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A FUNCTORIAL APPROACH TO MODULES OF G-DIMENSION ZERO

YUJI YOSHINO

Abstract. Let R be a commutative Noetherian ring and let $\mathcal{G}$ be the category of modules of G-dimension zero over R. We denote the associated stable category by $\underline{\mathcal{G}}$. We show that the functor category $\operatorname{mod}\underline{\mathcal{G}}$ is a Frobenius category and use this property to characterize $\mathcal{G}$ as a subcategory of $\operatorname{mod} R$.

1. Introduction

In this paper R always denotes a commutative Noetherian ring, and $\operatorname{mod} R$ is the category of finitely generated R-modules.

We say that an object $X \in \operatorname{mod} R$ is a module of G-dimension zero if it satisfies the conditions

$$(1) \qquad \operatorname{Ext}^i_R(X, R) = 0 \quad \text{and} \quad \operatorname{Ext}^i_R(\operatorname{Tr} X, R) = 0 \quad \text{for any} \quad i > 0,$$

as introduced in the paper [2] of Auslander and Bridger. Note that this is equivalent to saying that X is a reflexive module and, in addition, satisfies

$$\operatorname{Ext}^i_R(X, R) = \operatorname{Ext}^i_R(X^*, R) = 0 \quad \text{for any} \quad i > 0.$$

We remark that different terminology for modules of G-dimension zero is used by other authors. Actually, Avramov and Martsinkovsky [6] call such modules totally reflexive modules and Enochs and Jenda [7] call them Gorenstein projective modules.

In this paper we are mainly interested in the independence of the conditions in (1). Our motivating question is the following:

- Is the condition $\operatorname{Ext}^i_R(X, R) = 0$ for all $i > 0$ sufficient for X to be a module of G-dimension zero?

Very recently, Jorgensen and Sega [9] constructed an example in which the answer to this question is negative. However we still expect that the question will have an affirmative answer in sufficiently many cases. In fact, as one of the main theorems of this paper we shall show that this is the case if there

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are only a finite number of isomorphism classes of indecomposable modules X with $\mathrm{Ext}^i_R(X, R) = 0$ for all $i > 0$.

In this paper we introduce in a natural way two subcategories $\mathcal{G}$ and $\mathcal{H}$ of mod R, where $\mathcal{G}$ is the full subcategory of mod R consisting of all modules of G-dimension zero and $\mathcal{H}$ is the full subcategory consisting of all modules $X \in \mathrm{mod}\, R$ with $\mathrm{Ext}^i_R(X, R) = 0$ for all $i > 0$. Of course, we have the natural inclusion $\mathcal{G} \subseteq \mathcal{H}$, and we shall discuss the problem of how close $\mathcal{H}$ is to $\mathcal{G}$.

To this end, in the first half of this paper, we obtain functorial characterizations of $\mathcal{G}$ and $\mathcal{H}$ as subcategories of mod R. We need to recall several notations to make this more explicit. For any subcategory $\mathcal{C}$ of mod R, we denote by $\underline{\mathcal{C}}$ the associated stable category and by mod $\underline{\mathcal{C}}$ the category of finitely presented contravariant additive functors from $\underline{\mathcal{C}}$ to the category of Abelian groups. See §2 for the precise definitions of these associated categories, and the papers [1], [4] and [5] for a general discussion of categories of modules. As the first result of this paper we shall prove in §3 that the functor category mod $\underline{\mathcal{H}}$ is a quasi-Frobenius category, while mod $\underline{\mathcal{G}}$ is a Forbenius category; see Theorems 3.5 and 3.7.

By inspecting the proofs of these theorems, we see that the Frobenius and quasi-Frobenius property of the category mod $\underline{\mathcal{C}}$ will be a necessary condition for a general subcategory $\mathcal{C}$ of mod R to be contained in $\mathcal{G}$ and $\mathcal{H}$, respectively. We shall study this in detail in §4. To be more precise, let R be a henselian local ring, so that the category mod R is a Krull-Schmidt category. Then we shall prove in Theorem 4.2 that a resolving subcategory $\mathcal{C}$ of mod R is contained in $\mathcal{H}$ if and only if mod $\underline{\mathcal{C}}$ is a quasi-Frobenius category. In this sense, we obtain a functorial characterization of the subcategory $\mathcal{H}$ as the maximal subcategory $\mathcal{C}$ for which mod $\underline{\mathcal{C}}$ is a quasi-Frobenius category. For the subcategory $\mathcal{G}$ we give a similar functorial characterization using the Frobenius property instead of the quasi-Frobenius property, but with an additional assumption that the Auslander-Reiten conjecture is true. See Theorem 5.2 for precise statements.

In the final section, §5, we shall prove the main result of this paper, Theorem 5.5, which asserts that any resolving subcategory of finite type in $\mathcal{H}$ is contained in $\mathcal{G}$. In particular, if $\mathcal{H}$ itself is of finite type, then we deduce the equality $\mathcal{G} = \mathcal{H}$.

2. Preliminaries and notations

Let R be a commutative Noetherian ring, and let mod R be the category of finitely generated R-modules as defined in the introduction.

When we say that $\mathcal{C}$ is a subcategory of mod R, we always mean the following:

- $\mathcal{C}$ is essential in mod R, i.e., if $X \cong Y$ in mod R and if $X \in \mathcal{C}$, then $Y \in \mathcal{C}$.

- $\mathcal{C}$ is full in mod R, i.e., $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_R(X,Y)$ for $X, Y \in \mathcal{C}$.
- $\mathcal{C}$ is additive and additively closed in mod R, i.e., for any $X, Y \in$ mod R, $X \oplus Y \in \mathcal{C}$ if and only if $X \in \mathcal{C}$ and $Y \in \mathcal{C}$.

Furthermore, if all the projective modules in mod R belong to $\mathcal{C}$, then we say that $\mathcal{C}$ is a subcategory which contains the projectives.

The aim of this section is to settle the notation that will be used throughout this paper and to recall several notions of the categories associated to a given subcategory.

Let $\mathcal{C}$ be any subcategory of mod R. We first define the associated stable category $\underline{\mathcal{C}}$ as follows:

- The objects of $\underline{\mathcal{C}}$ are the same as those of $\mathcal{C}$.
- For $X, Y \in \underline{\mathcal{C}}$, the morphism set is an R-module

$$\underline{\mathrm{Hom}}_R(X,Y) = \mathrm{Hom}_R(X,Y)/P(X,Y),$$

where $P(X,Y)$ is the R-submodule of $\mathrm{Hom}_R(X,Y)$ consisting of all R-homomorphisms which factor through projective modules.

Of course, there is a natural functor $\mathcal{C} \to \underline{\mathcal{C}}$. For an object X and a morphism f in $\mathcal{C}$ we denote their images in $\underline{\mathcal{C}}$ under this natural functor by $\underline{X}$ and $\underline{f}$.

Definition 2.1. Let $\mathcal{C}$ be a subcategory of mod R. For a module X in $\mathcal{C}$, we take a finite presentation by finite projective modules

$$P_1 \to P_0 \to X \to 0,$$

and define the transpose $\mathrm{Tr}\, X$ of X as the cokernel of $\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R)$. Similarly, for a morphism $f : X \to Y$ in $\mathcal{C}$, since it induces a morphism between finite presentations

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & X & \longrightarrow & 0 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \\ Q_1 & \longrightarrow & Q_1 & \longrightarrow & Y & \longrightarrow & 0, \end{array}$$

we define the morphism $\mathrm{Tr}\, f : \mathrm{Tr}\, Y \to \mathrm{Tr}\, X$ as the morphism induced by $\mathrm{Hom}_R(f_1, R)$. It is easy to see that $\mathrm{Tr}\, X$ and $\mathrm{Tr}\, f$ are uniquely determined as an object and a morphism in the stable category $\underline{\mathcal{C}}$, so we have a well-defined functor

$$\mathrm{Tr} : (\underline{\mathcal{C}})^{op} \to \underline{\mathrm{mod}\ R}.$$

For a module $X \in \mathcal{C}$ its syzygy module ΩX is defined by the exact sequence

$$0 \to \Omega X \to P_0 \to X \to 0,$$

where P_0 is a projective module. It is also easy to see that Ω defines a functor

$$\Omega : \underline{\mathcal{C}} \to \underline{\mathrm{mod}\ R}.$$

We are interested in this paper in two particular subcategories and their associated stable categories.

Notation 2.2. We denote by $\mathcal{G}$ the subcategory of mod R consisting of all modules of G-dimension zero; that is, a module $X \in \operatorname{mod} R$ is an object in $\mathcal{G}$ if and only if

$$\operatorname{Ext}^i_R(X, R) = 0 \quad \text{and} \quad \operatorname{Ext}^i_R(\operatorname{Tr} X, R) = 0 \quad \text{for any} \quad i > 0.$$

We also denote by $\mathcal{H}$ the subcategory consisting of all modules satisfying the first half of these conditions; that is, a module $X \in \operatorname{mod} R$ is an object in $\mathcal{H}$ if and only if

$$\operatorname{Ext}^i_R(X, R) = 0 \quad \text{for any} \quad i > 0.$$

Of course we have $\mathcal{G} \subseteq \mathcal{H}$. The main motivation of this paper is to study how $\mathcal{G}$ and $\mathcal{H}$ are different by characterizing these subcategories by a functorial method.

Note that the definition of Tr already gives dualities on $\underline{\mathcal{G}}$ and $\underline{\operatorname{mod} R}$, namely, that the first and the third vertical arrows in the following diagram are isomorphisms of categories:

$$\begin{array}{ccccc} (\underline{\mathcal{G}})^{op} & \subseteq & (\underline{\mathcal{H}})^{op} & \subseteq & (\underline{\operatorname{mod} R})^{op} \\ {\scriptstyle \mathrm{Tr}}\downarrow & & {\scriptstyle \mathrm{Tr}}\downarrow & & {\scriptstyle \mathrm{Tr}}\downarrow \\ \underline{\mathcal{G}} & \subseteq & \operatorname{Tr}\underline{\mathcal{H}} & \subseteq & \underline{\operatorname{mod} R} \end{array}$$

Here we note that $\operatorname{Tr}\underline{\mathcal{H}}$ is the subcategory consisting of all modules X satisfying $\operatorname{Ext}^i_R(\operatorname{Tr} X, R) = 0$ for all $i > 0$. Hence we have the equality

$$\underline{\mathcal{G}} = \underline{\mathcal{H}} \cap \operatorname{Tr}\underline{\mathcal{H}}.$$

Therefore $\underline{\mathcal{G}} = \underline{\mathcal{H}}$ is equivalent to $\operatorname{Tr}\underline{\mathcal{H}} = \underline{\mathcal{H}}$, that is, $\underline{\mathcal{H}}$ is closed under Tr. Note also that $\underline{\mathcal{G}}$ and $\underline{\mathcal{H}}$ are closed under the syzygy functor, i.e., $\Omega\underline{\mathcal{G}} = \underline{\mathcal{G}}$ and $\Omega\underline{\mathcal{H}} \subseteq \underline{\mathcal{H}}$.

For an additive category $\mathcal{A}$, a contravariant additive functor from $\mathcal{A}$ to the category (Ab) of abelian groups is referred to as an $\mathcal{A}$-module, and a natural transform between two $\mathcal{A}$-modules is referred to as an $\mathcal{A}$-module morphism. We denote by $\operatorname{Mod}\mathcal{A}$ the category consisting of all $\mathcal{A}$-modules and all $\mathcal{A}$-module morphisms. Note that $\operatorname{Mod}\mathcal{A}$ is obviously an abelian category. An $\mathcal{A}$-module F is called finitely presented if there is an exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(\ \ , X_1) \to \operatorname{Hom}_{\mathcal{A}}(\ \ , X_0) \to F \to 0,$$

for some $X_0, X_1 \in \mathcal{A}$. We denote by $\operatorname{mod}\mathcal{A}$ the full subcategory of $\operatorname{Mod}\mathcal{A}$ consisting of all finitely presented $\mathcal{A}$-modules. For a general discussion of $\operatorname{mod}\mathcal{A}$, the reader should refer to the papers [1], [4] and [5].

Lemma 2.3 (Yoneda). *For any $X \in \mathcal{A}$ and any $F \in \operatorname{Mod}\mathcal{A}$, we have the following natural isomorphism:*

$$\operatorname{Hom}_{\operatorname{Mod}\mathcal{A}}(\operatorname{Hom}_{\mathcal{A}}(\ \ , X),\ F) \cong F(X).$$

Corollary 2.4. *An $\mathcal{A}$-module is projective in* $\operatorname{mod}\mathcal{A}$ *if and only if it is isomorphic to* $\operatorname{Hom}_{\mathcal{A}}(\ \ ,X)$ *for some* $X \in \mathcal{A}$.

Corollary 2.5. *The functor* $\mathcal{A}$ *to* $\operatorname{mod}\mathcal{A}$ *which sends* X *to* $\operatorname{Hom}_{\mathcal{A}}(\ \ ,X)$ *is a full embedding.*

Now let $\mathcal{C}$ be a subcategory of $\operatorname{mod} R$ and let $\underline{\mathcal{C}}$ be the associated stable category. Then the category of finitely presented $\mathcal{C}$-modules $\operatorname{mod}\mathcal{C}$ and the category of finitely presented $\underline{\mathcal{C}}$-modules $\operatorname{mod}\underline{\mathcal{C}}$ are defined as above. Note that for any $F \in \operatorname{mod}\mathcal{C}$ (resp. $G \in \operatorname{mod}\underline{\mathcal{C}}$) and for any $X \in \mathcal{C}$ (resp. $\underline{X} \in \underline{\mathcal{C}}$), the abelian group $F(X)$ (resp. $G(\underline{X})$) has a natural R-module structure. Hence F (resp. G) is in fact a contravariant additive functor from $\mathcal{C}$ (resp. $\underline{\mathcal{C}}$) to $\operatorname{mod} R$.

Remark 2.6. As we stated above, there is a natural functor $\mathcal{C} \to \underline{\mathcal{C}}$. We can define from this the functor $\iota : \operatorname{mod}\underline{\mathcal{C}} \to \operatorname{mod}\mathcal{C}$ by sending $F \in \operatorname{mod}\underline{\mathcal{C}}$ to the composition functor of $\mathcal{C} \to \underline{\mathcal{C}}$ with F. Then it is well known, and easy to prove, that ι gives an equivalence of categories between $\operatorname{mod}\underline{\mathcal{C}}$ and the full subcategory of $\operatorname{mod}\mathcal{C}$ consisting of all finitely presented $\mathcal{C}$-modules F with $F(R) = 0$.

We state the following lemma for a later use; the proof is straightforward and we leave it to the reader.

Lemma 2.7. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence in* $\operatorname{mod} R$. *Then we have the following:*

(1) *The induced sequence* $\underline{\operatorname{Hom}}_R(W,X) \to \underline{\operatorname{Hom}}_R(W,Y) \to \underline{\operatorname{Hom}}_R(W,Z)$ *is exact for any* $W \in \operatorname{mod} R$.
(2) *If* $\operatorname{Ext}^1_R(Z,R) = 0$, *then the induced sequence* $\underline{\operatorname{Hom}}_R(Z,W) \to \underline{\operatorname{Hom}}_R(Y,W) \to \underline{\operatorname{Hom}}_R(X,W)$ *is exact for any* $W \in \operatorname{mod} R$.

Corollary 2.8. *Let* W *be in* $\operatorname{mod} R$. *Then the covariant functor* $\underline{\operatorname{Hom}}_R(W,\ \ )$ *is a half-exact functor on* $\operatorname{mod} R$, *while the contravariant functor* $\underline{\operatorname{Hom}}_R(\ \ ,W)$ *is half-exact on* $\mathcal{H}$.

3. The Frobenius property of $\operatorname{mod}\underline{\mathcal{G}}$

Definition 3.1. Let $\mathcal{C}$ be a subcategory of $\operatorname{mod} R$.

(1) We say that $\mathcal{C}$ is closed under kernels of epimorphisms if it satisfies the following condition:

    If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\operatorname{mod} R$, and if $Y, Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

    (In Quillen's terminology, all epimorphisms from $\operatorname{mod} R$ in $\mathcal{C}$ are admissible.)

(2) We say that $\mathcal{C}$ is closed under extension or extension-closed if it satisfies the following condition:

If $0 \to X \to Y \to Z \to 0$ is an exact sequence in mod R, and if $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.

(3) We say that $\mathcal{C}$ is a resolving subcategory if $\mathcal{C}$ contains the projectives and if it is extension-closed and closed under kernels of epimorphisms.

(4) We say that $\mathcal{C}$ is closed under Ω if it satisfies the following condition:

If $0 \to X \to P \to Z \to 0$ is an exact sequence in mod R, where P is a projective module, and if $Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

Note that for a given $Z \in \mathcal{C}$, the module X in the above exact sequence is unique up to a projective summand. We denote X by ΩZ as an object in $\mathcal{C}$. Thus, $\mathcal{C}$ is closed under Ω if and only if $\Omega X \in \mathcal{C}$ whenever $X \in \mathcal{C}$.

(5) Closedness under Tr is defined similarly to (4). More precisely, we say that $\mathcal{C}$ is closed under Tr if $\operatorname{Tr} X \in \underline{\mathcal{C}}$ whenever $X \in \mathcal{C}$.

Note that the categories $\mathcal{G}$ and $\mathcal{H}$ satisfy any of the first four conditions above and that $\mathcal{G}$ is closed under Tr. We also note the following lemma.

Lemma 3.2. *Let $\mathcal{C}$ be a subcategory of* mod R *which contains the projectives.*

(1) *If $\mathcal{C}$ is closed under kernels of epimorphisms, then it is closed under Ω.*

(2) *If $\mathcal{C}$ is extension-closed and closed under Ω, then it is resolving.*

Proof. (1) Trivial.

(2) To show that $\mathcal{C}$ is closed under kernels of epimorphisms, let $0 \to X \to Y \to Z \to 0$ be an exact sequence in mod R and assume that $Y, Z \in \mathcal{C}$. Taking a projective cover $P \to Z$ and forming the pull-back diagram, we obtain the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega Z & = & \Omega Z & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathcal{C}$ is closed under Ω, we have $\Omega Z \in \mathcal{C}$. Then, since $\mathcal{C}$ is extension-closed, we have $E \in \mathcal{C}$. Noting that the middle row is a split exact sequence, we have $X \in \mathcal{C}$ since $\mathcal{C}$ is additively closed. □

We say that $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ is an exact sequence in a subcategory $\mathcal{C} \subseteq \operatorname{mod} R$ if it is an exact sequence in $\operatorname{mod} R$ and satisfies $X_i \in \mathcal{C}$ for all i.

Proposition 3.3. *Let $\mathcal{C}$ be a subcategory of* mod R *which contains the projectives and which is closed under kernels of epimorphisms.*

(1) *Then* mod $\underline{\mathcal{C}}$ *is an abelian category with enough projectives.*
(2) *For any $F \in \operatorname{mod} \underline{\mathcal{C}}$, there is a short exact sequence in $\mathcal{C}$*

$$0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$$

such that F has a projective resolution of the following type:

$$\begin{array}{l} \cdots \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,\Omega^2 X_2)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,\Omega^2 X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,\Omega^2 X_0)|_{\underline{\mathcal{C}}} \\ \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,\Omega X_2)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,\Omega X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,\Omega X_0)|_{\underline{\mathcal{C}}} \\ \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_2)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \\ \longrightarrow F \longrightarrow 0 \end{array}$$

Proof. (1) Note that mod $\underline{\mathcal{C}}$ is naturally embedded into an abelian category Mod $\underline{\mathcal{C}}$. Let $\varphi : F \to G$ be a morphism in mod $\underline{\mathcal{C}}$. It is easy to see from the definition that $\operatorname{Coker}(\varphi) \in \operatorname{mod} \underline{\mathcal{C}}$. If we prove that $\operatorname{Ker}(\varphi) \in \operatorname{mod} \underline{\mathcal{C}}$, then we see that mod $\underline{\mathcal{C}}$ is an abelian category, since it is a full subcategory of the abelian category Mod $\underline{\mathcal{C}}$ which is closed under kernels and cokernels. (See also [5, §2], in which Auslander and Reiten call this property the existence of pseudo-kernels and pseudo-cokernels.) We now prove that $\operatorname{Ker}(\varphi)$ is finitely presented.

(i) We first consider the case when F and G are projective. So let $\varphi : \underline{\mathrm{Hom}}_R(\ \ ,X_1) \to \underline{\mathrm{Hom}}_R(\ \ ,X_0)$. In this case, by Yoneda's lemma, φ is induced from $\underline{f} : \underline{X_1} \to \underline{X_0}$. If necessary, adding a projective summand to X_1, we may assume that $f : X_1 \to X_0$ is an epimorphism in mod R. Defining X_2 as the kernel of f, we have an exact sequence

$$0 \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{\ f\ } X_0 \longrightarrow 0.$$

Since $\mathcal{C}$ is closed under kernels of epimorphisms, we have $X_2 \in \mathcal{C}$. Then it follows from Lemma 2.7 that the sequence

$$\underline{\mathrm{Hom}}_R(\ \ ,X_2)|_{\underline{\mathcal{C}}} \xrightarrow{\ \psi\ } \underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \xrightarrow{\ \varphi\ } \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}}$$

is exact in mod $\underline{\mathcal{C}}$. Applying the same argument to $\operatorname{Ker}(\psi)$, we see that $\operatorname{Ker}(\varphi)$ is finitely presented as required.

(ii) We now consider a general case. The morphism $\varphi : F \to G$ induces the following commutative diagram whose horizontal sequences are finite presentations of F and G:

$$\begin{array}{ccccccc}
\underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} & \xrightarrow{a} & \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} & \xrightarrow{b} & F & \longrightarrow & 0 \\
{\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle \varphi}\downarrow & & \\
\underline{\mathrm{Hom}}_R(\ \ ,Y_1)|_{\underline{\mathcal{C}}} & \xrightarrow{c} & \underline{\mathrm{Hom}}_R(\ \ ,Y_0)|_{\underline{\mathcal{C}}} & \xrightarrow{d} & G & \longrightarrow & 0
\end{array}$$

We now define H by the following exact sequence:

$$0 \longrightarrow H \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \oplus \underline{\mathrm{Hom}}_R(\ \ ,Y_1)|_{\underline{\mathcal{C}}} \xrightarrow{(v,c)} \underline{\mathrm{Hom}}_R(\ \ ,Y_0)|_{\underline{\mathcal{C}}} \cdot$$

From the first step of this proof we have $H \in \mathrm{mod}\,\underline{\mathcal{C}}$. On the other hand, it is easy to see that there is an exact sequence:

$$\mathrm{Ker}(c) \oplus \underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \longrightarrow H \longrightarrow \mathrm{Ker}(\varphi) \longrightarrow 0\,.$$

Here, by the first step again, we have $\mathrm{Ker}(c) \in \mathrm{mod}\,\underline{\mathcal{C}}$. Since $\mathrm{mod}\,\underline{\mathcal{C}}$ is closed under cokernels in $\mathrm{Mod}\,\underline{\mathcal{C}}$, we finally obtain $\mathrm{Ker}(\varphi)$ as required.

(2) Let F be an arbitrary object in $\mathrm{mod}\,\underline{\mathcal{C}}$ with the finite presentation

$$\underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \xrightarrow{\varphi} \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \longrightarrow F \longrightarrow 0.$$

Then, as in the first step of the proof of (1), we may assume that there is a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{f} X_0 \longrightarrow 0$$

such that φ is induced by f. Applying Lemma 2.7, we obtain the following exact sequence:

$$\underline{\mathrm{Hom}}_R(\ \ ,X_2)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \xrightarrow{\varphi} \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \longrightarrow F \longrightarrow 0.$$

Similarly to the proof of Lemma 3.2(2), by taking a projective cover of X_0 and forming the pull-back, we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega X_0 & = & \Omega X_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_2 & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the second row is a split exact sequence, we get the exact sequence

$$0 \longrightarrow \Omega X_0 \longrightarrow X_2 \oplus P \longrightarrow X_1 \longrightarrow 0,$$

where P is a projective module. Then it follows from Lemma 2.7 that there is an exact sequence

$$\underline{\mathrm{Hom}}_R(\ \ ,\Omega X_0)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_2)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}}\,.$$

Continuing this procedure, we obtain the desired projective resolution of F in $\mathrm{mod}\,\underline{\mathcal{C}}$. □

Note that the proof of the first part of Theorem 3.3 is very similar to that of [11, Lemma (4.17)], where it is proved that $\mathrm{mod}\,\underline{\mathcal{C}}$ is an abelian category when R is a Cohen-Macaulay local ring and $\mathcal{C}$ is the category of maximal Cohen-Macaulay modules.

Definition 3.4. A category $\mathcal{A}$ is said to be a Frobenius category if it satisfies the following conditions:

(1) $\mathcal{A}$ is an abelian category with enough projectives and enough injectives.
(2) All projective objects in $\mathcal{A}$ are injective.
(3) All injective objects in $\mathcal{A}$ are projective.

Likewise, a category $\mathcal{A}$ is said to be a quasi-Frobenius category if it satisfies the following conditions:

(1) $\mathcal{A}$ is an abelian category with enough projectives.
(2) All projective objects in $\mathcal{A}$ are injective.

Theorem 3.5. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\,R$ *which contains the projectives and which is closed under kernels of epimorphisms. If $\mathcal{C} \subseteq \mathcal{H}$ then* $\mathrm{mod}\,\underline{\mathcal{C}}$ *is a quasi-Frobenius category.*

To prove this theorem, we first establish the following lemma. Here we recall that the full embedding $\iota : \mathrm{mod}\,\underline{\mathcal{C}} \to \mathrm{mod}\,\mathcal{C}$ is the functor induced by the natural functor $\mathcal{C} \to \underline{\mathcal{C}}$.

LEMMA 3.6. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\,R$ *which contains the projectives and which is closed under kernels of epimorphisms. Then the following conditions are equivalent for each $F \in \mathrm{mod}\,\underline{\mathcal{C}}$.*

(1) *F is an injective object in* $\mathrm{mod}\,\underline{\mathcal{C}}$.
(2) *$\iota F \in \mathrm{mod}\,\mathcal{C}$ is half-exact as a functor on $\mathcal{C}$.*

Proof. As we have shown in the previous proposition, the category $\mathrm{mod}\,\underline{\mathcal{C}}$ is an abelian category with enough projectives. Therefore an object $F \in \mathrm{mod}\,\underline{\mathcal{C}}$ is injective if and only if $\mathrm{Ext}^1_{\mathrm{mod}\,\underline{\mathcal{C}}}(G, F) = 0$ for any $G \in \mathrm{mod}\,\underline{\mathcal{C}}$. But for any given $G \in \mathrm{mod}\,\underline{\mathcal{C}}$ there is a short exact sequence in $\mathcal{C}$,

$$(*) \qquad 0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0,$$

such that G has a projective resolution

$$\underline{\mathrm{Hom}}_R(\ \ , X_2)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ , X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ , X_0)|_{\underline{\mathcal{C}}} \longrightarrow G \longrightarrow 0.$$

Conversely, for any short exact sequence in $\mathcal{C}$ such as $(*)$, the cokernel functor G of $\underline{\mathrm{Hom}}_R(\ \ , X_1)|_{\underline{\mathcal{C}}} \to \underline{\mathrm{Hom}}_R(\ \ , X_0)|_{\underline{\mathcal{C}}}$ is an object of $\mathrm{mod}\,\underline{\mathcal{C}}$. Therefore F is injective if and only if it satisfies the following condition:

The induced sequence

$$\mathrm{Hom}(\underline{\mathrm{Hom}}_R(\ \ , X_0)|_{\underline{\mathcal{C}}}, F) \to \mathrm{Hom}(\underline{\mathrm{Hom}}_R(\ \ , X_1)|_{\underline{\mathcal{C}}}, F) \to \mathrm{Hom}(\underline{\mathrm{Hom}}_R(\ \ , X_2)|_{\underline{\mathcal{C}}}, F)$$

is exact whenever $0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ *is a short exact sequence in $\mathcal{C}$.*

It follows from Yoneda's lemma that this is equivalent to saying that $F(X_0) \longrightarrow F(X_1) \longrightarrow F(X_2)$ is exact whenever $0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ is a short exact sequence in $\mathcal{C}$. This exactly means that ιF is half-exact as a functor on $\mathcal{C}$. □

Proof of Theorem 3.5. We have already shown that $\mathrm{mod}\,\underline{\mathcal{C}}$ is an abelian category with enough projectives. It remains to show that any projective module $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ $(X \in \mathcal{C})$ is an injective object in $\mathrm{mod}\,\underline{\mathcal{C}}$. Since $\mathcal{C}$ is a subcategory of $\mathcal{H}$, it follows from Corollary 2.8 that $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\mathcal{C}} = \iota(\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}})$ is a half-exact functor. Hence it is injective by the previous lemma. □

Before stating the next theorem, we remark that the syzygy functor Ω gives an automorphism on $\underline{\mathcal{G}}$.

THEOREM 3.7. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\,R$ *satisfying the following conditions:*

(1) *$\mathcal{C}$ is a resolving subcategory of* $\mathrm{mod}\,R$.

(2) $\mathcal{C} \subseteq \mathcal{H}$.
(3) *The functor* $\Omega : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ *yields a surjective map on the set of isomorphism classes of the objects in* $\underline{\mathcal{C}}$.

Then $\operatorname{mod} \underline{\mathcal{C}}$ *is a Frobenius category. In particular,* $\operatorname{mod} \underline{\mathcal{G}}$ *is a Frobenius category.*

Proof. Since $\mathcal{C}$ is subcategory of $\mathcal{H}$ that is closed under kernels of epimorphisms in $\operatorname{mod} R$, $\operatorname{mod} \underline{\mathcal{C}}$ is a quasi-Frobenius category by the previous theorem. It remains to prove that $\operatorname{mod} \underline{\mathcal{C}}$ has enough injectives and that all injectives are projective.

(i) For the first step of the proof we show that each $\underline{\mathcal{C}}$-module $F \in \operatorname{mod} \underline{\mathcal{C}}$ can be embedded into a projective $\underline{\mathcal{C}}$-module $\underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}}$ for some $Y \in \mathcal{C}$.

In fact, as we have shown in Proposition 3.3, for a given $F \in \operatorname{mod} \underline{\mathcal{C}}$, there is a short exact sequence in $\mathcal{C}$,

$$0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0,$$

such that F has a projective resolution

$$\underline{\operatorname{Hom}}_R(\ \ , X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\operatorname{Hom}}_R(\ \ , X_0)|_{\underline{\mathcal{C}}} \longrightarrow F \longrightarrow 0.$$

By the assumption (3), there is an exact sequence in $\mathcal{C}$,

$$0 \longrightarrow X_2 \longrightarrow P \longrightarrow Y \longrightarrow 0,$$

where P is a projective module. Then, similarly to the argument used in the proof of Lemma 3.2, by taking the push-out we obtain the following commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & E & \longrightarrow & X_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y & = & Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\operatorname{Ext}^1_R(X_0, P) = 0$, we see that the second row is splittable. Hence we have a short exact sequence of the type

$$0 \longrightarrow X_1 \longrightarrow X_0 \oplus P \longrightarrow Y \longrightarrow 0.$$

Therefore we obtain from Lemma 2.7 that there is an exact sequence

$$\underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}}.$$

Thus F can be embedded into $\underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}}$ as desired.

(ii) Since all projective modules in mod $\underline{\mathcal{C}}$ are injective, it follows from (i) that mod $\underline{\mathcal{C}}$ has enough injectives.

(iii) To show that every injective module in mod $\underline{\mathcal{C}}$ is projective, let F be an injective $\underline{\mathcal{C}}$-module in mod $\underline{\mathcal{C}}$. By (i), F is a $\underline{\mathcal{C}}$-submodule of $\underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}}$ for some $Y \in \mathcal{C}$. Hence F is a direct summand of $\underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}}$. Since it is a summand of a projective module, F is projective as well. □

4. Characterizing subcategories of $\mathcal{H}$

In this section we always assume that R is a henselian local ring with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. In the following, what we shall need from this assumption is the fact that $X \in \mathrm{mod}\, R$ is indecomposable only if $\mathrm{End}_R(X)$ is a (noncommutative) local ring. In fact, we will show the following lemma.

Lemma 4.1. *Let $(R, \mathfrak{m})$ be a henselian local ring. Then* mod $\underline{\mathcal{C}}$ *is a Krull-Schmidt category for any subcategory $\mathcal{C}$ of* mod R.

Proof. We only have to prove that $\mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$ is a local ring for any indecomposable $\underline{\mathcal{C}}$-module $F \in \mathrm{mod}\,\underline{\mathcal{C}}$.

First we note that $\mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$ is a module-finite algebra over R. In fact, since there is a finite presentation

$$\underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \longrightarrow F \longrightarrow 0,$$

$F(X_0)$ is a finite R-module. On the other hand, taking the dual by F of the above sequence and using Yoneda's lemma, we see that there is an exact sequence of R-modules

$$0 \longrightarrow \mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F) \longrightarrow F(X_0) \longrightarrow F(X_1).$$

As a submodule of a finite module, $\mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$ is finite over R.

Now suppose that $\mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$ is not a local ring. Then there is an element $e \in \mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$ such that both e and $1 - e$ are nonunits. Let $\overline{R}$ be the image of the natural ring homomorphism $R \to \mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$. We consider the subalgebra $\overline{R}[e]$ of $\mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$. Since $\overline{R}[e]$ is a commutative R-algebra which is finite over R, it is also henselian. Since $e, 1-e \in \overline{R}[e]$ are both nonunits, $\overline{R}[e]$ can be decomposed into a direct product of rings, by the henselian property. In particular, $\overline{R}[e]$ contains a nontrivial idempotent. This implies that $\mathrm{End}_{\mathrm{mod}\,\underline{\mathcal{C}}}(F)$ contains a nontrivial idempotent, and hence that F is decomposable in mod $\underline{\mathcal{C}}$. □

We next prove a converse of Theorem 3.5.

Theorem 4.2. *Let $\mathcal{C}$ be a resolving subcategory of* $\operatorname{mod} R$*. Suppose that* $\operatorname{mod} \underline{\mathcal{C}}$ *is a quasi-Frobenius category. Then* $\mathcal{C} \subseteq \mathcal{H}$*.*

Thus, in a sense, $\mathcal{H}$ is the largest resolving subcategory $\mathcal{C}$ of $\operatorname{mod} R$ for which $\operatorname{mod} \underline{\mathcal{C}}$ is a quasi-Frobenius category.

Proof. Let X be an indecomposable nonfree module in $\mathcal{C}$. It is sufficient to prove that $\operatorname{Ext}_R^1(X, R) = 0$.

In fact, if this is true for any $X \in \mathcal{C}$, then $\operatorname{Ext}_R^1(\Omega^i X, R) = 0$ for any $i \geq 0$, since $\mathcal{C}$ is closed under Ω. This implies that $\operatorname{Ext}_R^{i+1}(X, R) = 0$ for $i \geq 0$. Hence $X \in \mathcal{H}$.

We now assume that $\operatorname{Ext}_R^1(X, R) \neq 0$ and argue by contradiction. Let σ be a nonzero element of $\operatorname{Ext}_R^1(X, R)$ that corresponds to the nonsplit extension

$$\sigma \ : \ 0 \longrightarrow R \longrightarrow Y \xrightarrow{\ p\ } X \longrightarrow 0.$$

Since $\mathcal{C}$ is closed under extensions, we have $Y \in \mathcal{C}$. On the other hand, noting that R is the zero object in $\underline{\mathcal{C}}$, we have from Lemma 2.7 that there is an exact sequence of $\underline{\mathcal{C}}$-modules

$$0 \longrightarrow \underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} \xrightarrow{\ p_*\ } \underline{\operatorname{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}.$$

Since $\operatorname{mod} \underline{\mathcal{C}}$ is a quasi-Frobenius category, this monomorphism is a split one. Hence $\underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}}$ is a direct summand of $\underline{\operatorname{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ through p_*. Since the embedding $\underline{\mathcal{C}} \to \operatorname{mod} \underline{\mathcal{C}}$ is full, and since we assumed that X is indecomposable, $\underline{\operatorname{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ is indecomposable in $\operatorname{mod} \underline{\mathcal{C}}$ as well. Hence p_* is either an isomorphism or $p_* = 0$.

We first consider the case when p_* is an isomorphism. In this case, we can take a morphism $q \in \operatorname{Hom}_R(X, Y)$ such that $q_* : \underline{\operatorname{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}} \to \underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}}$ is the inverse of p_*. Then $p_* q_* = (pq)_*$ is the identity on $\underline{\operatorname{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$. Since $\underline{\operatorname{End}}_R(X) \cong \operatorname{End}(\underline{\operatorname{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}})$, we see that $\underline{pq} = \underline{1}$ in $\underline{\operatorname{End}}_R(X)$. Since $\operatorname{End}_R(X)$ is a local ring and since $\underline{\operatorname{End}}_R(X)$ is a residue ring of $\operatorname{End}_R(X)$, we see that $pq \in \operatorname{End}_R(X)$ is a unit. This shows that the extension σ splits, which is a contradiction. Hence this case never occurs.

We therefore have $p_* = 0$, which implies $\underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} = 0$. Since the embedding $\underline{\mathcal{C}} \to \operatorname{mod} \underline{\mathcal{C}}$ is full, this is equivalent to saying that Y is a projective, and hence free, module. Thus it follows from the extension σ that X has projective dimension exactly one. (We have assumed that X is nonfree.) Thus the extension σ is a minimal free resolution of X:

$$0 \longrightarrow R \xrightarrow{\ \alpha\ } R^r \xrightarrow{\ p\ } X \longrightarrow 0,$$

where $\alpha = (a_1, \cdots, a_r)$ is a matrix with entries in $\mathfrak{m}$. Now let $x \in \mathfrak{m}$ be any element and let us consider the extension corresponding to $x\sigma \in \operatorname{Ext}_R^1(X, R)$. By forming a push-out, we obtain this extension as the second row in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
\sigma & : & 0 & \longrightarrow & R & \xrightarrow{\alpha} & R^r & \xrightarrow{p} & X & \longrightarrow & 0 \\
 & & & & {\scriptstyle x}\downarrow & & \downarrow & & \| & & \\
x\sigma & : & 0 & \longrightarrow & R & \longrightarrow & Z & \xrightarrow{p'} & X & \longrightarrow & 0
\end{array}$$

Note that there is an exact sequence

$$R \xrightarrow{(x,\alpha)} R \oplus R^r \longrightarrow Z \longrightarrow 0,$$

where all entries of the matrix (x, α) are in $\mathfrak{m}$. Thus the R-module Z is not free, and therefore $p'_* : \underline{\mathrm{Hom}}_R(\ \ , Z)|_{\underline{\mathcal{C}}} \to \underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ is a nontrivial monomorphism. Then, applying the argument of the first case to the extension $x\sigma$, we see that $x\sigma = 0$ in $\mathrm{Ext}^1_R(X, R)$. Since this is true for any $x \in \mathfrak{m}$ and for any $\sigma \in \mathrm{Ext}^1_R(X, R)$, we obtain that $\mathfrak{m}\,\mathrm{Ext}^1_R(X, R) = 0$. On the other hand, by computation, we have $\mathrm{Ext}^1_R(X, R) \cong R/(a_1, \dots, a_r)$, and hence we must have $\mathfrak{m} = (a_1, \dots, a_r)R$. Since the residue field k has a free resolution of the form

$$R^r \xrightarrow{{}^t(a_1,\dots,a_r)} R \longrightarrow k \longrightarrow 0,$$

comparing this with the extension σ, we see that $X \cong \mathrm{Tr}\, k$. What we have proved so far is the following :

> Suppose there is an indecomposable nonfree module X in $\mathcal{C}$ which satisfies $\mathrm{Ext}^1_R(X, R) \neq 0$. Then X is isomorphic to $\mathrm{Tr}\, k$ as an object in $\underline{\mathcal{C}}$ and X has projective dimension one.

If R is a field then the theorem is obviously true. So we assume that the local ring R is not a field. Then we can find an indecomposable R-module L of length 2 and a nonsplit exact sequence

$$0 \longrightarrow k \longrightarrow L \longrightarrow k \longrightarrow 0.$$

Note that $L = R/I$ for some $\mathfrak{m}$-primary ideal I. Note also that $\operatorname{depth} R \geq 1$, since there is a module of projective dimension one. Therefore there is no nontrivial R-homomorphism from L to R, and thus we have $\underline{\mathrm{End}}_R(L) = \mathrm{End}_R(L) \cong R/I$. Similarly, we have $\underline{\mathrm{End}}_R(k) \cong k$.

It also follows from $\mathrm{Hom}_R(k, R) = 0$ that there is an exact sequence of the following type:

$$(*) \quad 0 \longrightarrow \mathrm{Tr}\, k \longrightarrow \mathrm{Tr}\, L \oplus P \longrightarrow \mathrm{Tr}\, k \longrightarrow 0,$$

where P is a suitable free module. Since $X = \mathrm{Tr}\, k$ is in $\mathcal{C}$, and since $\mathcal{C}$ is extension-closed, we have $\mathrm{Tr}\, L \in \mathcal{C}$ as well. Note that Tr is a duality on $\underline{\mathrm{mod}\ R}$. Hence we see that $\underline{\mathrm{End}}_R(\mathrm{Tr}\, L) \cong \underline{\mathrm{End}}_R(L)$, which is a local ring. As a consequence we see that $\mathrm{Tr}\, L$ is indecomposable in $\underline{\mathcal{C}}$.

We claim that $\mathrm{Tr}\, L$ has projective dimension exactly one, and hence, in particular, $\mathrm{Ext}^1_R(\mathrm{Tr}\, L, R) \neq 0$. In fact, we have from $(*)$ that $\mathrm{Tr}\, L$ has projective dimension at most one, and the same holds for $X = \mathrm{Tr}\, k$. If $\mathrm{Tr}\, L$

were free, then $X = \mathrm{Tr}\, k$ would be its own first syzygy by $(*)$ and hence free because X has projective dimension one. But this is a contradiction.

Thus it follows from the above claim that $\mathrm{Tr}\, L$ is isomorphic to $\mathrm{Tr}\, k$ in $\underline{\mathrm{mod}\, R}$. Taking the transpose again, we finally obtain that L is isomorphic to k in $\underline{\mathrm{mod}\, R}$. But this is absurd because $\underline{\mathrm{End}}_R(L) \cong R/I$ and $\underline{\mathrm{End}}_R(k) \cong k$. Thus the proof is complete. □

Corollary 4.3. *The following conditions are equivalent for a henselian local ring R.*

(1) $\mathrm{mod}\, (\underline{\mathrm{mod}\, R})$ *is a quasi-Frobenius category.*
(2) $\mathrm{mod}\, (\underline{\mathrm{mod}\, R})$ *is a Frobenius category.*
(3) *R is an artinian Gorenstein ring.*

Proof. (3) $\Rightarrow$ (2): If R is an artinian Gorenstein ring, then we have $\mathrm{mod}\, R = \mathcal{G}$. Hence this implication follows from Theorem 3.7.

(2) $\Rightarrow$ (1): Obvious.

(1) $\Rightarrow$ (3): Suppose that $\mathrm{mod}\, (\underline{\mathrm{mod}\, R})$ is a quasi-Frobenius category. Then, by Theorem 4.2, any indecomposable R-module X is in $\mathcal{H}$. In particular, the residue field $k = R/\mathfrak{m}$ is in $\mathcal{H}$. Hence, by definition, $\mathrm{Ext}^i_R(k, R) = 0$ for any $i > 0$. This happens only if R is an artinian Gorenstein ring. □

5. Characterizing subcategories of $\mathcal{G}$

Lemma 5.1. *Let R be a henselian local ring and let $\mathcal{C}$ be an extension-closed subcategory of* $\mathrm{mod}\, R$. *For objects $\underline{X}, \underline{Y} \in \underline{\mathcal{C}}$, we assume the following:*

(1) *There is a monomorphism φ in* $\mathrm{Mod}\, \underline{\mathcal{C}}$*:*

$$\varphi : \underline{\mathrm{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} \to \mathrm{Ext}^1(\ \ , X)|_{\underline{\mathcal{C}}}$$

(2) *$\underline{X}$ is indecomposable in $\underline{\mathcal{C}}$.*
(3) *$\underline{Y} \not\cong 0$ in $\underline{\mathcal{C}}$.*

Then the module X is isomorphic to a direct summand of ΩY.

Proof. We note from Yoneda's lemma that there is an element $\sigma \in \mathrm{Ext}^1_R(Y, X)$ which corresponds to the short exact sequence

$$\sigma : 0 \longrightarrow X \xrightarrow{\ a\ } L \xrightarrow{\ p\ } Y \longrightarrow 0$$

such that φ is induced by σ as follows:

For any $W \in \mathcal{C}$ and for any $\underline{f} \in \underline{\mathrm{Hom}}_R(W, Y)$, consider the pull-back diagram to get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & L & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

Defining the second exact sequence as $f^*\sigma$, we have $\varphi(\underline{f}) = f^*\sigma$.

Note that $L \in \mathcal{C}$, since $\mathcal{C}$ is extension-closed. Also note that σ is nonsplit. In fact, if σ splits, then φ is the zero map. Hence $\underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}} = 0$ from the assumption. Since the embedding $\underline{\mathcal{C}} \to \mathrm{mod}\,\underline{\mathcal{C}}$ is full, this implies that $\underline{Y} = \underline{0}$ in $\underline{\mathcal{C}}$, which is a contradiction.

Now let $P \to Y$ be a surjective R-module homomorphism, where P is a projective module. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a} & L & \xrightarrow{p} & Y & \longrightarrow & 0 \\ & & {\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow & & \| & & \\ 0 & \longrightarrow & \Omega Y & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & Y & \longrightarrow & 0 \end{array}$$

Note that the extension σ induces the following exact sequence of $\underline{\mathcal{C}}$-modules:

$$\underline{\mathrm{Hom}}_R(\ \ ,L)|_{\underline{\mathcal{C}}} \xrightarrow{p_*} \underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}} \xrightarrow{\varphi} \mathrm{Ext}^1_R(\ \ ,X)|_{\underline{\mathcal{C}}}$$

Since φ is a monomorphism, the morphism $\underline{\mathrm{Hom}}_R(\ \ ,L)|_{\underline{\mathcal{C}}} \to \underline{\mathrm{Hom}}_R(\ \ ,Y)|_{\underline{\mathcal{C}}}$ is the zero morphism. In particular, the map $\underline{p} \in \underline{\mathrm{Hom}}_R(L,Y)$ is the zero element by Yoneda's lemma. (Note that here we use the fact that $L \in \mathcal{C}$.) This is equivalent to saying that $p : L \to Y$ factors through a projective module, and hence through the map π. As a consequence, there are maps $k : L \to P$ and $\ell : X \to \Omega Y$ which make the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a} & L & \xrightarrow{p} & Y & \longrightarrow & 0 \\ & & {\scriptstyle \ell}\downarrow & & {\scriptstyle k}\downarrow & & \| & & \\ 0 & \longrightarrow & \Omega Y & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & Y & \longrightarrow & 0 \end{array}$$

Then, since $p(1-hk) = 0$, there is a map $b : L \to X$ such that $1 - hk = ab$. Likewise, since $\pi(1-kh) = 0$, there is a map $\beta : P \to \Omega Y$ such that $1 - kh = \alpha\beta$. Note that we have the following equalities:

$$a(1-g\ell) = a - ag\ell = a - h\alpha\ell = a - hka = (1-hk)a = aba.$$

Since a is a monomorphism, we have $1 - g\ell = ba$. Thus we finally obtain the equality $1 = ba + g\ell$ in the local ring $\mathrm{End}_R(X)$.

Since σ is a nonsplit sequence, $ba \in \mathrm{End}_R(X)$ can never be a unit, and it follows that $g\ell$ is a unit in $\mathrm{End}_R(X)$. This means that the map $g : \Omega Y \to X$ is a split epimorphism. Hence X is isomorphic to a direct summand of ΩY as desired. ☐

THEOREM 5.2. *Let R be a henselian local ring. Suppose that $\mathcal{C}$ satisfies the following conditions:*

(1) *$\mathcal{C}$ is a resolving subcategory of* $\mathrm{mod}\,R$.

(2) $\mathrm{mod}\,\underline{\mathcal{C}}$ *is a Frobenius category.*

(3) *There is no nonprojective module $X \in \mathcal{C}$ with* $\mathrm{Ext}^1_R(\ \ ,X)|_{\underline{\mathcal{C}}} = 0$.

Then $\mathcal{C} \subseteq \mathcal{G}$.

Proof. As the first step of the proof, we prove the following:

(i) For a nontrivial indecomposable object $\underline{X} \in \underline{\mathcal{C}}$ there is an object $\underline{Y} \in \underline{\mathcal{C}}$ such that X is isomorphic to a direct summand of ΩY.

To prove this, let $\underline{X} \in \underline{\mathcal{C}}$ be nontrivial and indecomposable. Consider the $\underline{\mathcal{C}}$-module $F := \operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}}$. The third assumption ensures that F is a nontrivial $\underline{\mathcal{C}}$-module. Hence there is an indecomposable module $W \in \underline{\mathcal{C}}$ such that $F(W) \neq 0$. Take a nonzero element σ in $F(W) = \operatorname{Ext}^1_R(W, X)$ that corresponds to an exact sequence

$$0 \longrightarrow X \longrightarrow E \longrightarrow W \longrightarrow 0.$$

Note that $E \in \mathcal{C}$, since $\mathcal{C}$ is extension-closed. Then we have an exact sequence of $\underline{\mathcal{C}}$-modules

$$\underline{\operatorname{Hom}}_R(\ \ , E)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\operatorname{Hom}}_R(\ \ , W)|_{\underline{\mathcal{C}}} \xrightarrow{\ \varphi\ } \operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}}\,.$$

We denote by F_σ the image of φ. Of course, F_σ is a nontrivial $\underline{\mathcal{C}}$-submodule of F which is finitely presented. Since we assume that $\operatorname{mod}\underline{\mathcal{C}}$ is a Frobenius category, we can take a minimal injective hull of F_σ that is projective as well, i.e., there is a monomorphism $i : F_\sigma \to \underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}}$ for some $Y \in \underline{\mathcal{C}}$ that is an essential extension. Since $\operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}}$ is half-exact as a functor on $\mathcal{C}$, we see as in the proof of Lemma 3.6 that $\operatorname{Hom}(\ \ , \operatorname{Ext}^1_R(\ \ , X))$ is an exact functor on $\operatorname{mod}\underline{\mathcal{C}}$. It follows from this that the natural embedding $F_\sigma \to F$ can be enlarged to the morphism $g : \underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} \to F$. Hence there is a commutative diagram

$$\begin{array}{ccc} F_\sigma & \xrightarrow[i]{\subset} & \underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}} \\ \cap\big\downarrow & & g\big\downarrow \\ F & = & \operatorname{Ext}^1_R(\ \ , X). \end{array}$$

Since i is an essential extension, we see that $\operatorname{Ker}\ g = 0$. Hence $\underline{\operatorname{Hom}}_R(\ \ , Y)|_{\underline{\mathcal{C}}}$ is a submodule of F. Therefore, by the previous lemma, X is isomorphic to a direct summand of ΩY. Thus claim (i) is proved.

We now prove the theorem. Since $\operatorname{mod}\underline{\mathcal{C}}$ is a quasi-Frobenius category, we know from Theorem 4.2 that $\mathcal{C} \subseteq \mathcal{H}$. To show that $\mathcal{C} \subseteq \mathcal{G}$, let $\underline{X}$ be a nontrivial indecomposable module in $\underline{\mathcal{C}}$. We want to prove that $\operatorname{Ext}^i_R(\operatorname{Tr} X, R) = 0$ for $i > 0$. It follows from claim (i) that there is $Y \in \underline{\mathcal{C}}$ such that X is a direct summand of ΩY. Note that $Y \in \mathcal{H}$. From the obvious sequence

$$0 \longrightarrow \Omega Y \longrightarrow P \longrightarrow Y \longrightarrow 0,$$

where P is a projective module, it is easy to see that there is an exact sequence of the type

$$(*) \qquad 0 \longrightarrow \operatorname{Tr} Y \longrightarrow P' \longrightarrow \operatorname{Tr}\Omega Y \longrightarrow 0,$$

where P' is projective. Since ΩY is a torsion-free module, it is obvious that $\mathrm{Ext}^1_R(\mathrm{Tr}\,\Omega Y, R) = 0$. Since X is a direct summand of ΩY, we have $\mathrm{Ext}^1_R(\mathrm{Tr}\,X, R) = 0$ as well. This is true for any indecomposable module in $\underline{\mathcal{C}}$, and hence for each indecomposable summand of Y. Therefore we have $\mathrm{Ext}^1_R(\mathrm{Tr}\,Y, R) = 0$. It follows from this together with $(*)$ that $\mathrm{Ext}^2_R(\mathrm{Tr}\,\Omega Y, R) = 0$. Hence we have $\mathrm{Ext}^2_R(\mathrm{Tr}\,X, R) = 0$. Continuing this procedure, we obtain $\mathrm{Ext}^i_R(\mathrm{Tr}\,X, R) = 0$ for any $i > 0$. □

Remark 5.3. We conjecture that $\mathcal{G}$ is the largest resolving subcategory $\mathcal{C}$ of mod R such that mod $\underline{\mathcal{C}}$ is a Frobenius category. Theorem 5.2 together with Theorem 3.7 say that this is true modulo the Auslander-Reiten conjecture:

(AR) If $\mathrm{Ext}^i_R(X, X \oplus R) = 0$ for any $i > 0$, then X is projective.

In fact, if the conjecture (AR) is true, then the third condition of the previous theorem is automatically satisfied.

Definition 5.4. Let $\mathcal{A}$ be any additive category. We denote by $\mathrm{Ind}(\mathcal{A})$ the set of nonisomorphic modules which represent all the isomorphism classes of indecomposable objects in $\mathcal{A}$. If $\mathrm{Ind}(\mathcal{A})$ is a finite set, then we say that $\mathcal{A}$ is a category of finite type.

The following theorem is the main result of this paper and shows that any resolving subcategory of finite type in $\mathcal{H}$ is contained in $\mathcal{G}$.

Theorem 5.5. *Let R be a henselian local ring and let $\mathcal{C}$ be a subcategory of* mod R *which satisfies the following conditions:*

(1) *$\mathcal{C}$ is a resolving subcategory of* mod R.
(2) $\mathcal{C} \subseteq \mathcal{H}$.
(3) *$\mathcal{C}$ is of finite type.*

Then mod $\underline{\mathcal{C}}$ *is a Frobenius category and* $\mathcal{C} \subseteq \mathcal{G}$.

Lemma 5.6. *Let $\mathcal{C}$ be a subcategory of* mod R *which contains the projectives and suppose that $\mathcal{C}$ is of finite representation type. Then the following conditions are equivalent for a contravariant additive functor F from $\underline{\mathcal{C}}$ to* mod R.

(1) *F is finitely presented, i.e.,* $F \in \mathrm{mod}\,\underline{\mathcal{C}}$.
(2) *$F(W)$ is a finitely generated R-module for each* $W \in \mathrm{Ind}(\underline{\mathcal{C}})$.

Proof. The implication (1) ⇒ (2) follows trivially from the definition. We prove (2) ⇒ (1). For this, note that for each $X, W \in \mathrm{Ind}(\underline{\mathcal{C}})$ and for each $\underline{f} \in \underline{\mathrm{Hom}}_R(X, W)$, the induced map $F(f) : F(W) \to F(X)$ is an R-module homomorphism and satisfies $F(af) = aF(f)$ for $a \in R$. Therefore the $\underline{\mathcal{C}}$-module homomorphism

$$\varphi_W : \underline{\mathrm{Hom}}_R(\ \ , W) \otimes_R F(W) \to F,$$

which sends $f \otimes x$ to $F(f)(x)$, is well-defined. Now let $\{W_1, \cdots, W_m\}$ be the complete list of elements in $\mathrm{Ind}(\underline{\mathcal{C}})$. Then the $\underline{\mathcal{C}}$-module homomorphism

$$\Phi = \oplus_{i=1}^{m} \varphi_{W_i} : \oplus_{i=1}^{m} \underline{\mathrm{Hom}}_R(\ \ , W_i) \otimes_R F(W_i) \to F$$

is defined, and it is clear that Φ is an epimorphism in $\mathrm{Mod}\,\underline{\mathcal{C}}$. Therefore F is finitely generated, and this is true for $\mathrm{Ker}(\Phi)$ as well. Hence F is finitely presented. □

LEMMA 5.7. *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\, R$ *which contains the projectives and which is of finite type, and let $\underline{X}$ be a nontrivial indecomposable module in $\underline{\mathcal{C}}$. Suppose that $\mathcal{C}$ is closed under kernels of epimorphisms. Then $\mathcal{C}$ admits an AR-sequence ending in X; that is, there is a nonsplit exact sequence in $\mathcal{C}$,*

$$0 \longrightarrow \tau X \longrightarrow L \xrightarrow{\ p\ } X \longrightarrow 0,$$

such that for any indecomposable $Y \in \mathcal{C}$ and for any morphism $f : Y \to X$ which is not a split epimorphism there is a morphism $g : Y \to L$ that makes the following diagram commutative:

$$\begin{array}{ccc} L & \xrightarrow{\ p\ } & X \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f} \\ Y & = & Y \end{array}$$

Proof. Let rad $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ be the radical functor of $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$; that is, for each $\underline{W} \in \mathrm{Ind}(\underline{\mathcal{C}})$, if $\underline{W} \not\cong \underline{X}$, then rad $\underline{\mathrm{Hom}}_R(W, X) = \underline{\mathrm{Hom}}_R(W, X)$, whereas if $\underline{W} = \underline{X}$, then rad $\underline{\mathrm{Hom}}_R(X, X)$ is the unique maximal ideal of $\underline{\mathrm{End}}_R(X)$. Since rad $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ is a $\underline{\mathcal{C}}$-submodule of $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$, it follows from the previous lemma that rad $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ is finitely presented. Hence there is an $\underline{L} \in \underline{\mathcal{C}}$ and a morphism $\underline{p} : \underline{L} \to \underline{X}$ such that $p_* : \underline{\mathrm{Hom}}_R(\ \ , L)|_{\underline{\mathcal{C}}} \to$ rad $\underline{\mathrm{Hom}}_R(\ \ , X)|_{\underline{\mathcal{C}}}$ is an epimorphism. Adding a projective summand to L if necessary, we may assume that the R-module homomorphism $p : L \to X$ is surjective. Defining $\tau X = \mathrm{Ker}(p)$, we see that $\tau X \in \mathcal{C}$, since $\mathcal{C}$ is closed under kernels of epimorphisms. It is clear that the obtained sequence $0 \to \tau X \to L \to X \to 0$ satisfies the condition defining an AR-sequence. □

See [1] and [4] for a discussion of AR-sequences.

LEMMA 5.8. *Let $\mathcal{C}$ be a resolving subcategory of* $\mathrm{mod}\, R$. *Suppose that $\mathcal{C}$ is of finite type. Then for any $X, Y \in \mathcal{C}$ the R-module* $\mathrm{Ext}_R^1(X, Y)$ *is of finite length.*

Proof. It is sufficient to prove the lemma in the case when X and Y are indecomposable. For any $x \in \mathfrak{m}$ and for any $\sigma \in \mathrm{Ext}_R^1(X, Y)$ it is enough to show that $x^n \sigma = 0$ for a large integer n.

Now suppose that $x^n\sigma \neq 0$ for any integer n. We shall show that this leads to a contradiction. Let us take an AR-sequence ending in X as in the previous lemma,

$$\alpha: \ 0 \longrightarrow \tau X \longrightarrow L \xrightarrow{\ p\ } X \longrightarrow 0,$$

and a short exact sequence that corresponds to each $x^n\sigma \in \operatorname{Ext}^1_R(X, Y)$,

$$x^n\sigma: \ 0 \longrightarrow Y \longrightarrow L_n \xrightarrow{\ p_n\ } X \longrightarrow 0.$$

Since p_n is not a split epimorphism, the following commutative diagram is induced:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & L_n & \xrightarrow{p_n} & X & \longrightarrow & 0 \\ & & {\scriptstyle h_n}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \tau X & \longrightarrow & L & \xrightarrow{p} & X & \longrightarrow & 0 \end{array}$$

The morphism h_n induces an R-module map

$$(h_n)_* : \operatorname{Ext}^1_R(X, Y) \longrightarrow \operatorname{Ext}^1_R(X, \tau X),$$

which sends $x^n\sigma$ to the AR-sequence α. Since $(h_n)_*$ is R-linear, we have $\alpha = x^n(h_n)_*(\sigma) \in x^n \operatorname{Ext}^1_R(X, \tau X)$. Note that this is true for any integer n and that $\bigcap_{i=1}^{\infty} x^n \operatorname{Ext}^1_R(X, \tau X) = (0)$. Therefore we must have $\alpha = 0$. This contradicts the fact that α is a nonsplit exact sequence. □

Remark 5.9. Compare the proof of the above lemma with that of [11, Theorem (3.4)]. We also note that in the case when $\mathcal{C}$ is the subcategory of maximal Cohen-Macaulay modules this lemma has been proved in Huneke and Leuschke [8].

We now proceed to the proof of Theorem 5.5. For this, let $\mathcal{C}$ be a subcategory of mod R that satisfies the three conditions of the theorem. The proof will be given in several steps.

For the first step we show:

Step 1. *The category* mod $\underline{\mathcal{C}}$ *is a quasi-Frobenius category.*

Proof. This follows from Theorem 3.5 since $\mathcal{C}$ is a resolving subcategory of $\mathcal{H}$. □

Next we prove the following claim:

Step 2. *Any $\underline{\mathcal{C}}$-module $F \in$* mod $\underline{\mathcal{C}}$ *can be embedded in an injective $\underline{\mathcal{C}}$-module of the form* $\operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}}$ *for some $\underline{X} \in \underline{\mathcal{C}}$. In particular,* mod $\underline{\mathcal{C}}$ *has enough injectives.*

Proof. As we have shown in Lemma 3.3, for a given $F \in$ mod $\underline{\mathcal{C}}$ there is a short exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$$

such that a projective resolution of F in $\mathrm{mod}\,\underline{\mathcal{C}}$ is given as in Lemma 3.3(2). It is easy to see from the above exact sequence that there is an exact sequence

$$\cdots \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_1)|_{\underline{\mathcal{C}}} \longrightarrow \underline{\mathrm{Hom}}_R(\ \ ,X_0)|_{\underline{\mathcal{C}}} \longrightarrow \mathrm{Ext}^1_R(\ \ ,X_2)|_{\underline{\mathcal{C}}} \longrightarrow \cdots$$

Hence there is a monomorphism $F \to \mathrm{Ext}^1_R(\ \ ,X_2)|_{\underline{\mathcal{C}}}$. Note that $\mathrm{Ext}^1_R(W,X_2)$ is a finitely generated R-module for each $\underline{W} \in \mathrm{Ind}(\underline{\mathcal{C}})$. Hence it follows from Lemma 5.6 that $\mathrm{Ext}^1_R(\ \ ,X_2)|_{\underline{\mathcal{C}}} \in \mathrm{mod}\,\underline{\mathcal{C}}$. On the other hand, since $\mathrm{Ext}^1_R(\ \ ,X_2)$ is a half-exact functor on $\mathcal{C}$, we see from Lemma 3.6 that $\mathrm{Ext}^1_R(\ \ ,X_2)|_{\underline{\mathcal{C}}}$ is an injective object in $\mathrm{mod}\,\underline{\mathcal{C}}$. □

STEP 3. *For each indecomposable module $X \in \mathcal{C}$, the $\underline{\mathcal{C}}$-module $\mathrm{Ext}^1_R(\ \ ,X)|_{\underline{\mathcal{C}}}$ is projective in* $\mathrm{mod}\,\underline{\mathcal{C}}$. *In particular,* $\mathrm{mod}\,\underline{\mathcal{C}}$ *is a Frobenius category.*

Proof. For the proof, we denote the finite set $\mathrm{Ind}(\underline{\mathcal{C}})$ by $\{\underline{W_1},\dots,\underline{W_m}\}$, where $m = |\,\mathrm{Ind}(\underline{\mathcal{C}})|$. Defining $E := \mathrm{Ext}^1_R(\ \ ,W_i)$ for any value of i $(1 \le i \le m)$, we need to show that E is projective in $\mathrm{mod}\,\underline{\mathcal{C}}$.

Firstly, we show that E is of finite length as an object in the abelian category $\mathrm{mod}\,\underline{\mathcal{C}}$; that is, there is no infinite sequence of strict submodules

$$E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots .$$

To show this, set $W = \oplus_{i=1}^m W_i$ and consider the sequence of R-submodules

$$E(W) \supset E_1(W) \supset E_2(W) \supset \cdots \supset E_n(W) \supset \cdots$$

Since we have shown in Lemma 5.8 that $E(W) = \mathrm{Ext}^1_R(W,X)$ is an R-module of finite length, this sequence will terminate, i.e., there is an integer n such that $E_n(W) = E_{n+1}(W) = E_{n+2}(W) = \cdots$. Since W contains every indecomposable module in $\underline{\mathcal{C}}$, this implies that $E_n = E_{n+1} = E_{n+2} = \cdots$ as functors on $\underline{\mathcal{C}}$. Therefore E is of finite length.

In particular, E contains a simple module in $\mathrm{mod}\,\underline{\mathcal{C}}$ as a submodule.

Now note that there are only m nonisomorphic indecomposable projective modules in $\mathrm{mod}\,\underline{\mathcal{C}}$; in fact, they are $\underline{\mathrm{Hom}}_R(\ \ ,W_i)|_{\underline{\mathcal{C}}}$ $(i = 1,2,\dots,m)$. Corresponding to these indecomposable projectives, there are only m nonisomorphic simple modules in $\mathrm{mod}\,\underline{\mathcal{C}}$, which are

$$S_i = \underline{\mathrm{Hom}}_R(\ \ ,W_i)|_{\underline{\mathcal{C}}} \ /\ \mathrm{rad}\ \underline{\mathrm{Hom}}_R(\ \ ,W_i)|_{\underline{\mathcal{C}}} \quad (i = 1,2,\dots,m).$$

Since we have shown in Steps 1 and 2 that $\mathrm{mod}\,\underline{\mathcal{C}}$ is an abelian category with enough projectives and enough injectives, each simple module S_i has the injective hull $I(S_i)$ for $i = 1,2,\dots,m$.

Since we have proved that E is an injective module of finite length, we see that E is a finite direct sum of $I(S_i)$ $(i = 1,2,\dots,m)$. Since any module in $\mathrm{mod}\,\underline{\mathcal{C}}$ can be embedded into a direct sum of injective modules of the form $E = \mathrm{Ext}^1_R(\ \ ,W_i)$, we conclude that all nonismorphic indecomposable injective modules in $\mathrm{mod}\,\underline{\mathcal{C}}$ are $I(S_i)$ $(i = 1,2,\dots,m)$. Note that these are exactly m in number.

Since $\operatorname{mod} \underline{\mathcal{C}}$ is a quasi-Frobenius category, all indecomposable projective modules in $\operatorname{mod} \underline{\mathcal{C}}$ are indecomposable injective. Hence the following two sets coincide:

$$\{\underline{\operatorname{Hom}}_R(\ \ , W_i)|_{\underline{\mathcal{C}}} \mid i = 1, 2, \ldots, m\} = \{I(S_i) \mid i = 1, 2, \ldots, m\}.$$

As a result, every injective module is projective. We have thus shown that $\operatorname{mod} \underline{\mathcal{C}}$ is a Frobenius category. □

Remark 5.10. We remark that the proof of Step 3 is the same as the proof of Nakayama's theorem, which states the following:

> *Let A be a finite dimensional algebra over a field. Then A is left selfinjective if and only A is right selfinjective. In particular,* $\operatorname{mod} A$ *is a quasi-Frobenius category if and only if so is* $\operatorname{mod} A^{op}$*, and in this case* $\operatorname{mod} A$ *is a Frobenius category.*

See [10], for example.

We now proceed to the final step of the proof. If we prove the following claim, then the category $\mathcal{C}$ satisfies all the assumptions in Theorem 5.2, and hence we obtain $\mathcal{C} \subseteq \mathcal{G}$, completing the proof.

Step 4. *If $X \in \mathcal{C}$ such that $\underline{X} \ncong \underline{0}$ in $\underline{\mathcal{C}}$, then we have* $\operatorname{Ext}^1_R(\ \ , X)|_{\underline{\mathcal{C}}} \neq 0$.

Proof. As in the proof of Step 3 we set $\operatorname{Ind}(\underline{\mathcal{C}}) = \{\underline{W_i} \mid i = 1, 2, \ldots, m\}$. It is enough to show that $\operatorname{Ext}^1_R(\ \ , W_i)|_{\underline{\mathcal{C}}} \neq 0$ for each i. Now assume that $\operatorname{Ext}^1_R(\ \ , W_1)|_{\underline{\mathcal{C}}} = 0$. We shall show that this leads to a contradiction. In this case, it follows from Step 2 that any module in $\operatorname{mod} \underline{\mathcal{C}}$ can be embedded into a direct sum of copies of $(m-1)$ modules $\operatorname{Ext}^1_R(\ \ , W_2)|_{\underline{\mathcal{C}}}, \ldots, \operatorname{Ext}^1_R(\ \ , W_m)|_{\underline{\mathcal{C}}}$. In particular, any indecomposable injective modules appear in these $(m-1)$ modules as direct summands. But we have shown in the proof of Step 3 that there are m indecomposable injective modules $I(S_i)$ $(i = 1, 2, \ldots, m)$. Hence at least one of $\operatorname{Ext}^1_R(\ \ , W_2)|_{\underline{\mathcal{C}}}, \ldots, \operatorname{Ext}^1_R(\ \ , W_m)|_{\underline{\mathcal{C}}}$ contains two different indecomposable injective modules as direct summands. Since $\operatorname{mod} \underline{\mathcal{C}}$ is a Frobenius category, we see in particular that it is decomposed nontrivially into a direct sum of projective modules in $\operatorname{mod} \underline{\mathcal{C}}$. We may assume that $\operatorname{Ext}^1_R(\ \ , W_2)|_{\underline{\mathcal{C}}}$ is decomposed as

$$\operatorname{Ext}^1_R(\ \ , W_2)|_{\underline{\mathcal{C}}} \cong \underline{\operatorname{Hom}}_R(\ \ , Z_1)|_{\underline{\mathcal{C}}} \oplus \underline{\operatorname{Hom}}_R(\ \ , Z_2)|_{\underline{\mathcal{C}}},$$

where $Z_1, Z_2 (\ncong 0) \in \underline{\mathcal{C}}$. Then it follows from Lemma 5.1 that W_2 is isomorphic to a direct summand of $\Omega Z_1 \oplus \Omega Z_2$. But since W_2 is indecomposable, we may assume that W_2 is isomorphic to a direct summand of ΩZ_1. Then $\operatorname{Ext}^1_R(\ \ , W_2)|_{\underline{\mathcal{C}}}$ is a direct summand of $\operatorname{Ext}^1_R(\ \ , \Omega Z_1)|_{\underline{\mathcal{C}}} \cong \underline{\operatorname{Hom}}_R(\ \ , Z_1)|_{\underline{\mathcal{C}}}$. This is a contradiction because $\operatorname{mod} \underline{\mathcal{C}}$ is a Krull-Schmidt category by Lemma 4.1. □

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Math. Department, Faculty of Science, Okayama University, Okayama 700-8530, Japan

E-mail address: yoshino@math.okayama-u.ac.jp