdots \\ x_k &:= y_{1k} + y_{2k} + y_{3k} + \cdots \in X_k, \\ &\vdots \end{aligned}$$

then clearly

$$x = \sum_{k=0}^{\infty} x_k$$

and

$$\sum_{k=0}^{\infty} \|x_k\|_k \leq \sum_{n,k} \|y_{nk}\|_k < |||\, x \,||| + \epsilon,$$

as desired. □

We are now ready to prove the theorem. (Actually, as we shall see, the proof does not even require the full power of the lemma.)

Proof of Theorem 3.4. If $0 \neq \varphi \in X$, we aim to show that $\|I(\varphi)\|_p \neq 0$. Without loss of generality, we may assume $|||\, \varphi \,||| = 1$ and use Lemma 3.5 to find $\langle x_n \rangle$ such that each $x_n \in X_n$ and

$$\sum_n x_n = \varphi \quad \text{with} \quad \sum_n \|x_n\|_n < \infty.$$

Now choose an index n_1 so that if $n \geq n_1$ then

$$\sum_{n+1}^{\infty} \|x_n\|_n < \frac{1}{4},$$

and define

$$\varphi_1 := x_1 + \cdots + x_{n_1}.$$

Note that $|||\,\varphi_1\,||| \geq 3/4$, for otherwise

$$|||\,\varphi\,||| \leq |||\,\varphi_1\,||| + \sum_{n_1+1}^{\infty} \|x_n\|_n < \frac{3}{4} + \frac{1}{4} = 1,$$

a contradiction. Therefore, since $\varphi_1 \in X_{n_1}$, we have

$$c_{n_1}\|\varphi_1\|_p = \|\varphi_1\|_{n_1} \geq |||\,\varphi_1\,||| \geq \frac{3}{4},$$

and hence

$$\|\varphi_1\|_p \geq \frac{3}{4c_{n_1}}. \tag{1}$$

We shall use this shortly. Next, define

$$\varphi_2 := \sum_{n_1+1}^{\infty} x_n,$$

and observe that

$$\begin{aligned} c_{n_1}\|\varphi_2\|_p = c_{n_1}\left\|\sum_{n_1+1}^{\infty} x_n\right\|_p &\leq c_{n_1}\sum_{n_1+1}^{\infty}\|x_n\|_p \\ &\leq \sum_{n_1+1}^{\infty} c_n\|x_n\|_p = \sum_{n_1+1}^{\infty}\|x_n\|_n < \frac{1}{4}, \end{aligned}$$

and so

$$\|\varphi_2\|_p < \frac{1}{4c_{n_1}}. \tag{2}$$

Thus, if

$$\|\varphi\|_p = \|\varphi_1 + \varphi_2\|_p = 0,$$

then, in L_p, we have $\varphi_1 = -\varphi_2$, i.e., $\|\varphi_1\|_p = \|\varphi_2\|_p$, which contradicts (1) and (2) above. □

4. The "repellent nature" of the spaces

In this section we will answer a question of Roberts [8], by showing that uniform $L_p(w)$ spaces are "hard to map into" by continuous linear operators.

We begin by introducing a generalization of the concept of uniform $L_p(w)$ spaces. We say that an $L_p(w)$ space is *biuniform* if there is a sequence $\langle A_n, B_n\rangle$ such that $A_1 \leq B_1 \leq A_2 \leq B_2 \leq \cdots$ and a sequence of intervals $\langle I_n\rangle$ satisfying:

(1) $I_1 \in \Pi_1 = \{[0,1/2],[1/2,1]\}$ and $I_n \in \bigcup_{k=1}^{n-1}\Pi_k$ when $n \geq 2$.
(2) The intervals from Π_n that are in I_n are all of the same size, and the intervals from Π_n that are in the complement of I_n are all of the same size. (These two sizes may be different.)
(3) If $x \in X_n$, then $\|x\|_n = A_n\|x1_{I_n}\|_p + B_n\|x1_{I_n^c}\|_p$.

Observe that if $A_n = B_n$ then the space is uniform. A biuniform $L_p(w)$ space is called *unbalanced* if there is a sequence of positive numbers $\epsilon_n \downarrow 0$ satisfying:

(1) For each $I \in \bigcup_{j=1}^{\infty} \Pi_j$, $I_n = I$ for infinitely many n.
(2) For $n \geq 2$, $\{I \in \Pi_n : I \subset I_n\} = \{J_1, \dots J_{N_n}\}$, where $\|N_n 1_{J_k}\|_n < \delta_n$ (i.e., $N_n^p w(J_k) < \delta_n$), with $\delta_n > 0$ being chosen such that $\operatorname{co} B_{2\delta_n} \cap S_{n-1} \subset B_{\epsilon_n} \cap S_{n-1}$.
(3) For all $x \in S_{n-1}$, we have $B_n \|1_{I_n^c} x\|_p \geq M_n \|1_{I_n^c} x\|_{n-1}$, where $M_n > (N_n^{1-p}\delta_n + 1){\epsilon_n}^{-1}$.

The construction of such spaces is discussed in [8]. In the same paper, using biuniform spaces, Roberts obtains the following result, which inspired our work in this section.

THEOREM 4.1 (ROBERTS). *Let $Y \in \mathcal{B}$. There is an unbalanced biuniform $L_p(w)$ space such that if $X = L_p(w)$ or $X = L_p(w)/\mathbf{R}1$, then $\mathcal{L}(Y, X) = \{0\}$ and Y is a quotient of X.*

CONJECTURE (ROBERTS). *Let $Y \in \mathcal{B}$. There is a uniform $L_p(w)$ space X with $\mathcal{L}(Y, X) = \{0\}$.*

The corollary to our next theorem proves this conjecture, by showing that, in some sense, uniform $L_p(w)$ spaces "repel" continuous linear operators.

THEOREM 4.2. *Let $Y \in \mathcal{B}$. There exists a uniform $L_p(w)$ space X and a one-to-one continuous linear operator $T \colon X \to Y$.*

COROLLARY. *If Z is a separable, trivial-dual p-Banach space, then there is a uniform $L_p(w)$ space X such that there are no non-zero continuous linear operators from Z into X; i.e., $\mathcal{L}(Z, X) = \{0\}$.*

Proof of the corollary, assuming the theorem. Let $Z \in \mathcal{B}$ be arbitrary. Use Theorem 4.1 to find Y, a biuniform unbalanced $L_p(w)$ space such that $\mathcal{L}(Z, Y) = \{0\}$. Find a uniform $L_p(w)$ space X and an operator T as in Theorem 4.2. Then also $\mathcal{L}(Z, X) = \{0\}$, for otherwise, if $S \in \mathcal{L}(Z, X)$ and $S \neq 0$, then $T \circ S$ is a non-trivial continuous linear operator from Z into Y, contradicting the choice of Y. $\square$

Proof of Theorem 4.2. Pick positive sequences $s_n \uparrow \infty$, $\delta_n \downarrow 0$ and $z_k \downarrow 0$ ($s_0 = \delta_0 = z_0 = 1$) such that $\langle z_k \rangle$ satisfies

$$z_n > \sum_{k>n} z_k \quad \text{for all } n.$$

Our goal is to construct X and $T \colon X \to Y$ such that:

(1) Each Π_n is obtained by dividing each interval I of Π_{n-1} into N_n equal parts. (We will denote $\prod_0^n N_j$ by M_n.)

(2) If $I \in \Pi_n$, then $w(I) := \delta_n M_n^{-p}$.
(3) Each N_n is large enough so that $N_n^{1-p} \geq 2\delta_{n-1}/\delta_n$.
(4) $\inf\{\|T(x)\|_Y : x \in S_{n-1}\} > z_{k_n} =: v_n$, where $S_n = \{x \in X_n : |||\,x\,||| = 1\}$ for all n.
(5) If $I \in \Pi_n$, then $\|T(1_I)\|_Y \leq \delta_n v_n M_n^{-p}/2$.
(6) For all n, the set of vectors $\{T(1_I) : I \in \Pi_n\}$ is linearly independent.

NOTES. (i) Since $c_n = \delta_n M_n^{1-p} = \delta_n M_{n-1}^{1-p} N_n^{1-p} \geq 2\delta_{n-1} M_{n-1}^{1-p} = 2c_{n-1}$, property (3) above ensures that $c_n \uparrow \infty$ (see Theorem 3.1).

(ii) Since $\langle v_n \rangle$ is a sub-sequence of $\langle z_k \rangle$, the sequence $\langle v_n \rangle$ also has the property that $v_m > \sum_{n>m} v_n$, for all m.

(iii) Property (4) implies that $\|T(\varphi)\|_Y > |||\,\varphi\,|||\, v_n$, for any $\varphi \in X_{n-1}$.

(iv) Property (5) ensures that $\|T(\varphi)\|_Y \leq (1/2)\, |||\,\varphi\,|||$, whenever $\varphi \in X_\infty$, so that T is bounded. To see this, take first Φ to be of the form

$$\Phi = \sum_{k=1}^{M_n} \alpha_k 1_{I_k} \in X_n.$$

For such Φ we have

$$\begin{aligned}\|T(\Phi)\|_Y &= \left\| \sum_{k=1}^{M_n} \alpha_k T(1_{I_k}) \right\|_Y \leq \sum_{k=1}^{M_n} |\alpha_k|^p \|T(1_{I_k})\|_Y \\ &\leq \frac{1}{2} v_n \sum_{k=1}^{M_n} |\alpha_k|^p \delta_n M_n^{-p} = \frac{1}{2} v_n \|\Phi\|_n \leq \frac{1}{2} \|\Phi\|_n.\end{aligned}$$

Now, if $\Phi \in X_\infty$ and $\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_n$, where $\Phi_k \in X_k$, then

$$\begin{aligned}\|T(\Phi)\|_Y &= \|T(\Phi_1) + \cdots + T(\Phi_n)\|_Y \leq \|T(\Phi_1)\|_Y + \cdots + \|T(\Phi_n)\|_Y \\ &\leq \frac{1}{2} \left(\|\Phi_1\|_1 + \cdots + \|\Phi_n\|_n \right).\end{aligned}$$

Taking the infimum we obtain

$$\|T(\Phi)\|_Y \leq \frac{1}{2}\, |||\,\Phi\,|||\,.$$

Assume now that X and T are constructed to satisfy (1)–(6) above and let us show that T is one-to-one. If $\ker T \neq \{0\}$, choose $x \in \ker T$ with $|||\,x\,||| = 1$. By a theorem of Roberts (see the remark following Lemma 3.3), x is a sum of norm-attaining elements, i.e.,

$$x = \sum_{n=0}^{\infty} x_n, \qquad x_n \in X_n,$$

and we have

$$|||\,x\,||| = \sum_{n=0}^{\infty} \|x_n\|_n$$

Now choose an index n_0 so that

$$\sum_{n>n_0} \|x_n\|_n < \frac{1}{4}.$$

Note that, as in the proof of Theorem 3.4, this implies

$$\sum_{n=0}^{n_0} \|x_n\|_n \geq \frac{3}{4}.$$

By (iii) above and since $\sum_{n=0}^{n_0} x_n \in X_{n_0}$, we have

$$\left\| T\left(\sum_{n=0}^{n_0} x_n\right)\right\|_Y > v_{n_0+1} \left|\left|\left| \sum_{n=0}^{n_0} x_n \right|\right|\right|,$$

and since

$$\left|\left|\left| \sum_{n=0}^{n_0} x_n \right|\right|\right| = \sum_{n=0}^{n_0} \|x_n\|_n \geq \frac{3}{4},$$

we obtain

$$\left\| T\left(\sum_{n=0}^{n_0} x_n\right)\right\|_Y \geq \frac{3}{4} v_{n_0+1}.$$

Next, we claim that $\|T(x_n)\| \leq v_n/8$ for any $n > n_0$. Indeed, an argument similar to the one in (iv) above shows that

$$\|T(x_n)\|_Y \leq \frac{1}{2} v_n \|x_n\|_n < \frac{1}{8} v_n,$$

since $\|x_n\|_n < 1/4$. But $T(x) = 0$ by assumption, so we have

$$T\left(\sum_{n=0}^{n_0} x_n\right) = -T\left(\sum_{n=n_0+1}^{\infty} x_n\right),$$

which implies

$$\begin{aligned} \frac{3}{4} v_{n_0+1} &\leq \left\| T\left(\sum_{n=0}^{n_0} x_n\right)\right\|_Y = \left\| T\left(\sum_{n=n_0+1}^{\infty} x_n\right)\right\|_Y \\ &\leq \sum_{n=n_0+1}^{\infty} \|T(x_n)\|_Y \leq \frac{1}{8} v_{n+1} + \frac{1}{8} \sum_{n=n_0+2}^{\infty} v_n \\ &< \frac{1}{8} v_{n_0+1} + \frac{1}{8} v_{n_0+1} = \frac{1}{4} v_{n_0+1}, \end{aligned}$$

a contradiction. This shows that $\ker T = \{0\}$ and completes the argument.

To complete the proof of Theorem 4.2, we construct inductively X and T satisfying (1)–(6). We choose $y_0 \in Y$ so that

$$\|y_0\|_Y \leq \frac{1}{2},$$

and let

$$\Pi_0 := \{[0,1]\}.$$

Note that $N_0 = w([0,1]) = 1$. Next, define

$$T_0(1_{[0,1]}) = y_0.$$

Properties (1), (3), and (4) are vacuous, while (2), (5) and (6) are satisfied with $\delta_0 = z_0 = v_0 = 1$.

Suppose now that we have constructed $X_0, \dots, X_{n-1}$ along with $T_0, \dots, T_{n-1}$ and $\langle v_j \rangle_{j=0}^{n-1}$ so that (1)–(6) hold. For the induction step, assume that

$$\Pi_{n-1} = \{I_1, \dots, I_l\}, \quad \text{where} \quad l = M_{n-1},$$

and

$$T_{n-1}(1_{I_i}) = y_i \quad \text{for} \quad 1 \le i \le l.$$

Choose $v_n = z_{k_n}$ so that

$$\inf \{\|T_{n-1}(x)\| : x \in S_{n-1}\} > z_{k_n}.$$

Note that this is possible, since S_{n-1} is compact; furthermore, because of (6), T_{n-1} is one-to-one on X_{n-1} (i.e., nonzero on S_{n-1}), so

$$m = \min \{\|T_{n-1}(x)\| : x \in S_{n-1}\}$$

exists. Since $z_k \downarrow 0$, we can assume $z_{k_n} \le m$.

Now, by trivial-duality, each y_i can be written as the average of vectors of arbitrarily small norm; i.e.,

$$y_i = \frac{1}{M_i} \sum_{j=1}^{M_i} y_{ij},$$

where we require that

$$\|y_{ij}\|_Y < \frac{1}{4} v_n \frac{\delta_n}{(N_1 N_2 \dots N_{n-1})^p}.$$

Let $M = N_n$ be a common multiple of $M_1, M_2, \dots, M_l$, large enough so that

$$N_n^{1-p} \ge 2 \frac{\delta_{n-1}}{\delta_n}.$$

Now for each i we have

$$y_i = \frac{1}{M} \sum_{j=1}^{M} w_{ij},$$

where each y_{ij} appears in the finite sequence $w_{i1}, \dots, w_{iM}$ exactly M/M_i many times.

We now modify the elements w_{ij} in order to achieve (6). For ease of notation we set

$$\nu := \frac{v_n \delta_n}{(N_1 N_2 \dots N_{n-1})^p},$$

and recall that $\|w_{ij}\|_Y < \nu/4$ for all i, j. Choose $l(M-1)$ linearly independent vectors, $\{t_j^i\}_{ij}$, each of norm less than or equal to $\nu/(4M^{1+p})$, so that

$$\operatorname{span}\left(\{w_{ij}\}_{ij}\right) \cap \operatorname{span}\left(\{t_j^i\}_{ij}\right) = \{0\},$$

where the elements t_j^i are indexed as follows:

$$t_1^{(1)}, \ldots, t_{M-1}^{(1)}, t_1^{(2)}, \ldots, t_{M-1}^{(2)}, \ldots, t_1^{(l)}, \ldots, t_{M-1}^{(l)}.$$

For each i, define

$$u_{ij} := w_{ij} + M t_j^{(i)} \quad \text{if} \quad j < M,$$

and set

$$u_{iM} := w_{iM} - M\left(\sum_{s=1}^{M-1} t_s^{(i)}\right).$$

Clearly,

$$\frac{1}{M}\sum_{j=1}^{M} u_{ij} = y_i,$$

and furthermore,

$$\|u_{ij}\|_Y \leq \|w_{ij}\|_Y + M^{1+p}\|t_j^{(i)}\|_Y \leq \frac{\nu}{4} + M^{1+p}\frac{\nu}{4M^{1+p}} = \frac{\nu}{2}.$$

The next lemma will show that the system $\{u_{ij}\}_{ij}$ is linearly independent. We define

$$\Pi_n = \{I_{ij}\}_{\substack{i=1,\ldots,l \\ j=1,\ldots,M}},$$

where, for each i, the intervals I_{ij} are obtained by dividing I_i into M intervals of equal length. Let

$$M_n = M_{n-1}M$$

and

$$w(I_{ij}) = \delta_n M_n^{-p}$$

for each i, j. Define

$$T_n(1_{I_{ij}}) = \frac{1}{M} u_{ij}$$

and note that

$$\|T_n(1_{I_{ij}})\|_Y = \frac{1}{M^p}\|u_{ij}\|_Y = N_n^{-p}\|u_{ij}\|_Y \leq N_n^{-p}\frac{\nu}{2} = \frac{1}{2}\delta_n v_n M_n^{-p}.$$

Now extend each T_n linearly to all of X_n. It is clear from the construction and from the following lemma (Lemma 4.3) that T_n satisfies (1)–(6). It is easy to check that $T_n|_{X_{n-1}} = T_{n-1}$ for all n, and we let T_∞ denote the common extension of the maps T_n on $X_\infty = \bigcup_n X_n$. Finally, we extend T_∞ to the completion of X_∞, that is, to all of X. □

LEMMA 4.3. *The system $\{u_{ij}\}_{ij}$ defined in the proof of Theorem 4.2 is linearly independent.*

Proof. Suppose

$$\sum_{i=1}^{l}\sum_{j=1}^{M}\alpha_j^{(i)}u_{ij}=0.$$

By the definition of u_{ij}, we have

$$\sum_{i=1}^{l}\left(\sum_{j=1}^{M-1}\alpha_j^{(i)}(w_{ij}+Mt_j^{(i)})+\alpha_M^{(i)}\left(w_{iM}-M\left(\sum_{s=1}^{M-1}t_s^{(i)}\right)\right)\right)=0.$$

Rearranging the terms we obtain

$$\sum_{i=1}^{l}\sum_{j=1}^{M}\alpha_j^{(i)}w_{ij}=M\sum_{i=1}^{l}\sum_{j=1}^{M-1}(\alpha_M^{(i)}-\alpha_j^{(i)})t_j^{(i)}.$$

Observe that the left hand side is now in span $\left(\{w_{ij}\}_{ij}\right)$ while the vector on the right hand side belongs to span $\left(\left\{t_j^{(i)}\right\}_{ij}\right)$, so both are zero because of the choice of the $t_j^{(i)}$. But the vectors $t_j^{(i)}$ are linearly independent, so we have, for all i,

$$\alpha_M^{(i)}=\alpha_j^{(i)}\quad(j=1,\ldots,M-1).$$

Introducing $\xi_i:=\alpha_j^{(i)}$ $(j=1,\ldots,M)$, we have

$$\sum_{i=1}^{l}\xi_i\sum_{j=1}^{M}w_{ij}=0,$$

and using

$$\sum_{j=1}^{M}w_{ij}=My_i,$$

we obtain

$$M\sum_{i=1}^{l}\xi_iy_i=0.$$

By the induction hypothesis, the vectors $\{y_i\}_i$ are linearly independent, so $\xi_i=0$ for all i. Hence $\alpha_j^{(i)}=0$ as well, and the proof is complete. □

The following result is a corollary to the previous theorem; a corresponding result for biuniform spaces was obtained by Roberts [8].

THEOREM 4.4. *Let Λ be the first uncountable ordinal. There exists a family $\{Y_\alpha:\alpha\in\Lambda\}$ such that each $Y_\alpha=L_p(w_\alpha)$ is a uniform $L_p(w)$ space, and if $\alpha<\beta$, $\mathcal{L}(Y_\alpha,Y_\beta)=\{0\}$.*

Proof. Starting with a uniform $L_p(w)$ space Y_1, we construct $\{Y_\alpha : \alpha \in \Lambda\}$ inductively as follows.

If $\beta \in \Lambda$ such that $\beta > 1$ and $\{Y_\alpha\}_\alpha$ is already defined for $\{\alpha \in \Lambda : \alpha < \beta\}$, then, since $\{\alpha \in \Lambda : \alpha < \beta\}$ is a countable collection, there is a Y_β so that

$$\mathcal{L}(Y_\alpha, Y_\beta) = \{0\} \quad \text{for all} \quad \alpha < \beta.$$

Now set

$$\bigoplus_\alpha Y_\alpha = X.$$

Then X is a separable trivial-dual p-Banach space with p-norm defined by

$$\|x\| = \sum \|x_\alpha\|_{X_\alpha}.$$

for $\langle x_\alpha \rangle \in X$. By the previous theorem there exists a uniform space Y_β with $\mathcal{L}(X, Y_\beta) = \{0\}$. This concludes the induction step, and our argument is therefore complete. □

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