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EXTENSIONS OF HIGGS BUNDLES

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Abstract. We prove a Hitchin-Kobayashi correspondence for extensions of Higgs bundles. The results generalize known results for extensions of holomorphic bundles. Using Simpson's methods, we construct moduli spaces of stable objects. In an appendix we construct Bott-Chern forms for Higgs bundles.

1. Introduction

The underlying principle at work in this paper is that, when approached in the right way, results about holomorphic bundles can be made applicable to Higgs bundles.

The type of results we have in mind fall under the general heading of the Hitchin-Kobayashi Correspondence, i.e., they concern notions of stability, construction of moduli spaces, and the relation of these to solutions of gauge theoretic equations. Originally established for holomorphic bundles, results of this sort have been extended to Higgs bundles and also to a host of so-called 'augmented holomorphic bundles', i.e., holomorphic bundles with some kind of prescribed additional structure (see [BDGW] for a survey). Indeed a Higgs bundle can be treated as an augmented holomorphic bundle in which the augmentation is the Higgs field. However this is not always the best point of view—and is not the one we have in mind. The better approach is the one developed by Simpson in [S1], [S2] and [S3].

In Simpson's approach, instead of treating the Higgs structure as an augmentation, it is encoded in a more fundamental way. In fact there are two versions of this approach, one differential geometric and one algebraic. In the first (described in Section 4), the extra structure on a Higgs bundle is encoded as a modification of the partial differential operator which defines the holomorphic structure on the underlying complex bundle. From the algebraic point of view (cf. Section 7), locally free coherent analytic sheaves on a variety X are replaced by sheaves of pure dimension on T^*X, and the Higgs structure is encoded in the $\mathcal{O}_{T^*X}$-module structure. Having made these

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adjustments, a proof designed for holomorphic bundles or coherent analytic sheaves re-emerges as a proof for Higgs bundles or Higgs sheaves!

In this paper we apply these principles to extensions of holomorphic bundles. A Hitchin-Kobayashi correspondence for such extensions was investigated in [BGP] and [DUW]; natural gauge-theoretic conditions for special metrics, and a notion of stability were formulated, and the correspondence between them established. In [DUW], GIT methods were used to construct the moduli spaces. The main results in this paper thus show how, after the appropriate modifications, these ideas can be carried over to Higgs bundles. We set up and prove the Hitchin-Kobayashi correspondence for extensions of Higgs bundles (Theorems 5.1 and 5.13), and we give (in Section 7) a GIT construction for the associated moduli spaces.

We also use the gauge-theoretic equations to deduce Bogomolov-type inequalities on the Chern classes of stable Higgs extensions. Our results (in Section 6) generalize the corresponding results described in [DUW] for extensions of holomorphic bundles, with the proofs being one more illustration of how results for holomorphic bundles can be recast as results for Higgs bundles. Going one step further than in [DUW], we describe in detail the implications of attaining equality in the Bogomolov inequalities.

Finally, in the Appendix, we extend to Higgs bundles the construction of Bott-Chern forms. These forms play an important role in the proof of the Hitchin-Kobayashi correspondence. In fact our proof uses only two special cases and all the requisite results can be extracted from the literature. The available treatments are however all somewhat ad hoc. We have thus undertaken a more systematic and general discussion, but have confined it to an Appendix. Our results show how the original constructions of Bott and Chern for holomorphic bundles go over in their entirety to the case of Higgs bundles. This can be viewed as yet another illustration of the main underlying principle of this paper.

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2. The objects

Let X be a closed Kähler manifold of dimension d and with Kähler form ω. A Higgs sheaf (cf. [S1], [S2], [S3], [S4]) on X is a pair $(\mathcal{E}, \Theta)$, where $\mathcal{E}$ is a coherent sheaf on X and Θ is a morphism $\Theta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X$ (where Ω^1_X

is the sheaf of holomorphic sections of the cotangent bundle T^*X) such that $\Theta \wedge \Theta = 0$. If $\mathcal{E}$ is locally free, Θ can be thought of as a holomorphic section of $\mathrm{End}(\mathcal{E}) \otimes \Omega^1_X$. A morphism of Higgs sheaves $f : (\mathcal{E}, \Theta) \longrightarrow (\mathcal{F}, \Psi)$ is a morphism of sheaves $\overline{f} : \mathcal{E} \longrightarrow \mathcal{F}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Theta} & \mathcal{E} \otimes \Omega^1_X \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{f}\otimes \mathrm{id}} \\ \mathcal{F} & \xrightarrow{\Psi} & \mathcal{F} \otimes \Omega^1_X \end{array} \tag{2.1}$$

Since the category of Higgs sheaves is abelian, the notion of exact sequence makes sense.

Definition 2.1. An extension of Higgs sheaves (or Higgs extension) is a short exact sequence

$$0 \longrightarrow (\mathcal{E}_1, \Theta_1) \xrightarrow{i} (\mathcal{E}, \Theta) \xrightarrow{q} (\mathcal{E}_2, \Theta_2) \longrightarrow 0\,. \tag{2.2}$$

A morphism between extensions of Higgs sheaves is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{E}'_1, \Theta'_1) & \longrightarrow & (\mathcal{E}', \Theta') & \longrightarrow & (\mathcal{E}'_2, \Theta'_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & (\mathcal{E}_1, \Theta_1) & \longrightarrow & (\mathcal{E}, \Theta) & \longrightarrow & (\mathcal{E}_2, \Theta_2) & \longrightarrow & 0 \end{array} \tag{2.3}$$

It follows that a morphism of Higgs extensions is an isomorphism if and only if the three morphisms f_1, f and f_2 are isomorphisms of Higgs bundles.

3. Stability

The notions of stability for holomorphic bundles adapt straightforwardly to define both slope- and Gieseker stability for Higgs bundles (cf. [S1], [S2], [S3], [S4] and [H]). In [BGP] and [DUW] these notions are defined for extensions of holomorphic bundles (or more generally, extensions of coherent sheaves). In this section we combine both of these to define stability for extensions of Higgs sheaves. As usual, the definition involves a numerical criterion on all subobjects. We must thus first define subobjects.

Definition 3.1. Consider a morphism of Higgs extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{E}'_1, \Theta'_1) & \longrightarrow & (\mathcal{E}', \Theta') & \longrightarrow & (\mathcal{E}'_2, \Theta'_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & (\mathcal{E}_1, \Theta_1) & \longrightarrow & (\mathcal{E}, \Theta) & \xrightarrow{q} & (\mathcal{E}_2, \Theta_2) & \longrightarrow & 0 \end{array} \tag{3.1}$$

If f_1, f and f_2 are injective, then the extension in the first row is called a subextension of the extension in the second row. A subextension is called proper if $\mathcal{E}'$ is a proper subsheaf of $\mathcal{E}$.

Remark 3.2. Note that giving a proper subextension is the same thing as giving a proper subsheaf $\mathcal{E}'$ of $\mathcal{E}$ that is invariant under Θ, in the sense that the image of $\Theta(\mathcal{E}')$ is in $\mathcal{E}' \otimes \Omega^1_X \subset \mathcal{E} \otimes \Omega^1_X$. Indeed, if $\mathcal{E}'$ is invariant under Θ, it defines a Higgs subbundle $(\mathcal{E}', \Theta')$, and we can recover $(\mathcal{E}'_2, \Theta'_2)$ as the image of $\mathcal{E}'$ under q, and $(\mathcal{E}'_1, \Theta'_1)$ is recovered as the kernel.

We can now define the notion of slope (or Mumford) stability.

Definition 3.3 (Slope stability). Fix $\alpha < 0$. Given a Higgs extension

$$0 \longrightarrow (\mathcal{E}_1, \Theta_1) \longrightarrow (\mathcal{E}, \Theta) \longrightarrow (\mathcal{E}_2, \Theta_2) \longrightarrow 0, \tag{3.2}$$

define its α-slope as

$$\mu_\alpha(\mathcal{E}) = \mu(\mathcal{E}) + \alpha \frac{\mathrm{rk}(\mathcal{E}_2)}{\mathrm{rk}(\mathcal{E})} \ , \tag{3.3}$$

We say that a Higgs extension is α-slope stable (resp. semistable), if for all proper subextensions, we have

$$\mu_\alpha(\mathcal{E}') < \mu_\alpha(\mathcal{E}) \qquad \text{(resp. } \leq \text{)}. \tag{3.4}$$

Remarks 3.4.

(1) While the definition of stability seems to make sense for all real values of α, there are both algebraic and analytic motivations for insisting that α be negative. The algebraic explanation has its roots in the relation between the above notion of α-stability and stability in the sense of geometric invariant theory (GIT). As discussed in Section 7, the negativity of α is required to guarantee the ampleness of a line bundle used in the GIT construction (cf. the proof of Theorem 7.4). From the analytic point of view, the sign of α is required in order to ensure the convexity of the functional defined in Definition 5.2, without which the existence and uniqueness results for solutions to the α-Higgs-Hermitian-Einstein equations (4.21) break down. The relation between the sign of α and the convexity of the functional is evident in equation (5.24) in Proposition 5.3.

(2) In addition to having zero as an upper bound, the range for α is also bounded below. Indeed, it is an immediate consequence of the definition that if $(\mathcal{E}, \Theta)$ is α-stable then $\mu_\alpha(\mathcal{E}_1) < \mu_\alpha(\mathcal{E})$, and hence $\alpha > \mu(\mathcal{E}_1) - \mu(\mathcal{E}_2)$. The allowed range for the parameter α is thus

$$\mu(\mathcal{E}_1) - \mu(\mathcal{E}_2) < \alpha < 0 \ . \tag{3.5}$$

In Section 7, where we construct moduli spaces, we will need a notion of Gieseker (semi-)stability for Higgs extensions.

Definition 3.5 (Gieseker stability). Fix $\alpha < 0$. Let $P(\mathcal{E}, m)$ denote the Hilbert polynomial of $\mathcal{E}$. A Higgs extension is called α-Gieseker stable (resp. semistable) if all proper subextensions $\mathcal{E}'$ satisfy:

(i) $\mu_\alpha(\mathcal{E}') \leq \mu_\alpha(\mathcal{E})$.

(ii) If equality holds in (i), then

$$\frac{P(\mathcal{E}',m)}{\mathrm{rk}(\mathcal{E}')} \leq \frac{P(\mathcal{E},m)}{\mathrm{rk}(\mathcal{E})} \quad \text{for } m \gg 0. \tag{3.6}$$

(iii) If equality holds in (i) and (ii), then

$$\frac{P(\mathcal{E}_2',m)}{\mathrm{rk}(\mathcal{E}_2')} > \frac{P(\mathcal{E}_2,m)}{\mathrm{rk}(\mathcal{E}_2)} \quad (\text{resp. } \geq) \text{ for } m \gg 0. \tag{3.7}$$

As usual, we have the following implications:

α-slope stable $\Longrightarrow$ α-Gieseker stable
$\Longrightarrow$ α-Gieseker semistable $\Longrightarrow$ α-slope semistable.

4. Differential geometric description and metric equations

All the essential differential geometric machinery for Higgs bundles can be found in [S3], [S4] and [H]. We thus give only a brief summary, emphasizing the aspects needed later in this paper. Denoting the underlying smooth bundle of a holomorphic bundle $\mathcal{E}$ by E, we can describe the holomorphic structure on $\mathcal{E}$ by an integrable partial connection, i.e., by a **C**-linear map

$$\overline{\partial}_E : \Omega^0(E) \longrightarrow \Omega^{0,1}(E) \tag{4.1}$$

which satisfies the $\overline{\partial}$-Leibniz formula and also the integrability condition

$$\overline{\partial}_E \circ \overline{\partial}_E = \overline{\partial}_E{}^2 = 0. \tag{4.2}$$

A Higgs bundle $(\mathcal{E},\Theta)$ can thus be specified by a triple $(E,\overline{\partial}_E,\Theta)$, where

- E is a smooth complex bundle on X,
- $\overline{\partial}_E : \Omega^0(E) \longrightarrow \Omega^{0,1}(E)$ satisfies the $\overline{\partial}$-Leibniz formula and $\overline{\partial}_E{}^2 = 0$,
- $\Theta \in \Omega^{1,0}(\mathrm{End}(E))$ satisfies $\overline{\partial}_E(\Theta) = 0$ and $\Theta \wedge \Theta = 0$.

Instead of treating the holomorphic structure ($\overline{\partial}_E$) and the Higgs field (Θ) as separate, we can combine them to define the Higgs operator

$$\nabla'' = \overline{\partial}_E + \Theta : \Omega^0(E) \longrightarrow \Omega^{0,1}(E) \oplus \Omega^{1,0}(E). \tag{4.3}$$

Notice that this differs from the partial connection $\overline{\partial}_E$ in that its image is not confined to $\Omega^{0,1}(E)$. However, like $\overline{\partial}_E$, it satisfies the $\overline{\partial}$-Leibniz formula and extends in the usual way to an operator on $\Omega^p(E)$. Conversely, given any **C**-linear map $\nabla'' : \Omega^0(E) \longrightarrow \Omega^1(E)$ which satisfies the $\overline{\partial}$-Leibniz formula, we can separate it into $\nabla'' = \overline{\partial}_E + \Theta$, corresponding to the splitting $\Omega(E)^1 = \Omega^{0,1}(E) \oplus \Omega^{1,0}(E)$. The integrability condition,

$$(\nabla'')^2 = 0 \ , \tag{4.4}$$

is clearly equivalent to the defining conditions of a Higgs bundle, viz.

$$(\overline{\partial}_E)^2 = 0 \ , \ \overline{\partial}_E(\Theta) = 0 \ , \ \Theta \wedge \Theta = 0 \ .$$

We thus arrive at the following description of a Higgs bundle, formally identical to the differential geometric description of a holomorphic bundle, but with the operator $\overline{\partial}_E$ replaced by the operator ∇''.

Definition 4.1 (Higgs operator description). A Higgs bundle on $X$is a pair (E, ∇'') in which E is a smooth bundle on X and $\nabla'' : \Omega^0(E) \longrightarrow \Omega^1(E)$ is a $\mathbf{C}$-linear map which satisfies the $\overline{\partial}$-Leibniz formula and the integrability condition (4.4).

Given a Hermitian bundle metric, H, on E, we can complete ∇'' so as to define a connection. To do so, we first define the adjoint $\Theta^*_H \in \Omega^{0,1}(\operatorname{End} E)$ by the condition that for all sections $s, t \in \Omega^0(E)$

$$(\Theta s, t)_H = (s, \Theta^*_H t)_H \ . \tag{4.5}$$

If we fix a local frame $\{e_i\}$ for E, and define the Hermitian matrix

$$H_{ji} = (e_i, e_j)_H \ , \tag{4.6}$$

then Θ^*_H is represented by the matrix

$$\Theta^*_H = H^{-1}\overline{\Theta}^T H \ . \tag{4.7}$$

More explicitly, if we write

$$\Theta = \sum_\alpha [\Theta^\alpha]_{ij} \otimes \omega_\alpha \ , \tag{4.8}$$

where the ω_α are $(1,0)$-forms and the matrices $[\Theta^\alpha]_{ij}$ are local descriptions (with respect to the frame $\{e_i\}$) of bundle endomorphisms, then

$$\Theta^*_H = \sum_\alpha [\Theta^{*;\alpha}_H]_{ij} \otimes \overline{\omega}_\alpha \ , \tag{4.9}$$

where

$$[\Theta^{*;\alpha}_H]_{ij} = H^{-1}_{ip} \overline{[\Theta^{*;\alpha}_H]}^T_{pq} H_{qj} \ . \tag{4.10}$$

Definition 4.2. Define

$$\nabla'_H = D'_H + \Theta^*_H \ , \tag{4.11}$$

where $D(\overline{\partial}_E, H) = \overline{\partial}_E + D'_H$ is the Chern connection compatible with $\overline{\partial}_E$ and H. The Higgs Connection is then defined by

$$\nabla = \nabla'' + \nabla'_H \ . \tag{4.12}$$

The curvature of this connection

$$F^\nabla_H = \nabla^2 \ , \tag{4.13}$$

is called the Higgs curvature.

REMARK 4.3. The Higgs curvature, like the curvature of any connection, is a section of $\Omega^2(M, \mathrm{End}\, E)$. Unlike in the case of the Chern connection, F_H^{∇} does not have complex form type $(1,1)$. The Higgs connection and its curvature do however have the following two crucial features:

- (Kähler identities)

$$i[\Lambda, \nabla''] = (\nabla'_H)^* \ , \ i[\Lambda, \nabla'_H] = -(\nabla'')^* \ , \tag{4.14}$$

where the adjoints are taken with respect to the metric H and

$$\Lambda : \Omega^{p,q}(E) \longrightarrow \Omega^{p-1,q-1}(E) \tag{4.15}$$

is the adjoint of wedging with the Kähler form on X.

- (Bianchi identity)

$$\nabla'_H(F_H^{\nabla}) = 0 = \nabla''(F_H^{\nabla}) \ . \tag{4.16}$$

Notice that these are direct analogs of the properties enjoyed by the Chern connection, with ∇'' and ∇'_H playing the role here that $\overline{\partial}_E$ and D'_H play for the Chern connection. This formal correspondence, which leads directly to the underlying principle mentioned in the Introduction, is summarized in Table 1.

We now consider an extension of Higgs bundles,

$$0 \longrightarrow (\mathcal{E}_1, \Theta_1) \longrightarrow (\mathcal{E}, \Theta) \longrightarrow (\mathcal{E}_2, \Theta_2) \longrightarrow 0,$$

i.e., a Higgs extension as in Definition 2.1 but in which the sheaves are locally free. If we denote the underlying smooth bundle of $\mathcal{E}$ by E, then we can fix a smooth splitting $E = E_1 \oplus E_2$, where the summands are the underlying smooth bundles for $\mathcal{E}_1$ and $\mathcal{E}_2$. Thus the sub-Higgs bundle in the extension is described by the triple $(E_1, \overline{\partial}_1, \Theta_1)$, and the quotient Higgs bundle by $(E_2, \overline{\partial}_2, \Theta_2)$. The Higgs extension is then specified by the triple $(E, \overline{\partial}_E, \Theta)$, where

- the holomorphic structure is of the form

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_1 & \beta \\ 0 & \overline{\partial}_2 \end{pmatrix} \ , \ \beta \text{ a holomorphic section in } \Omega^{0,1}(\mathrm{Hom}(E_2, E_1)) \ , \tag{4.17}$$

- and the Higgs field is of the form

$$\Theta = \begin{pmatrix} \Theta_1 & b \\ 0 & \Theta_2 \end{pmatrix} \ , \ b \text{ a holomorphic section in } \Omega^{1,0}(\mathrm{Hom}(E_2, E_1)) \ . \tag{4.18}$$

Here the holomorphic structure on $\mathrm{Hom}(E_2, E_1)$ is that induced by $\overline{\partial}_1$ and $\overline{\partial}_2$. Alternatively, using Higgs operators to describe the Higgs bundles, we have

$$0 \longrightarrow (E_1, \nabla''_1) \longrightarrow (E, \nabla'') \longrightarrow (E_2, \nabla''_2) \longrightarrow 0,$$

where, with respect to a smooth splitting $E = E_1 \oplus E_2$, the Higgs operator on E is of the form

$$\nabla'' = \begin{pmatrix} \nabla''_1 & b + \beta \\ 0 & \nabla''_2 \end{pmatrix} . \tag{4.19}$$

	Holomorphic bundle	Higgs bundle
underlying smooth bundle	E	E
differential operator	$\overline{\partial}_E : \Omega^0(E) \longrightarrow \Omega^{0,1}(E)$	$\nabla'' : \Omega^0(E) \longrightarrow \Omega^1(E)$
integrability condition	${\overline{\partial}_E}^2 = 0$	$(\nabla'')^2 = 0$
complementary operator	$(D'_H)^* = i[\Lambda, \overline{\partial}_E]$	$(\nabla'_H)^* = i[\Lambda, \nabla'']$
connection	$D = \overline{\partial}_E + D'_H$	$\nabla = \nabla'' + \nabla'_H$
gauge theory equations for special metrics	$i\Lambda F^D_H = \frac{2\pi\mu}{V}\mathbf{I}$	$i\Lambda F^{\nabla}_H = \frac{2\pi\mu}{V}\mathbf{I}$
(other) Kähler identity	$(\overline{\partial}_E)^* = -i[\Lambda, D'_H]$	$(\nabla'')^* = -i[\Lambda, \nabla'_H]$
Bianchi curvature identities	$\overline{\partial}_E(F^D_H) = D'_H(F^D_H) = 0$	$\nabla''(F^{\nabla}_H) = \nabla'_H(F^{\nabla}_H) = 0$

Table 1

Differential Geometric Dictionary, illustrating the formal similarity resulting from using the Higgs operator $\nabla'' = \overline{\partial}_E + \Theta$ to encode the Higgs structure in a Higgs bundle

Suppose now that we have a metric H on the middle bundle in the extension. It then makes sense to talk of an orthogonal splitting $E = E_1 \oplus E_2$. We can thus define a bundle automorphism $\mathbf{T} : E \longrightarrow E$ which, with respect to the H-orthogonal splitting, is given by the matrix

$$\mathbf{T} = \begin{pmatrix} \frac{n_2}{n}\mathbf{I}_1 & 0 \\ 0 & -\frac{n_1}{n}\mathbf{I}_2 \end{pmatrix} . \tag{4.20}$$

Here $n = \mathrm{rk}(E)$ and $n_i = \mathrm{rk}(E_i)$. We can now formulate the following gauge theoretic equations:

Definition 4.4. Fix the real number α. We say the metric H satisfies the α-Higgs-Hermitian-Einstein (αHHE) condition if

$$i\Lambda F_H^{\nabla} = \frac{2\pi\mu}{V}\mathbf{I} + \frac{2\pi\alpha}{V}\mathbf{T} , \tag{4.21}$$

where F_H^{∇} is the Higgs curvature as in (4.13), Λ is as in (4.15), T is the bundle automorphism defined in (4.20), $V = \int_X \frac{\omega^d}{d!}$ is the volume of X, and $\mu = \mu(\mathcal{E})$ is the slope of $\mathcal{E}$.

Remark 4.5.

- In the case $\Theta = 0$, when $\nabla'' = \overline{\partial}_E$ and thus the Higgs curvature F_H^{∇} reduces to F_H^D (the curvature of the Chern connection compatible with H and $\overline{\partial}_E$ on E), equation (4.21) becomes the deformed Hermitian-Einstein equation defined in [BGP] on extensions of holomorphic bundles.
- If we set $\alpha = 0$ then we recover the usual Higgs equation (defined by Simpson and Hitchin) for a metric on the Higgs bundle $(\mathcal{E}, \Theta)$.
- Using the fact that $(\nabla'')^2 = 0$, we can express ΛF_H^{∇} as

$$\Lambda F_H^{\nabla} = \Lambda(F_H^D + [\Theta, \Theta^*]) , \tag{4.22}$$

where F_H^D is the curvature of the Chern connection. The α-Higgs-Hermitian-Einstein equation can thus also be written in the form

$$i\Lambda(F_H^D + [\Theta, \Theta^*]) = \frac{2\pi\mu}{V}\mathbf{I} + \frac{2\pi\alpha}{V}\mathbf{T} . \tag{4.23}$$

5. The Hitchin-Kobayashi correspondence

In this section we investigate the relation between the α-stability of a Higgs extension and the existence of a metric satisfying the αHHE condition. As in Section 4, we fix an extension of Higgs bundles

$$0 \longrightarrow (\mathcal{E}_1, \Theta_1) \longrightarrow (\mathcal{E}, \Theta) \longrightarrow (\mathcal{E}_2, \Theta_2) \longrightarrow 0. \tag{5.1}$$

The underlying smooth bundles are denoted, as usual, by E_1, E_2, and E. With Higgs operators defined as in (4.3) we can thus equivalently describe the extension as

$$0 \longrightarrow (E_1, \nabla_1'') \longrightarrow (E, \nabla'') \longrightarrow (E_2, \nabla_2'') \longrightarrow 0. \tag{5.2}$$

The Hitchin-Kobayashi correspondence asserts that α-stability is equivalent to the existence of an αHHE metric. In Section 5.1 we prove that existence of an αHHE metric implies α-(poly)stability. The converse is proved in Section 5.2. In both cases we see the advantage of encoding the Higgs structure in the Higgs operator; having done so, the proofs amount to little more than using the dictionary provided in Table 1 to adapt the corresponding proofs for extensions of holomorphic bundles (as in [BGP]).

5.1. The easy direction.

THEOREM 5.1. *Fix $\alpha < 0$. Suppose that the Higgs extension* (5.1) *supports a metric with respect to which the smooth splitting $E = E_1 \oplus E_2$ is orthogonal, and satisfying the αHHE condition* (4.21). *Then either the Higgs extension is α-stable or it splits as a direct sum of α-stable Higgs extensions, all with the same α-slope.*

Proof. Suppose that the metric $H = H_1 \oplus H_2$ on E satisfies (4.21). Let $\nabla = \nabla'' + \nabla'_H$ be the Higgs connection determined by H and the Higgs operator on E, and let F_H^{∇} be its curvature (as in Definition 4.2). Let $\mathcal{E}' \subset \mathcal{E}$ be any Higgs subsheaf, with corresponding Higgs subextension

$$0 \longrightarrow (\mathcal{E}_1', \Theta_1') \longrightarrow (\mathcal{E}', \Theta') \longrightarrow (\mathcal{E}_2', \Theta_2') \longrightarrow 0. \tag{5.3}$$

If $\mathcal{E}'$ is a saturated subsheaf then it is locally free outside of a codimension two subset, say Σ, in X. We can thus define a projection $\pi : \mathcal{E}|_{X-\Sigma} \longrightarrow \mathcal{E}'|_{X-\Sigma}$. Since $(\mathcal{E}', \Theta')$ is a Higgs subsheaf, we can compute the degree of $\mathcal{E}'$ by the formula (cf. [S3, Lemma 3.2])

$$2\pi \deg(\mathcal{E}') = i \int_X \mathrm{Tr}(\Lambda \pi F_H^{\nabla}) - \int_X |\nabla'' \pi|_H^2 \ . \tag{5.4}$$

But by (4.21)

$$\frac{iV}{2\pi} \Lambda F_H^{\nabla} = \begin{pmatrix} \tau_1 \mathbf{I}_1 & 0 \\ 0 & \tau_2 \mathbf{I}_2 \end{pmatrix} \ , \tag{5.5}$$

where

$$\tau_1 = \mu + \alpha \frac{n_2}{n}, \quad \tau_2 = \mu - \alpha \frac{n_1}{n}. \tag{5.6}$$

It follows (precisely as in Proposition 3.8 of [BGP]) that

$$\frac{i}{2\pi} \int_X \mathrm{Tr}(\Lambda \pi F_H^{\nabla}) = n_1' \tau_1 + n_2' \tau_2 \ , \tag{5.7}$$

where $n_1' = \mathrm{rank}(\mathcal{E}_1')$ and $n_2' = \mathrm{rank}(\mathcal{E}_2')$. Notice that the first of the relations in (5.6) can be written as $\tau_1 = \mu_\alpha(\mathcal{E})$, and that together they imply $\alpha = \tau_1 - \tau_2$. Combining (5.7) and (5.4) thus leads to

$$\mu_\alpha(\mathcal{E}') = \mu_\alpha(\mathcal{E}) - \frac{1}{2\pi(n_1' + n_2')} \int_X |\nabla'' \pi|_H^2 \ , \tag{5.8}$$

from which the conclusion follows in the usual way. □

5.2. The hard direction. We now consider the converse of Theorem 5.1. Keeping the notation of Section 5.1, we show that if a Higgs extension (5.1) is α-stable, then $\mathcal{E}$ admits a metric with respect to which the smooth splitting $E = E_1 \oplus E_2$ is orthogonal and which satisfies the αHHE equation (4.21), i.e., such that

$$i\Lambda F_H^{\nabla} = \frac{2\pi\mu}{V}\mathbf{I} + \frac{2\pi\alpha}{V}\mathbf{T} \ .$$

As in [S3] and [BGP], we can separate the trace and trace-free parts of this equation. We can always fix $\det(H)$ so that

$$i\Lambda \,\mathrm{Tr}(F_H^{\nabla}) = n\frac{2\pi\mu}{V} \ . \tag{5.9}$$

In fact, since $[\Theta, \Theta^*] = 0$ has zero trace, $i\Lambda \,\mathrm{Tr}(F_H^{\nabla})$ is the same for the Higgs connection as it is for the (metric) Chern connection. The above equation is thus satisfied if $\det(H)$ is the Hermitian-Einstein metric on the determinant line bundle $\det(\mathcal{E})$. Henceforth, we assume that we have fixed a background metric, K, such that $i\Lambda \,\mathrm{Tr}(F_K) = n\frac{2\pi\mu}{V}$. It remains therefore to prove that E admits a metric satisfying

$$i\Lambda F_H^{\perp} = \frac{2\pi\alpha}{V}\mathbf{T} \ , \tag{5.10}$$

where $F_H^{\perp} = F_H^{\nabla} - \frac{1}{n}\mathrm{Tr}(F_H^{\nabla})\mathbf{I}$ is the trace-free part of F_H.

The proof follows the standard pattern for Hitchin-Kobayashi correspondences. The method we use is essentially that of Simpson, with modifications as in [BGP] to accommodate the features arising from the extension structure (i.e., the non-zero right hand side in the equation). We thus give only a sketch of the proof, in which we fully describe all novel modifications, but do not repeat the details that can be found in [BGP], [S3] and [Do1]. Let

$$S(K) = \{s \in \Omega^0(X, \mathrm{End}\, E) | s^{*_K} = s \ , \ \mathrm{Tr}(s) = 0\} \ . \tag{5.11}$$

Then any other metric with the same determinant as K can be described by Ke^s, with $s \in S(K)$. Fix an integer $p > 2n$, and define

$$\mathrm{Met}_2^p = \{H = Ke^s \ | s \in L_2^p(S(K))\} \ . \tag{5.12}$$

We now define a Donaldson functional on Met whose critical points are solutions to (5.10). The original Donaldson functional was defined using Bott-Chern forms for pairs of metrics, and had Hermitian-Einstein metrics on holomorphic bundles as its critical points. The generalization for metrics on Higgs bundles is due to Simpson, while the adaptation for extensions of holomorphic bundles can be found in [BGP]. Here we must combine both of these modifications.

Given metrics H and K, we denote the functional defined by Donaldson by $M_D(K,H)$. Its definition in terms of Bott-Chern classes is

$$M_D(H,K) = \int_X R_2(H,K) \wedge \omega^{d-1} \ , \tag{5.13}$$

where R_2 is the Bott-Chern form associated with the polynomial $-\frac{1}{2}\operatorname{Tr}(AB+BA)$. Donaldson also gave a more explicit formula which applies for pairs (H,K) when $H = Ke^s$ with $s \in S(K)$. Simpson's generalization of M_D can be obtained directly from this formula: one simply replaces the Chern connection by the Higgs connection. We will denote Simpson's functional by $M_S(H,K)$. Though it is not needed in this proof, and was not formulated in this way by Simpson, this modification can be put in a more general framework. In the Appendix we show how it can be seen as the result of a modification of the Bott-Chern forms themselves. The functional used in [BGP] for metrics on $E = E_1 \oplus E_2$ can be defined as

$$M_{\tau_1,\tau_2}(H,K) = M_D(H,K) - \frac{4\pi(\tau_1-\tau_2)}{V}\int_X R_1(H_1,K_1)\wedge\omega^d \ , \tag{5.14}$$

where H_1 and K_1 are the induced metrics on E_1 and the Bott-Chern form R_1 is given by

$$R_1(H,K) = \log\det(K^{-1}H) = \operatorname{Tr}(\log K^{-1}H) \ . \tag{5.15}$$

We can combine this with Simpson's generalization if we replace M_D by M_S. We then get the following, which is the appropriate functional for extensions of Higgs bundles:

Definition 5.2. Let

$$M^{\mathrm{Higgs}}_{\tau_1,\tau_2}(H,K) = M_S(H,K) - \frac{4\pi(\tau_1-\tau_2)}{V}\int_X R_1(H_1,K_1)\wedge\omega^d \ , \tag{5.16}$$

or, setting $\alpha = \tau_1 - \tau_2$,

$$M^{\mathrm{Higgs}}_{\alpha}(H,K) = M_S(H,K) - \frac{4\pi\alpha}{V}\int_X R_1(H_1,K_1)\wedge\omega^d \ . \tag{5.17}$$

If we fix one of the metrics, say K, we can define

$$M^{\mathrm{Higgs}}_{\alpha}(H) = M^{\mathrm{Higgs}}_{\alpha}(H,K). \tag{5.18}$$

Following [BGP], we now define $m^0_\alpha : \mathrm{Met} \longrightarrow \Omega^0(X, \operatorname{End} E)$ by

$$m^0_\alpha(H) = \Lambda F^{\perp}_H + \frac{2\pi i\alpha}{V}\mathbf{T}_H \ , \tag{5.19}$$

where, with respect to the H-orthogonal splitting $E = E_1 \oplus E_2$,

$$\mathbf{T}_H = \begin{pmatrix} \frac{n_2}{n}\mathbf{I}_1 & 0 \\ 0 & -\frac{n_1}{n}\mathbf{I}_2 \end{pmatrix} . \tag{5.20}$$

The crucial properties of $M_\alpha^{\mathrm{Higgs}}$ and m_α^0 are described in the next proposition.

Proposition 5.3.

(1) *Given any three metrics H, K, J, we have*

$$M_\alpha^{\mathrm{Higgs}}(H,K) + M_\alpha^{\mathrm{Higgs}}(K,J) = M_\alpha^{\mathrm{Higgs}}(H,J) \ . \tag{5.21}$$

(2) *If $H(t) = He^{ts}$ with $s \in S(H)$, then*

$$\frac{d}{dt} M_\alpha^{\mathrm{Higgs}}(H(t)) = 2i \int_X \mathrm{Tr}\left(s m_\alpha^0(H(t))\right) \ . \tag{5.22}$$

(3) *Define the operator L on $L_2^p(S(H))$ by*

$$L(s) = \frac{d}{dt} m_\alpha^0(H(t))|_{t=0} \ . \tag{5.23}$$

If $s \in S(H)$ is given by $s = \begin{pmatrix} s_1 & u \\ u^* & s_2 \end{pmatrix}$ with respect to the H-orthogonal splitting $E = E_1 \oplus E_2$, and $H(t) = He^{ts}$, then

$$\begin{aligned} 2i\langle s, L(s)\rangle_H &= \frac{d^2}{dt^2} M_\alpha^{\mathrm{Higgs}}(H(t))|_{t=0} \\ &= \| \nabla''(s) \|_H^2 - \frac{4\pi\alpha}{V} \| u \|_H^2 \end{aligned} \tag{5.24}$$

(4) *If $s \in S(H)$ and $K = He^s$, then*

$$\Delta |s| \le 2(|m_\alpha^0(H)|_H + |m_\alpha^0(K)|_K) \ , \tag{5.25}$$

where the norm on $|s|$ can be with respect to either H or K.

Proof. We start by proving items (1) and (2). When $\alpha = 0$, these results follow as in §5 of [S3] and [Do2] (or, equivalently, follow from the properties of Bott-Chern forms, as described in the Appendix). The modification required when $\alpha < 0$ is exactly the same as described in the proof of Proposition 3.11 in [BGP].

We now prove (3). The proof is formally identical to that in Proposition 3.11 in [BGP], except that we replace the result about the second variation of M_D with the corresponding result for M_S, viz.

$$\frac{d^2}{dt^2} M_S(H(t))|_{t=0} = \| \nabla''(s) \|_H^2 \ . \tag{5.26}$$

This result can be found in [S3]. It can also be derived directly from the properties of Bott-Chern forms, as in Proposition A.16 of the Appendix.

Finally, we prove (4). When $\alpha = 0$, this is part (d) of Lemma 3.1 in [S3]. In general we have

$$m^0_\alpha(H) - m^0_\alpha(K) = (m^0_0(H) - m^0_0(K)) + \frac{2\pi i\alpha}{V}(\mathbf{T}_H - \mathbf{T}_K) \ . \tag{5.27}$$

This changes the computation in Simpson's proof by the introduction of an extra term of the form

$$\frac{2\pi\alpha}{V}\operatorname{Tr}\left(e^s(\mathbf{T}_H - \mathbf{T}_K)\right) \ . \tag{5.28}$$

But $\operatorname{Tr}(e^s\mathbf{T}_H) = \operatorname{Tr}(e^s\mathbf{T}_K)$, so the extra term does not affect the result. □

Corollary 5.4. *Suppose that $\alpha < 0$ and (5.1) is an α-stable extension. Then*

$$\operatorname{Ker}(L) = 0 \ , \tag{5.29}$$

where L is the operator defined above on $L^p_2(S(H))$.

Proof. Suppose that $L(s) = 0$ for some non-zero $s \in L^p_2(S(H))$. Then by (5.24) we have $\nabla''(s) = 0 = u$, where $u \in \Omega^0(X, \operatorname{Hom}(E_2, E_1))$ comes from writing $s = \begin{pmatrix} s_1 & u \\ u^* & s_2 \end{pmatrix}$, with $s_i \in L^p_2(S(K_i))$. Recall that with respect to the H-orthogonal splitting $E = E_1 \oplus E_2$, the holomorphic structure and Higgs field on E are given by (4.17) and (4.18). Thus

$$\nabla'' = \begin{pmatrix} \nabla''_1 & \beta + b \\ 0 & \nabla''_2 \end{pmatrix} \tag{5.30}$$

and we can conclude that $\nabla''_1(s_1) = \nabla''_2(s_2) = 0$. But $\nabla''_i(s_i) = 0$ is equivalent to

$$\overline{\partial}_i(s_i) = 0 \ \text{ and } \ [\Theta_i, s_i] = 0. \tag{5.31}$$

The eigenspaces of s thus split the extension (5.1) into a direct sum of Higgs extensions. Since $\operatorname{Tr}(s) = 0$ there must be at least two such summands. But this violates the stability criterion, since the α-slope inequality cannot be satisfied by both summands. □

Remark 5.5. This same computation shows that for any path $H(t) = He^{ts}$ with $s \in S(H)$, we get

$$\frac{d^2}{dt^2}M^{\text{Higgs}}_\alpha(H(t)) > 0 \ , \tag{5.32}$$

i.e., M^{Higgs}_α is a convex functional.

Next, we fix a positive real number B such that $\| m^0_\alpha(K) \|^p_{L^p} \leq B$, where

$$\| m^0_\alpha(K) \|^p_{L^p} = \int_X |m^0_\alpha(K)|^p_K d\,\text{vol} \tag{5.33}$$

and define

$$\text{Met}_2^p(B) = \{H \in \text{Met}_2^p \mid \ \| m_\alpha^0(H) \|_{L^p}^p \leq B \ \} . \tag{5.34}$$

LEMMA 5.6. *If the extension* (5.1) *is α-stable, then there are no extrema of M_α^{Higgs} on the boundary of this constrained space, and the minima occur at solutions to the metric equation $m_\alpha^0(H) = 0$.*

Proof. The proof is the same as in [B1, Lemma 3.4.2], in which the relation $\text{Ker}(L) = 0$ is the key. □

We thus look for minima of $M_\alpha^{\text{Higgs}}(H)$ on $\text{Met}_2^p(B)$. To show that minima do occur, we need

PROPOSITION 5.7 ([BGP, 3.14]). *Either* (5.1) *is not α-stable or we can find positive constants C_1 and C_2 such that*

$$\sup |s| < C_1 M_\alpha^{\text{Higgs}}(Ke^s) + C_2 \tag{5.35}$$

for all $Ke^s \in \text{Met}_2^p(B)$.

REMARK 5.8. This proposition motivates what might be called the Donaldson-Uhlenbeck-Simpson-Yau (DUSY) Alternative: either one can produce a minimizing sequence for the functional M_α^{Higgs}—and hence a solution to the metric equation—or one can use the functional to produce a sequence which in the limit destabilizes the extension (5.1).

Sketch of proof. The first step is to show that for metrics in the constrained set $\text{Met}_2^p(B)$, the C^0 estimate given above is equivalent to a C^1 estimate of the same type. The proof of this uses (5.25) in Proposition 5.3, but is otherwise identical to that in [S3] or [B1]. One then supposes that no such C^1 estimate holds. It follows that one may find an unbounded sequence of constants C_i and metrics $Ke_i^s \in \text{Met}_2^p(B)$ such that the estimate is violated. After normalizing the s_i, this produces a sequence $\{u_i\} \subset L_2^P(S(K))$ such that $\| u_i \|_{L^1} = 1$. This has a weakly convergent subsequence in $L_1^2(S(K))$, with non-trivial limit denoted by u_∞. One then shows that the eigenvalues of u_∞ are constant almost everywhere. This is done, as in [S3, §5], by making use of an estimate of the following form:

LEMMA 5.9 ([BGP, Lemma 3.13]). *Suppose that $\alpha < 0$ and let $H = Ke^s$ with $s \in L_2^p(S(K))$. Let $s = \begin{pmatrix} s_1 & u \\ u^* & s_2 \end{pmatrix}$ be the block decomposition of s with respect to the K-orthogonal splitting $E = E_1 \oplus E_2$. Let $\Psi : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be* *the smooth function as in* [B1] (*or* [S3]). *Then*

$$\begin{aligned}\frac{1}{2}M_\alpha^{\text{Higgs}}(H) &= i\int_X \operatorname{Tr}(s\Lambda F_K) + \int_x (\Psi(s)\nabla''s, \nabla''s)_K - \frac{2\pi\alpha}{V}R_1(H_1, K_1)\\ &\geq i\int_X \operatorname{Tr}(s\Lambda F_K) + \int_x (\Psi(s)\nabla''s, \nabla''s)_K - \frac{2\pi\alpha}{V}\int_x \operatorname{Tr}(s_1), \qquad (5.36)\end{aligned}$$

where the meaning of $\Psi(s)$ *is as in* [B1] *or* [S3].

Proof. As in [BGP], the first line follows from the computations in [S3]. The second line uses the convexity properties of the function $R_1(H(t)_1, K_1)$, and the fact that its first derivative at $t = 0$ is given by $\int_X \operatorname{Tr}(s_1)$. □

REMARK 5.10. The astute reader will notice a minor difference between the formula (5.36) and the corresponding one given in Lemma 3.13 in [BGP]. The difference involves the placement of factors of 2; the version in the first line of (3.3.18) in [BGP] is incorrect, but the errors do not affect any of the results in that paper.

Following the analysis in [S3, Lemma 5.4], this leads to the following result:

PROPOSITION 5.11 ([BGP, 3.15]). *Let* $\mathcal{F} : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ *be any smooth positive function which satisfies* $\mathcal{F}(x, y) \leq 1/(x - y)$ *whenever* $x > y$. *Then*

$$\begin{aligned} i\int_X \operatorname{Tr}(u_\infty \Lambda F_K) + \int_x (\mathcal{F}(u_\infty)\nabla''u_\infty, \nabla''u_\infty)_K &\\ - \frac{2\pi\alpha}{V}\int_x \operatorname{Tr}(u_{\infty,1}) \leq 0 \ , & \qquad (5.37)\end{aligned}$$

where $u_\infty = \begin{pmatrix} u_{\infty,1} & * \\ * & * \end{pmatrix}$ *with respect to the* K*-orthogonal splitting of* E.

Since $\operatorname{Tr}(u_\infty) = 0$, there are at least two distinct eigenvalues. Let $\lambda_1 < \lambda_2, \ldots, < \lambda_k$ denote the distinct eigenvalues. Setting $a_i = \lambda_{i+1} - \lambda_i$, one can thus define projections $\pi_i \in L_1^2(S(K))$ such that

$$u_\infty = \lambda_r \mathbf{I} - \sum_i^{k-1} a_i \pi_i. \qquad (5.38)$$

LEMMA 5.12. *The projections* π_i *satisfy*

(1) $\pi_i \in L_1^2(S(K))$,
(2) $\pi_i^2 = \pi_i$,
(3) $(1 - \pi_1)\nabla''(\pi_i) = 0$.

Proof. The case $\alpha = 0$ is proved in [S3] (see Lemma 5.6 and the succeeding remarks). The presence of the extra term depending on α in (5.37) does not affect the method of proof. □

Each π_i thus defines a weak Higgs subbundle in the sense of Uhlenbeck and Yau [UY], as adapted by Simpson [S3] for Higgs bundles, and hence produces a filtration of $\mathcal{E}$ by reflexive Higgs subsheaves

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}. \tag{5.39}$$

Each Higgs subsheaf $\mathcal{E}_j$ determines a Higgs subextension

$$0 \longrightarrow \mathcal{E}_{1,j} \longrightarrow \mathcal{E}_j \longrightarrow \mathcal{E}_{2,j} \longrightarrow 0 \ . \tag{5.40}$$

Now define the numerical quantity

$$Q = \lambda_k(r\mu(\mathcal{E}) - r_1\tau_1 - r_2\tau_2) - \sum_i^{k_1} a_i(r_i\mu(\mathcal{E}_i) - r_{1,i}\tau_1 - r_{2,i}\tau_2) \ , \tag{5.41}$$

where $\mu(\mathcal{E}_i)$ is the slope of $\mathcal{E}_j$, and $r_{a,i}$ is the rank of $\mathcal{E}_{a,i}$. Using Lemma 5.9 and the fact that $u_\infty = \lambda_r \mathbf{I} - \sum_i^{k-1} a_i\pi_i$, one shows (by precisely the method in [S3]) that $Q \leq 0$. On the other hand, τ_1 and τ_2 are related by $r\mu(\mathcal{E}) - r_1\tau_1 - r_2\tau_2 = 0$, and if (5.1) is α-stable, then

$$r_i\mu(\mathcal{E}_i) - r_{1,i}\tau_1 - r_{2,i}\tau_2 < 0 \tag{5.42}$$

for all $i = 1, \ldots, k-1$. Thus Q must be strictly positive if (5.1) is α-stable. We conclude therefore that if (5.1) is α-stable then there must be constants C_1 and C_2 such that the estimate (5.35) holds. This completes the proof of Proposition (5.7). □

We can now prove:

THEOREM 5.13. *Fix $\alpha < 0$ and suppose that the Higgs extension* (5.1) *is α-stable. Then E admits a unique metric H with respect to which the smooth splitting $E = E_1 \oplus E_2$ is orthogonal, with* $\det(H) = \det(K)$*, and such that*

$$i\Lambda F_H^{\perp} = \frac{2\pi\alpha}{V}\mathbf{T} \ . \tag{5.43}$$

Proof. By Proposition 5.7, there is an estimate of the form in (5.35) and hence the functional $M_\alpha^{\mathrm{Higgs}}$ is bounded below. By Lemma 5.6, a minimizing sequence produces a solution in $\mathrm{Met}_2^p(B)$ to the equation $m_\alpha^0(H) = 0$. The smoothness and uniqueness of the solution follows in exactly the same way as in [Do1], [S3] or [B1]. The smoothness is a result of elliptic regularity, while the uniqueness is a consequence of the convexity properties of $M_\alpha^{\mathrm{Higgs}}$. □

6. Bogomolov inequality

The existence of a solution to the α-Higgs-Hermitian-Einstein equations on an α-stable Higgs extension can be used to deduce topological constraints. The constraints are expressed as inequalities involving the Chern classes of the underlying bundles. As such, they are direct generalizations of the Bogomolov inequalities for stable holomorphic bundles. The notation in this section is as follows:

- As in Section 5, (E, ∇'') is a Higgs bundle which has the structure of an extension of Higgs bundles as in (5.2), i.e., which can be written as
$$0 \longrightarrow (E_1, \nabla''_1) \longrightarrow (E, \nabla'') \longrightarrow (E_2, \nabla''_2) \longrightarrow 0 \ .$$
- The ranks of the underlying smooth bundles E_1, E_2 and E are denoted by n_1, n_2 and n, respectively.
- The base space is the Kähler manifold (X, ω). The dimension of X is d, and its volume is V.
- Using the Kähler form ω and the Chern classes $c_1(E)$, $c_2(E)$, we define the characteristic numbers

$$C_2(E) = \int_X c_2(E) \wedge \omega^{d-2} \quad , \ C_1^2(E) = \int_X c_1^2(E) \wedge \omega^{d-2}. \tag{6.1}$$

With this notation, we prove the following results:

THEOREM 6.1 (Bogomolov Inequality). *Let (E, ∇'') be a Higgs bundle which has the structure of an extension of Higgs bundles as in* (5.1)*, i.e., which can be written as*

$$0 \longrightarrow (E_1, \nabla''_1) \longrightarrow (E, \nabla'') \longrightarrow (E_2, \nabla''_2) \longrightarrow 0 \ .$$

Suppose that (E, ∇'') is α-polystable as an extension of Higgs bundles, for some $\alpha < 0$. Then

$$2C_2(E) - \frac{n-1}{n} C_1^2(E) + \frac{\alpha^2}{V} \left(\frac{n_1 n_2}{n}\right) \frac{(d-1)!}{d} \geq 0 \ . \tag{6.2}$$

THEOREM 6.2. *Let (E, ∇'') be as in Theorem 6.1. Suppose that (E, ∇'') is α-polystable as an extension of Higgs bundles and that equality holds in* (6.2)*, i.e., its Chern classes satisfy*

$$2C_2(E) - \frac{n-1}{n} C_1^2(E) + \frac{\alpha^2}{V} \left(\frac{n_1 n_2}{n}\right) \frac{(d-1)!}{d} = 0 \ . \tag{6.3}$$

Then:

(1) *With respect to the splitting $E = E_1 \oplus E_2$ we have*

$$\nabla'' = \begin{pmatrix} \nabla''_1 & 0 \\ 0 & \nabla''_2 \end{pmatrix} \ , \ \textit{i.e.,}\ \overline{\partial}_E = \begin{pmatrix} \overline{\partial}_1 & 0 \\ 0 & \overline{\partial}_2 \end{pmatrix} \ \textit{and}\ \Theta = \begin{pmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{pmatrix} \ . \tag{6.4}$$

(2) *There is a metric $H = H_1 \oplus H_2$ on E such that for $i = 1, 2$ we have*

$$F_{H_i}^{\perp} = 0 \ , \tag{6.5}$$

where $F^{\perp}_{H_i}$ is the trace-free part of the Higgs connection determined by $(\overline{\partial}_i, \Theta_i, H_i)$ on E_i, and furthermore,

$$\frac{\mathrm{Tr}(F^{\nabla_1}_{H_1})}{n_1} - \frac{\mathrm{Tr}(F^{\nabla_2}_{H_2})}{n_2} = \frac{2\pi i\alpha}{V}\frac{\omega}{d} \tag{6.6}$$

(3) *The parameter α has the value*

$$\alpha = \mu_1 - \mu_2 \ , \tag{6.7}$$

where

$$\mu_i = \frac{\int_X \Lambda c_1(E_i)\omega^d/d!}{n_i} \ . \tag{6.8}$$

Conversely, if conditions (1)–(3) *apply, then the Higgs extension is α-polystable and its Chern classes satisfy the equality* (6.3).

Remark 6.3. Conditions (1) and (2) in Theorem 6.2 together imply that (E, ∇'') splits as a direct sum of polystable Higgs bundles.

We require the following key technical result:

Proposition 6.4 ([S3, §3]). *If F^{∇}_H is the curvature of the Higgs connection determined by metric H on (E, ∇''), then*

$$\mathrm{Tr}(F^{\nabla}_H \wedge F^{\nabla}_H \wedge \omega^{d-2}) = \left| F^{\nabla}_H - \frac{1}{d}(\Lambda F^{\nabla}_H)\omega \right|^2 \frac{\omega^d}{d(d-1)} - |\Lambda F^{\nabla}_H|^2 \frac{\omega^d}{d^2} \ , \tag{6.9}$$

where $d = \dim(X)$. Similarly, if $F^{\perp} = F^{\nabla}_H - \frac{1}{d}\mathrm{Tr}(F^{\nabla}_H)\mathbf{I}$, then

$$(F^{\perp}_H \wedge F^{\perp}_H \wedge \omega^{d-2}) = \left| F^{\perp}_H - \frac{1}{n}(\Lambda F^{\perp}_H)\omega \right|^2 \frac{\omega^d}{d(d-1)} - |\Lambda F^{\perp}_H|^2 \frac{\omega^d}{d^2} \ . \tag{6.10}$$

Proof. This uses the following features of Higgs connections:

$$(F^{\nabla}_H)^{1,1} + ((F^{\nabla}_H)^{1,1})^{*_H} = 0, \tag{6.11}$$

$$(F^{\nabla}_H)^{2,0} = ((F^{\nabla}_H)^{0,2})^{*_H}. \tag{6.12}$$

□

Proof of Theorem 6.1. If (E, ∇'') is α-polystable, then (by Theorem 5.13) it has a metric satisfying the αHHE equation (4.21). Taking the trace-free part (given in (5.10)), we get

$$\begin{aligned} ||\Lambda F^{\perp}_H||^2 &= \int_X |\Lambda F^{\perp}_H|^2 \ \frac{\omega^d}{d!} \\ &= \frac{4\pi^2}{V^2}\int_X |\alpha \mathbf{T}|^2 \ \frac{\omega^d}{d!} \\ &= \frac{4\pi^2}{V}\alpha^2 \frac{n_1 n_2}{n} 2\pi \ . \end{aligned} \tag{6.13}$$

Using the Chern-Weil formulae for $\mathrm{ch}_2(E)$ and $c_1(E)$, plus the identity $\mathrm{ch}_2 = \frac{1}{2}c_1^2 - c_2$, we get

$$\begin{aligned} &\frac{1}{4\pi^2}\int_X \mathrm{Tr}(F_H^{\perp} \wedge F_H^{\perp} \wedge \omega^{d-2}) \\ &= \frac{1}{4\pi^2}\int_X (\mathrm{Tr}(F_H^{\nabla} \wedge F_H^{\nabla}) - \frac{1}{n}\mathrm{Tr}(F_H^{\nabla}) \wedge \mathrm{Tr}(F_H^{\nabla})) \wedge \omega^{d-2} \\ &= \int_X (-2\,\mathrm{ch}_2(E) + \frac{1}{n}c_1^2(E)) \wedge \omega^{d-2} \\ &= \int_X (2c_2(E) - \frac{n-1}{n}c_1^2(E)) \wedge \omega^{d-2}\ . \end{aligned} \tag{6.14}$$

Equation (6.10) thus yields

$$\begin{aligned} 2C_2(E) - \frac{n-1}{n}C_1^2(E) + \frac{\alpha^2}{V}\left(\frac{n_1 n_2}{n}\right)\frac{(d-1)!}{d} \\ = \frac{(d-2)!}{4\pi^2}||F_H^{\perp} - \frac{1}{d}(\Lambda F_H^{\perp})\omega||^2\ , \end{aligned} \tag{6.15}$$

where $C_2(E)$ and $C_1^2(E)$ are as in (6.1). Theorem 6.1 follows directly from this. □

Proof of Theorem 6.2. Suppose that (E, ∇'') is α-polystable as an extension of Higgs bundles, and that (6.3) holds. As in the previous proof, we may thus assume that E supports a metric $H = H_1 \oplus H_2$ which satisfies the trace-free αHHE equation (5.10). It then follows from (6.15) that the trace free part of the curvature, i.e., $F_H^{\perp}$, satisfies

$$F_H^{\perp} = -\frac{2\pi i\alpha}{V}\mathbf{T}\frac{\omega}{d}\ . \tag{6.16}$$

Applying the Bianchi identity, viz. $\nabla(F_H^{\nabla}) = 0$, and the fact that (cf. Lemma A.11) $d\,\mathrm{Tr}(F_H^{\nabla}) = \mathrm{Tr}\,\nabla(F_H^{\nabla})$, we get

$$\nabla(T) = 0\ . \tag{6.17}$$

It follows from this that the subbundles corresponding to eigenvalues n_2/n and $-n_1/n$ of $\mathbf{T}$ both give rise to Higgs subbundles of (E, ∇''). Alternatively, one can compute the covariant derivative $\nabla(T)$ and observe directly from (6.17) that ∇'' (and hence $\overline{\partial}_E$ and Θ) must be as in (6.4). Either way, we have

$$F_H^{\nabla} = \begin{pmatrix} F_{H_1}^{\nabla_1} & 0 \\ 0 & F_{H_2}^{\nabla_2} \end{pmatrix} \tag{6.18}$$

and hence

$$F_H^{\perp} = \begin{pmatrix} F_{H_1}^{\perp} & 0 \\ 0 & F_{H_2}^{\perp} \end{pmatrix} + \left(\frac{\mathrm{Tr}(F_{H_1}^{\nabla_1})}{n_1} - \frac{\mathrm{Tr}(F_{H_2}^{\nabla_2})}{n_2}\right)\mathbf{T}\ , \tag{6.19}$$

where

$$F_{H_1}^{\perp} = F_{H_1}^{\nabla_1} - \frac{\mathrm{Tr}(F_{H_1}^{\nabla_1})}{n_1}\mathbf{I}_1,$$

and similarly for $F_{H_2}^{\perp}$. Combining this with (6.16), we see that

$$\begin{pmatrix} F_{H_1}^{\perp} & 0 \\ 0 & F_{H_2}^{\perp} \end{pmatrix} = \left(\frac{\mathrm{Tr}(F_{H_2}^{\nabla_2})}{n_2} - \frac{\mathrm{Tr}(F_{H_1}^{\nabla_1})}{n_1} - \frac{2\pi i\alpha}{V}\frac{\omega}{d} \right) \mathbf{T} \ , \tag{6.20}$$

i.e.,

$$\begin{aligned} F_{H_1}^{\perp} &= \frac{n_2}{n}\left(\frac{\mathrm{Tr}(F_{H_2}^{\nabla_2})}{n_2} - \frac{\mathrm{Tr}(F_{H_1}^{\nabla_1})}{n_1} - \frac{2\pi i\alpha}{V}\frac{\omega}{d} \right) \mathbf{I}_1 \ , \\ F_{H_2}^{\perp} &= -\frac{n_1}{n}\left(\frac{\mathrm{Tr}(F_{H_2}^{\nabla_2})}{n_2} - \frac{\mathrm{Tr}(F_{H_1}^{\nabla_1})}{n_1} - \frac{2\pi i\alpha}{V}\frac{\omega}{d} \right) \mathbf{I}_2 \ . \end{aligned} \tag{6.21}$$

Taking the trace of either of these equations yields (6.6). Contracting with ω and integrating over X then yields (6.7). Conversely, suppose that (1)–(3) apply. Then (6.19) implies

$$F_H^{\perp} = -\frac{2\pi i\alpha}{V}\mathbf{T}\frac{\omega}{d} = \Lambda F_H^{\perp}\frac{\omega}{d} \ , \tag{6.22}$$

and hence that the right hand side of (6.15) vanishes. Thus, with $H = H_1 \oplus H_2$, we see that $i\Lambda F_H^{\nabla} = \frac{2\pi\mu}{V}\mathbf{I} + \frac{2\pi\alpha}{V}\mathbf{T}$, as required. It remains to verify (6.3). We write, for $i = 1, 2$,

$$c_1(E_i) = \delta_i\omega + \beta_i \ , \tag{6.23}$$

$$c_2(E_i) = a_i\omega^2 + b_i \wedge \omega + c_i, \tag{6.24}$$

where $\delta_i, a_i \in \mathbf{R}$ and $\beta_i, b_i \in \Omega^{(1,1)}(X, \mathbf{R})$ are primitive forms, and $c_i \wedge \omega^{(d-2)} = 0$. The condition in (6.6) then becomes

$$\frac{\beta_1}{n_1} - \frac{\beta_2}{n_2} = 0 \ . \tag{6.25}$$

Using the identities

$$c_2(E_1 \oplus E_2) = c_2(E_1) + c_2(E_2) + c_1(E_1) \wedge c_1(E_2) \ , \tag{6.26}$$

and

$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2) \ , \tag{6.27}$$

we thus compute

$$
\begin{aligned}
&(2c_2(E) - \frac{(n-1)}{n} c_1^2(E)) \wedge \omega^{d-2} \\
&= (2(a_1 + a_2 + \delta_1\delta_2) - \frac{n-1}{n}(\delta_1 + \delta_2)^2)\omega^d \\
&\quad + 2(\beta_1 \wedge \beta_2 - \frac{n-1}{n}(\beta_1 + \beta_2)^2) \wedge \omega^{d-2} \\
&= \sum_{i=1,2} (2c_2(E_i) - \frac{n_i - 1}{n_i} c_1^2(E_i)) \wedge \omega^{d-2} \\
&\quad - \frac{n_1 n_2}{n} \left(\frac{\delta_1}{n_1} - \frac{\delta_2}{n_2} \right)^2 \wedge \omega^d - \frac{n_1 n_2}{n} \left(\frac{\beta_1}{n_1} - \frac{\beta_2}{n_2} \right)^2 \wedge \omega^{d-2}.
\end{aligned}
\tag{6.28}
$$

But $F^{\perp}_{H_1} = F^{\perp}_{H_2} = 0$. Thus by (6.14) applied to E_1 and E_2 we have

$$
2C_2(E_i) - \frac{n_i - 1}{n_i} C_1^2(E_i) = 0 \ . \tag{6.29}
$$

Together with (6.25), equation (6.28) thus reduces to

$$
\begin{aligned}
2C_2(E) - \frac{(n-1)}{n} C_1^2(E) &= -\frac{n_1 n_2}{n} \left(\frac{\delta_1}{n_1} - \frac{\delta_2}{n_2} \right)^2 \int_X \omega^d \\
&= -\frac{\alpha^2}{V} \left(\frac{n_1 n_2}{n} \right) \frac{(d-1)!}{2\pi d} \ ,
\end{aligned}
\tag{6.30}
$$

where in the last line we have used $\alpha = \mu_1 - \mu_2$ and $\int_X \omega^d = V d!$. □

REMARKS 6.5.

(1) The condition (6.16) makes sense for connections on complex bundles over symplectic manifolds, where ω is then the symplectic form. It is thus tempting to view this as the definition a symplectic version of a stable Higgs extension, in much the same way that flat bundles provide the real versions of a stable Higgs bundles (under suitable restrictions on Chern classes). However, as the above proof shows, the condition forces the Higgs extension to be a direct sum of polystable Higgs bundles, so no new phenomena emerge. It is also worth noting that, by (6.7), the equation $F^{\perp}_H = -\frac{2\pi i \alpha}{V} \mathbf{T}\omega$ can apply only if α is at the extreme lower bound of its range.

(2) In the case where $\Theta = 0$, or equivalently $\nabla'' = \overline{\partial}_E$, Theorem 6.1 yields a Bogomolov inequality for α-stable extensions. This is equivalent to Theorem 3.11 in [DUW]. Taking $\nabla'' = \overline{\partial}_E$ in Theorem 6.2 similarly yields a result for extensions of bundles. It provides the necessary and sufficient conditions under which equality can be attained in the Bogomolov inequality for an α-stable extension. As far as we are aware, this result has not previously appeared anywhere.

7. Algebro-geometric description and GIT construction

We now return to the algebraic setting and consider Higgs sheaves and extensions of Higgs sheaves as defined in Section 2. In [DUW], Daskalopoulos, Uhlenbeck and Wentworth have constructed the moduli space of extensions of torsion free sheaves, following ideas of Simpson. In this section we will show how basically the same construction also gives the moduli space of extensions of Higgs sheaves. The main modification required is to use sheaves of pure dimension, rather than torsion free sheaves.

We will start by recalling Simpson's identification between Higgs sheaves on X and sheaves on the cotangent bundle T^*X. Let Z be the usual projective completion of the cotangent bundle T^*X, extending the projection $\pi : T^*X \longrightarrow X$ to a projective bundle $\overline{\pi} : Z \longrightarrow X$. Let $D = Z - T^*X$ be the divisor at infinity. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, and choose b such that $\mathcal{O}_Z(1) := \overline{\pi}^*\mathcal{O}_X(b) \otimes \mathcal{O}_Z(D)$ is an ample line bundle on Z. In [S2] Simpson shows (cf. Lemma 6.8) that a Higgs sheaf $(\mathcal{E}, \Theta)$ on X is the same thing as a sheaf $\mathfrak{E}$ on Z such that $\mathrm{Supp}(\mathfrak{E}) \cap D = \emptyset$. In fact, $\mathcal{E} = \overline{\pi}_*\mathfrak{E}$, and the homomorphism Θ (with $\Theta \wedge \Theta = 0$) is equivalent to giving the $\mathcal{O}_{T^*X}$-module structure. This identification is also called the spectral cover construction. Set $S = \mathrm{Supp}(\mathfrak{E})$, and consider the projection $\pi_S : S \longrightarrow X$. The fiber over a point $x \in X$ is a length $n = \mathrm{rk}(\mathcal{E})$, zero-dimensional subscheme of $T^*_xX = \Omega^1_x$. Hence $\pi_S : S \longrightarrow X$ is an n-to-1 cover of X. If X is a curve, then S is the spectral curve studied in [BNR]. The reason for this name is that if we restrict the Higgs field Θ to a point $x \in X$, we obtain an endomorphism of the fiber E_x with values in $\Omega^1_x \cong \mathbb{C}$,

$$\Theta_x : E_x \longrightarrow E_x \otimes \Omega^1_x,$$

and hence the eigenvalues of Θ_x give a set of n points (counted with multiplicity) of T^*_xX. This set is precisely the fiber of S over $x \in X$.

This identification between Higgs sheaves $(\mathcal{E}, \Theta)$ on X and torsion sheaves $\mathfrak{E}$ on T^*_xX is compatible with morphisms, giving an equivalence of categories. The sheaf $\mathcal{E}$ is torsion free if and only if $\mathfrak{E}$ is of pure dimension $d = \dim(X)$ (i.e., if $\mathfrak{E}$ is torsion free when restricted to its support and every irreducible component of its support has dimension d). Since $\mathcal{O}_{T^*X}(1) = \pi^*\mathcal{O}_X(b)$, the Hilbert polynomials of $\mathfrak{E}$ and $\mathcal{E} = \overline{\pi}_*\mathfrak{E}$ are related by

$$P(\mathfrak{E}, m) = P(\mathcal{E}, bm) =: \widetilde{P}(\mathcal{E}, m),$$

and hence $\mathfrak{E}$ is (semi)stable with respect to $\mathcal{O}_X(1)$ if and only if $\mathcal{E}$ is (semi)stable with respect to $\mathcal{O}_Z(1)$ [S2, Cor. 6.9]. These correspondences between the Higgs sheaf and the sheaf of pure dimension are summarized in Table 2.

Simpson then gives a method to construct the (projective) moduli space $M_{\mathrm{pure}}(Z, \widetilde{P})$ of semistable (with respect to $\mathcal{O}_Z(1)$) sheaves with pure dimension on Z and with Hilbert polynomial $\widetilde{P}$. Using the previous identification,

	$(\mathcal{E},\Theta)$ Higgs sheaf on X	$\mathfrak{E}$ sheaf on T^*X
support	X	$S \subset T^*X$ spectral cover of$X$
Higgs structure	Θ	$\mathcal{O}_{T^*X}$-module structure
sheaf type	torsion free	of pure dimension $\dim(X)$
ample line bundle	$\mathcal{O}_X(1)$	$\mathcal{O}_Z(1) :=$ $\overline{\pi}^*\mathcal{O}_X(b)\otimes\mathcal{O}_Z(D)$
Hilbert polynomial	$P(\mathcal{E},bm)$	$P(\mathfrak{E},m)$
Gieseker stability	w.r.t. $\mathcal{O}_X(1)$	w.r.t. $\mathcal{O}_Z(1)$

Table 2

Algebro-Geometric Dictionary, giving the correspondence between Higgs sheaves on X and sheaves of pure dimension on $T^*X \subset Z$

plus the openness of the condition that $\mathrm{Supp}(\mathfrak{E})$ does not intersect D, one is thus able to identify $M_{\mathrm{Higgs}}(X,P)$, the moduli space of semistable Higgs sheaves with Hilbert polynomial P, as an open subset of $M_{\mathrm{pure}}(Z,\widetilde{P})$.

As in [DUW], instead of considering extensions, it is more convenient to take the equivalent point of view of considering *quotient pairs* of Higgs sheaves.

Definition 7.1. A quotient pair of Higgs sheaves is a surjective morphism of Higgs sheaves

$$(\mathcal{E},\Theta) \xrightarrow{\ q\ } (\mathcal{F},\Psi) \longrightarrow 0,$$

and it will be denoted by q or by $(\mathcal{E},\Theta;\mathcal{F},\Psi)$. A morphism between quotient pairs of Higgs sheaves is a commutative diagram

$$\begin{array}{ccccc} (\mathcal{E}',\Theta') & \xrightarrow{q'} & (\mathcal{F}',\Psi') & \longrightarrow & 0 \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ (\mathcal{E},\Theta) & \xrightarrow{q} & (\mathcal{F},\Psi) & \longrightarrow & 0 \end{array} \tag{7.1}$$

REMARK 7.2. Clearly, isomorphism classes of quotient pairs are the same thing as isomorphism classes of extensions. Indeed, using the notation of Section 2, we take $(\mathcal{E}_1,\Theta_1)=\ker q$, and $(\mathcal{E}_2,\Theta_2)=(\mathcal{F},\Psi)$. We say that a quotient pair is stable if the corresponding Higgs extension is stable. A quotient pair $(\mathcal{E},\Theta;\mathcal{F},\Psi)$ is called torsion free if $\mathcal{E}$ is a torsion free sheaf (note that $\mathcal{F}$ might have torsion).

PROPOSITION 7.3 (Jordan-Hölder filtration). *If $(\mathcal{E},\Theta;\mathcal{F},\Psi)$ is an α-Gieseker semistable torsion free quotient pair, then there exists a filtration*

$$\begin{array}{ccccccccccc} (0,0) & = & (\mathcal{E}_0,\Theta_0) & \subset & (\mathcal{E}_1,\Theta_1) & \subset & \cdots & \subset & (\mathcal{E}_l,\Theta_l) & = & (\mathcal{E},\Theta) \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ (0,0) & = & (\mathcal{F}_0,\Psi_0) & \subset & (\mathcal{F}_1,\Psi_1) & \subset & \cdots & \subset & (\mathcal{F}_l,\Psi_l) & = & (\mathcal{F},\Psi) \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ & & 0 & & 0 & & & & 0 & & 0 \end{array}$$

such that $\mathcal{E}_{i-1}$ is saturated in $\mathcal{E}_i$, the induced quotients

$$\overline{q}_i:(\mathcal{E}_i/\mathcal{E}_{i-1},\overline{\Theta}_i)\longrightarrow(\mathcal{F}_i/\mathcal{F}_{i-1},\overline{\Psi}_i)$$

are α-Gieseker stable, and

$$\begin{aligned} \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})-\alpha\,\mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})}{\mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})} &= \frac{\deg(\mathcal{E})-\alpha\,\mathrm{rk}(\mathcal{F})}{\mathrm{rk}(\mathcal{E})}, \\ \frac{P(\mathcal{E}_i/\mathcal{E}_{i-1},m)}{\mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})} &= \frac{P(\mathcal{E},m)}{\mathrm{rk}(\mathcal{E})} \quad \textit{for all } m, \\ \frac{P(\mathcal{F}_i/\mathcal{F}_{i-1},m)}{\mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})} &= \frac{P(\mathcal{F},m)}{\mathrm{rk}(\mathcal{F})} \quad \textit{for all } m. \end{aligned}$$

Moreover, the direct sum of these quotient pairs, denoted by

$$\mathrm{gr}(q)=\bigoplus_{i=1}^{l}\overline{q}_i,$$

is unique up to isomorphism.

Proof. Analogous to [HL, Prop. 1.5.2] or [DUW, Prop. 2.13]. □

REMARK. Two quotient pairs q and q' are called S-equivalent if $\mathrm{gr}(q)\cong\mathrm{gr}(q')$. If q is α-Gieseker stable, then $\mathrm{gr}(q)\cong q$.

Theorem 7.4. *Fix Hilbert polynomials P and P''. There exists a quasi-projective scheme $M^{\alpha}_{\mathrm{Higgs}}(X,P,P'')$ whose points correspond to S-equivalence classes of quotient pairs of α-Gieseker semistable torsion free Higgs sheaves with the given Hilbert polynomials.*

Proof. The moduli space $M^{\alpha}_{tf}(X,P,P'')$ of quotient pairs of torsion free sheaves has been constructed in [DUW], but since the authors use Simpson's method, their proof works not only for torsion free sheaves, but also for quotient pairs of sheaves of pure dimension. Let $M^{\alpha}_{\mathrm{pure}}(Z,\widetilde{P},\widetilde{P}'')$ be the moduli space of quotient pairs $\mathfrak{E} \longrightarrow \mathfrak{F} \to 0$ of sheaves on Z with $\mathfrak{E}$ of pure dimension. Since the condition that $\mathrm{Supp}(\mathfrak{E})$ does not intersect D is open, using Simpson's identification we finally conclude that $M^{\alpha}_{\mathrm{Higgs}}(X,P,P'')$ is an open subset of $M^{\alpha}_{\mathrm{pure}}(Z,\widetilde{P},\widetilde{P}'')$.

Now we will briefly recall the construction in [DUW, Section 5], indicating what has to be changed to consider sheaves of pure dimension. For any coherent sheaf $\mathfrak{E}$ on Z, its Hilbert polynomial can be written as

$$\chi(\mathfrak{E}(m)) = r(\mathfrak{E}) \deg(\mathrm{Supp}\,\mathfrak{E}) \frac{m^d}{d!} + a(\mathfrak{E}) \frac{m^{d-1}}{(d-1)!} + \cdots,$$

where d is the dimension of the support of $\mathfrak{E}$. Following Simpson [S1, p. 55], we call $r(\mathfrak{E})$ the rank of $\mathfrak{E}$, and $a(\mathfrak{E})$ the degree of $\mathfrak{E}$ with respect to $\mathcal{O}_Z(1)$.

Using these new definitions for rank and degree, the GIT construction in [DUW] goes through for quotient pairs of pure dimension. First one proves that the set of semistable quotient pairs (with fixed Hilbert polynomials $\widetilde{P}$ and $\widetilde{P}''$) is bounded, and then that there is an integer K_0 such that if $k \geq K_0$, for all semistable quotient pairs $q : \mathfrak{E} \longrightarrow \mathfrak{F}$ (with $\mathfrak{E}$ of pure dimension), $\mathfrak{E}(k)$ is generated by global sections and $h^0(\mathfrak{E}(k)) = \chi(\mathfrak{E}(k)) =: N$.

Let $V = \mathbb{C}^N$ be a fixed vector space of dimension N. Consider pairs (q,ϕ), where q is a semistable quotient pair and $\phi : V \longrightarrow H^0(\mathfrak{E}(k))$ is an isomorphism. A pair (q,ϕ) is the same thing as a commutative diagram

$$\begin{array}{ccccc} V \otimes \mathcal{O}_Z & \xrightarrow{q_1} & \mathfrak{E}(k) & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle q} & & \\ V \otimes \mathcal{O}_Z & \xrightarrow{q_2} & \mathfrak{F}(k) & \longrightarrow & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array} \tag{7.2}$$

such that q_1 induces an isomorphism $V \cong H^0(\mathfrak{E}(k))$. Hence for each pair (q,ϕ) we get a point (q_1,q_2) in

$$\mathrm{Quot}(V \otimes \mathcal{O}_Z, \widetilde{P}_m) \times \mathrm{Quot}(V \otimes \mathcal{O}_Z, \widetilde{P}''_m), \tag{7.3}$$

where $\mathrm{Quot}(V\otimes\mathcal{O}_Z,\widetilde{P}_m)$ (resp. $\mathrm{Quot}(V\otimes\mathcal{O}_Z,\widetilde{P}''_m)$) is Grothendieck's quotient scheme, parameterizing quotients of $V\otimes\mathcal{O}_Z$ with Hilbert polynomials $\widetilde{P}_m(i)=\widetilde{P}(m+i)$ (resp. $\widetilde{P}''_m(i)=\widetilde{P}''(m+i)$).

Let $\widehat{Q}_k$ be the closed subset of (3), where $\ker q_1\subset\ker q_2$ (i.e., q_2 factors through q_1), let $Q_k\subset\widehat{Q}_k$ be the subscheme where $\mathfrak{E}$ is of pure dimension, and let $\overline{Q}_k\subset\widehat{Q}_k$ be its closure. The projective scheme $\overline{Q}_k$ parameterizes commutative diagrams like (7.2). Now we have to get rid of the choice of isomorphism ϕ. The group SL(V) acts on (7.3) and hence on $\overline{Q}_k$ (since this is invariant). From the point of view of pairs (q,ϕ), this action corresponds to $(q,\phi)\mapsto(q,g\circ\phi)$ for $g\in\mathrm{SL(V)}$, so to get rid of the choice of the isomorphism ϕ we only need to take the quotient by SL(V). Note that it is enough to use SL(V), and we do not need to use GL(V), because scalar multiplication acts trivially on (7.3). This is done by taking the GIT quotient of $\overline{Q}_k$ by SL(V), but to do this, first we have to linearize the action of SL(V) on an ample line bundle on $\overline{Q}_k$. Following Grothendieck, by tensoring with $\mathcal{O}_Z(j)$ for high enough j, and taking sections, we embed (7.3) (and hence $\overline{Q}_k$) into a product of Grassmanians

$$\mathrm{Gr}(V\otimes W,\widetilde{P}(k+j))\times\mathrm{Gr}(V\otimes W,\widetilde{P}''(k+j)),$$

where $W=H^0(\mathcal{O}_Z(j))$. Using Plücker coordinates we get an embedding in

$$P=\mathbb{P}\Big(\bigwedge{}^{\widetilde{P}(k+j)}(V\otimes W)^\vee\Big)\times\mathbb{P}\Big(\bigwedge{}^{\widetilde{P}''(k+j)}(V\otimes W)^\vee\Big). \tag{7.4}$$

The natural action of SL(V) on (7.4) has a natural linearization on $\mathcal{O}_P(r,s)$ for any r and s, and by restriction we obtain a linearization on the line bundle $\mathcal{O}_P(r,s)|_{\overline{Q}_k}$ on $\overline{Q}_k$.

We choose r and s depending on α as in [DUW, p. 511]. Namely, consider the set of α-semistable quotient pairs $q:\mathfrak{E}\to\mathfrak{F}$. Then consider the set of subobjects $q':\mathfrak{E}'\to\mathfrak{F}'$ with $\mu_\alpha(\mathfrak{E}')=\mu_\alpha(\mathfrak{E})$, where $\mu_\alpha(\mathfrak{E})=(a(\mathfrak{E})+\alpha a(\mathfrak{F}))/r(\mathfrak{E})$. We may assume that $\mathfrak{E}'$ is saturated in $\mathfrak{E}$. The set of such $\mathfrak{E}'$ is bounded, and the set of polynomials of such $\mathfrak{E}'$ is finite. Let C be the maximum of the absolute value of the coefficients of $k^{n-2}/(n-2)!$ in the polynomials $P(\mathfrak{E}',k)/r(\mathfrak{E}')-P(\mathfrak{E},k)/r(\mathfrak{E})$ as $\mathfrak{E}'$ varies over this set. Then choose M large enough so that

$$\Big(M-\alpha\frac{r(\mathfrak{F})}{r(\mathfrak{E})}-\frac{a(\mathfrak{E})}{r(\mathfrak{E})}\Big)\frac{(-\alpha)}{r(\mathfrak{E})^2}-\frac{\deg(\operatorname{Supp}\mathfrak{E})}{d}(d-1)C\geq 1.$$

Finally choose r and s to be positive integers such that

$$\frac{r}{s}=\frac{k\deg(\operatorname{Supp}\mathfrak{E})+Md}{-\alpha d}$$

Note that r and s have to be positive, so that $\mathcal{O}_P(r,s)$ is ample, and this forces $\alpha<0$.

Next one proves that GIT-semistable (resp. stable) points on $\overline{Q}_k$ correspond to α-Gieseker semistable (resp. stable) quotient pairs, and then the moduli space is obtained as the GIT quotient

$$M^{\alpha}_{\mathrm{pure}}(Z, \widetilde{P}, \widetilde{P}'') = \overline{Q}_k /\!\!/ \mathrm{SL(V)}\,.$$

Finally one checks that points of $M^{\alpha}_{\mathrm{pure}}(Z, \widetilde{P}, \widetilde{P}'')$ correspond to S-equivalence classes. □

Appendix: Bott-Chern forms for Higgs bundles

A.1. Introduction. In this Appendix we adapt the computations of Bott and Chern (in their paper [BC]) to construct Bott-Chern forms for Higgs Bundles. We recall the notation of Section 4:

- $\mathcal{E} \longrightarrow X$ is a rank n holomorphic bundle with underlying smooth complex bundle E and holomorphic structure determined by an integrable partial connection $\overline{\partial}_E$ (as in 4.1),
- A Higgs field on E is denoted by Θ. $\nabla'' = \overline{\partial}_E + \Theta$ is the Higgs operator. As in Definition 4.1, a Higgs bundle on X is a pair (E, ∇'') in which $(\nabla'')^2 = 0$,

Definition A.1. Let ϕ be any symmetric $\mathrm{GL}(n, \mathbf{C})$-invariant, k-linear function on Mat_n, the space of $n \times n$ matrices. We extend ϕ to a k-linear map on Mat_n-valued forms as follows: if $a_i \otimes \alpha_i \in \mathrm{Mat}_n \otimes \Omega^{p_i}(X)$, then

$$\phi(a_1 \otimes \alpha_1, \ldots, a_k \otimes \alpha_k) = \phi(a_1, \ldots, a_n)\alpha_1 \wedge \cdots \wedge \alpha_k\ . \tag{A.1}$$

Each $\mathrm{GL}(n, \mathbf{C})$-invariant polynomial ϕ defines a characteristic class for E. This class, denoted by $[\phi] \in H^{2k}(X, \mathbf{C})$, can be represented by the closed $2k$-form

$$\left(\frac{i}{2\pi}\right)^k \phi(F_D) \equiv \left(\frac{i}{2\pi}\right)^k \phi(F_D, F_D, \ldots, F_D)\ , \tag{A.2}$$

where D is any $\mathrm{GL}(n, \mathbf{C})$ connection on E, and F_D is the $\mathrm{GL}(n, \mathbf{C})$-valued 2-form which represents the curvature of D with respect to a local frame. Suppose now that E is the underlying smooth bundle of a holomorphic bundle $\mathcal{E} = (E, \overline{\partial}_E)$. Then any Hermitian bundle metric, say H, determines a unique Chern connection. Denoting the curvature of this connection by F^D_H, we thus get a representative $2k$-form

$$\left(\frac{i}{2\pi}\right)^k \phi(H) = \left(\frac{i}{2\pi}\right)^k \phi(F^D_H)\ , \tag{A.3}$$

corresponding to each metric. If K is any other metric, then $\phi(K)$ and $\phi(H)$ must differ by a closed form since they represent the same class in cohomology. The Bott-Chern forms give a more refined measure of this difference between $\phi(K)$ and $\phi(H)$, for any pair of metrics.

The essential ingredient in this construction is the Chern connection, which uses the defining structure of the holomorphic bundle (i.e., the operator $\overline{\partial}_E$) to associate a unique connection to each metric on $\mathcal{E} = (E, \overline{\partial}_E)$. Suppose now that we add a Higgs field Θ to $\mathcal{E}$ and, as outlined in Section 4, replace $\overline{\partial}_E$ by the Higgs operator $\nabla'' = \overline{\partial}_E + \Theta$. Each metric then produces a unique connection determined by the defining data of the Higgs bundle, i.e., determined by ∇'' (or, equivalently, by $\overline{\partial}_E$ and Θ). Given a $\mathrm{GL}(n, \mathbf{C})$-invariant polynomial we can use these Higgs connections to associate to each metric, H, a Higgs representative for the corresponding characteristic class:

Definition A.2. Let H be a Hermitian metric on the Higgs bundle (E, ∇''). Let ∇_H be the corresponding Higgs connection, and let F_H^{∇} be the curvature of this connection. Let ϕ be any $\mathrm{GL}(n, \mathbf{C})$-invariant, k-linear, symmetric function on M_n. We define

$$\phi_{\mathrm{Higgs}}(H) = \phi(F_H^{\nabla}, F_H^{\nabla}, \ldots, F_H^{\nabla}) \ . \tag{A.4}$$

The Higgs-Bott-Chern forms measure the difference between the closed forms $\phi_{\mathrm{Higgs}}(H)$ and $\phi_{\mathrm{Higgs}}(K)$, for any two metrics H and K. Our main result is as follows:

Theorem A.3. *Corresponding to each* $\mathrm{GL}(n, \mathbf{C})$*-invariant, k-linear function ϕ there is a function of pairs of metrics,* $R_{\mathrm{Higgs}}(H, K)$*, such that:*

(i) $R_{\mathrm{Higgs}}(H, K)$ *takes its values in* $\Omega^{2k-2}(X, \mathbf{C})$.

(ii) $R_{\mathrm{Higgs}}(H, K)$ *is well defined modulo* $\mathrm{Im}\,\overline{\partial} + \mathrm{Im}\,\partial$*, where* $\mathrm{Im}\,\overline{\partial}$ *and* $\mathrm{Im}\,\partial$ *denote the images* $\overline{\partial}(\Omega^{2k-3}(X, \mathbf{C}))$ *and* $\partial(\Omega^{2k-3}(X, \mathbf{C}))$*, respectively, in* $\Omega^{2k-2}(X, \mathbf{C})$.

(iii) *We have*

$$\phi_{\mathrm{Higgs}}(H) - \phi_{\mathrm{Higgs}}(K) = i\overline{\partial}\partial R_{\mathrm{Higgs}}(H, K) \ . \tag{A.5}$$

The forms $R_{\mathrm{Higgs}}(H, K)$ are the analogs for Higgs bundles of the Bott-Chern forms associated to pairs of metrics on a holomorphic bundle. We will thus refer to these as *Higgs Bott-Chern forms*. Notice that unlike on holomorphic bundles, for which the Bott-Chern forms take their values in $\Omega^{(p,p)}(X, \mathbf{C})$, the Higgs Bott-Chern forms need not have holomorphic type (p, p). This difference does not play any role in the proof of Theorem A.3. Indeed, the main ingredients in the proof are formally identical to those of Proposition 3.15 in [BC], the difference being that in place of the Chern connections used in [BC], here we use Higgs connections.

A.2. Definition of $R_{\mathrm{Higgs}}(H, K)$. Fix ϕ, a symmetric $\mathrm{GL}(n, \mathbf{C})$-invariant k-linear function on Mat_n as in Definition A.1.

Notice that though ϕ is symmetric, its extension to Mat_n-valued forms on X is not in general symmetric because of the skew-symmetry of the wedge product on forms. The symmetry will, however, be preserved if *at most one* of the forms has odd degree. Since we will need them later, we record the following basic properties:

LEMMA A.4. *Let ϕ be any $\mathrm{GL}(n,\mathbf{C})$-invariant, k-linear function on Mat_n. For any matrix-valued forms $A_i = a_i \otimes \alpha_i \in \mathrm{Mat}_n \otimes \Omega^{p_i}(X)$ $(i = 1, \ldots, k)$,*

$$d\phi(A_1, \ldots, A_k) = \sum_j (-1)^{p_1 + \cdots + p_{j-1}} \phi(A_1, \ldots, d(A_j), \ldots, A_k) \ , \tag{A.6}$$

If $B = b \otimes \beta \in \mathrm{Mat}_n \, \Omega^q(X)$, then

$$\sum_j (-1)^{p_{j+1} + \cdots + p_k} \phi(A_1, \ldots, [A_j, B], \ldots, A_k) = 0 \ , \tag{A.7}$$

where $[A_i, B] = [a_i, b]\alpha_i \wedge \beta$.

Given two metrics H and K we can pick a 1-parameter family of metrics, $H(t)$, such that $H(0) = H$ and $H(1) = K$, and so that it corresponds to a smooth path from H to K in the space of metrics. We can compute derivatives with respect to the parameter t and thus define L_t by

$$(L_t \eta, \nu)_{H(t)} = \frac{d}{dt} (\eta, \nu)_{H(t)} \tag{A.8}$$

for any smooth sections $\eta, \nu \in \Omega^0(E)$.

LEMMA A.5 ([BC]). *Defined as above, L_t is a bundle endomorphism, i.e., a global section in $\Omega^0(\mathrm{End}\, E)$. If $[H]$ denotes the matrix representing H with respect to the local frame $\{e_i\}$, then the matrix representing L_t is given by*

$$[L_t] = [H(t)]^{-1} [\dot{H}(t)] \ , \tag{A.9}$$

where $[\dot{H}(t)] = \frac{d}{dt}[H(t)]$.

Henceforth, where no confusion can arise, we drop the square braces and denote the matrix representing H by H, etc. Corresponding to the path of metrics $H(t)$ we get (cf. Definition 4.2) a family

$$\nabla'_t = D'_{H(t)} + \Theta^*_{H(t)} \tag{A.10}$$

and thus a family of Higgs connections given by

$$\nabla_t = \nabla'' + \nabla'_t \ . \tag{A.11}$$

Viewing the space of connections as an affine space, and identifying the tangent space at ∇_t with $\Omega^1(X, \mathrm{End}\, E)$, we can compute the derivative with respect to t. This yields an element $\dot{\nabla}_t \in \Omega^1(X, \mathrm{End}\, E)$.

LEMMA A.6 ([BC]). *We have*

$$\frac{d}{dt} \nabla_t = \dot{\nabla}_t = \nabla'_t(L_t) \ , \tag{A.12}$$

where

$$\nabla'_t(L_t) = \nabla'_t \circ L_t - L_t \circ \nabla'_t \ , \tag{A.13}$$

i.e., where $\nabla'_t(L_t)$ *is the contribution to the covariant derivative* $\nabla_t(L_t)$ *resulting from the decomposition of* ∇_t *as* $\nabla'' + \nabla'_t$.

We denote by F_t the curvature of the Higgs connection determined by $H(t)$, and define

$$\phi'_{\text{Higgs}}(F_t, L_t) = \sum_{j=1}^{k} \phi(F_t, \ldots, F_t, L_t, F_t \ldots, F_t) \ . \tag{A.14}$$

We compute

$$\begin{aligned} \partial\phi'_{\text{Higgs}}(F_t, L_t) \quad &= \quad \sum_{j=1}^{k} \sum_{i<j} \phi(F_t, \ldots \partial F_t, \ldots, F_t, L_t, F_t \ldots, F_t) \\ &\quad + \sum_{j=1}^{k} \phi(F_t, \ldots, F_t, \partial L_t, F_t \ldots, F_t) \\ &\quad - \sum_{j=1}^{k} \sum_{i>j} \phi(F_t, \ldots, F_t, L_t, F_t \ldots, \partial F_t, \ldots, F_t) \ . \end{aligned}$$

But by the Bianchi identities for Higgs connections,

$$\nabla'_t(F_t) = 0 = \partial F_t + [F_t, A_t] + [F_t, \Theta_t] \ , \tag{A.15}$$

where $\partial + A_t$ is the $(1,0)$ part of the Chern connection corresponding to $H(t)$. Together with the invariance of ϕ (cf. equations (A.7) and (A.12)), this leads to the expression

$$\begin{aligned} \partial\phi'_{\text{Higgs}}(F_t, L_t) \quad &= \quad \sum_{j=1}^{k} \phi(F_t, \ldots, F_t, \partial L_t - [L_t, A_t] - [L_t, \Theta_t], F_t, \ldots, F_t) \\ &= \quad \sum_{j=1}^{k} \phi(F_t, \ldots, F_t, \nabla'_t(L_t), F_t, \ldots, F_t) \\ &= \quad \phi'_{\text{Higgs}}(F_t, \dot{\nabla}_t) \ . \end{aligned} \tag{A.16}$$

But (cf. Proposition 2.18 in [BC], or any standard discussion of the Chern-Weil homomorphism) $\int_0^1 \phi'_{\text{Higgs}}(F_t, \dot{\nabla}_t)dt$ is precisely the transgression term relating $\phi_{\text{Higgs}}(H)$ and $\phi_{\text{Higgs}}(K)$, i.e.,

$$\phi_{\text{Higgs}}(K) - \phi_{\text{Higgs}}(H) = d\left(\int_0^1 \phi'_{\text{Higgs}}(F_t, \dot{\nabla}_t)dt\right) \ . \tag{A.17}$$

It thus follows from (A.16) that

$$\phi_{\text{Higgs}}(K) - \phi_{\text{Higgs}}(H) = \overline{\partial}\partial \left(\int_0^1 \phi'_{\text{Higgs}}(F_t, L_t) dt \right) . \tag{A.18}$$

We make therefore the following definition.

Definition A.7. Given metrics H and K, and given a path $H(t)$ from H to K, set

$$R_{\text{Higgs}}(H, K) = -i \int_0^1 \phi'_{\text{Higgs}}(F_t, L_t) dt . \tag{A.19}$$

Remark A.8. In particular, (A.18) implies that $\overline{\partial}\partial R_{\text{Higgs}}(H, K)$ is independent of the path H_t joining H and K.

A.3. Independence of the path $H(t)$. To prove that $R_{\text{Higgs}}(H, K)$ is well defined, i.e., is independent of the choice of path $H(t)$, we reformulate the definition in terms of a 1-form on $\text{Met}(E)$, the space of Hermitian metrics on E, and appeal to Stokes' Theorem. Recall (cf. [Ko]) that $\text{Met}(E)$ is a convex domain in an infinite dimensional vector space, and that the tangent space at any point $H \in \text{Met}(E)$ can be identified with hermitian sections of $\text{End}(E)$, i.e.,

$$T_H \,\text{Met}(E) = \text{Herm}_H(E) = \{ u \in \Omega^0(\text{End}\, E) \mid u^{*_H} = u \} . \tag{A.20}$$

Definition A.9. Let U_H be a tangent vector in $T_H \,\text{Met}(E)$, and let $H(t)$ be a path in $\text{Met}(E)$ with $H(0) = H$ and $\dot{H}(0) = U_H$. Define

$$\theta_H(U_H) = \phi'_{\text{Higgs}}(F_H^{\nabla}, L_0) , \tag{A.21}$$

where, as before, $L_t = H(t)^{-1}\dot{H}(t)$.

Given a curve $\gamma = H(t)$ which joins H and K in $\text{Met}(E)$, our definition of $R_{\text{Higgs}}(H, K)$ thus becomes

$$R_{\text{Higgs}}(H, K) = -i \int_\gamma \theta . \tag{A.22}$$

Expressed in this way, it becomes apparent that we can show the independence of the path γ by computing $d\theta$ and applying Stokes' Theorem. Suppose therefore that U_H, V_H are vectors in $T_H \,\text{Met}(H)$. Let $h(s, t)$ be a smooth map from a neighborhood of the origin in $\mathbf{R}^2$ to $\text{Met}(E)$, such that

$$h(0, 0) = H, \quad h_* \left(\frac{\partial}{\partial s} \right) = U_{h(s,t)} , \quad h_* \left(\frac{\partial}{\partial t} \right) = V_{h(s,t)} , \tag{A.23}$$

where $U_{h(s,t)}$ and $V_{h(s,t)}$ are vector fields which extend U_H and V_H, respectively. Then

$$d\theta_H(U,V) = h^*(d\theta)\left(\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial s}\theta\left(h_*\frac{\partial}{\partial t}\right) - \frac{\partial}{\partial t}\theta\left(h_*\frac{\partial}{\partial s}\right) = U_H(\theta_H(V_H)) - V_H(\theta_H(U_H)). \tag{A.24}$$

LEMMA A.10. *Under the identification of tangent spaces of* $\mathrm{Met}(E)$ *with hermitian sections of* $\mathrm{End}(E)$*, as in (A.20) we get*

$$\frac{\partial}{\partial s}(h^{-1}(s,t)V_{h(s,t)})\Big|_{s=t=0} = -U_HV_H + H^{-1}\frac{\partial^2 h}{\partial s\partial t}\Big|_{s=t=0}, \tag{A.25}$$

$$\frac{\partial}{\partial s}F_{h(s,t)} = \nabla''\nabla'_{h(s,t)}(h^{-1}(s,t)U_{h(s,t)}). \tag{A.26}$$

Using (A.21), (A.25) and (A.26) we thus get from (A.24) that

$$\begin{aligned} d\theta_H(U,V) \;=\; &\phi\left([H^{-1}V_H, H^{-1}U_H], F_H^\nabla,\dots,F_H^\nabla\right) \\ &-\sum_{j=2}^k \phi\left(H^{-1}U_H, F_H^\nabla,\dots,F_H^\nabla, \nabla''\nabla'_H(H^{-1}V_H), F_H^\nabla,\dots,F_H^\nabla\right) \\ &+\sum_{j=2}^k \phi\left(H^{-1}V_H, F_H^\nabla,\dots,F_H^\nabla, \nabla''\nabla'_H(H^{-1}U_H), F_H^\nabla,\dots,F_H^\nabla\right) . \end{aligned} \tag{A.27}$$

To simplify the notation, we set $u = H^{-1}U_H$ and $v = H^{-1}V_H$. The first term in (A.27) is then

(A.28)

$$\begin{aligned} \phi\left([v,u], F_H^\nabla,\dots,F_H^\nabla\right) &= -\sum_{j=2}^k \phi\left(v, F_H^\nabla,\dots,F_H^\nabla,[F_H^\nabla,u],F_H^\nabla,\dots,F_H^\nabla\right) \\ &= -\sum_{j=2}^k \phi\left(v, F_H^\nabla,\dots,F_H^\nabla,\nabla''\nabla'_H(u),F_H^\nabla,\dots,F_H^\nabla\right) \\ &\quad -\sum_{j=2}^k \phi\left(v, F_H^\nabla,\dots,F_H^\nabla,\nabla'_H\nabla''(u),F_H^\nabla,\dots,F_H^\nabla\right) . \end{aligned}$$

Here the first equality follows by (A.7) and the second equality follows from the fact that

$$[F_H^\nabla, u] = F_H^\nabla(u) = \nabla''\nabla'_H(u) + \nabla'_H\nabla''(u) \ ,$$

where the F_H^∇ in the expression $F_H^\nabla(u)$ refers to the curvature of the induced connection on End E. Hence (A.27) becomes

$$\begin{aligned} d\theta_H(U,V) = &-\sum_{j=2}^{k}\phi\left(u, F_H^\nabla,\dots,F_H^\nabla,\nabla''\nabla'_H(v),F_H^\nabla,\dots,F_H^\nabla\right) \\ &-\sum_{j=2}^{k}\phi\left(v, F_H^\nabla,\dots,F_H^\nabla,\nabla'_H\nabla''(u),F_H^\nabla,\dots,F_H^\nabla\right). \end{aligned} \tag{A.29}$$

LEMMA A.11. *For any connection D on E, any (symmetric), invariant k-linear function ϕ, and any collection $A_i \in \Omega^{p_i}(\mathrm{End}(E))$ $(i = 1,\dots,k)$, we have*

$$d\phi(A_1,\dots,A_k) = \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,DA_j,\dots,A_k)\ . \tag{A.30}$$

Proof. We fix a local frame for E and write $D = d + A$, where A is the connection 1-form. Thus $DA_j = dA_j + (-1)^{p_j}[A_j,A]$. Using both parts of Lemma A.4 we get

$$\begin{aligned} d\phi(A_1,\dots,A_k) &= \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,dA_j,\dots,A_k) \\ &= \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,DA_j,\dots,A_k) \\ &\quad - \sum_j (-1)^{p_1+\cdots+p_{j-1}+p_j}\phi(A_1,\dots,[A_j,A],\dots,A_k) \\ &= \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,DA_j,\dots,A_k). \end{aligned} \tag{A.31}$$

□

COROLLARY A.12. *If $\nabla'' = \overline{\partial}_E + \Theta$ is the Higgs operator, then*

$$\overline{\partial}\phi(A_1,\dots,A_k) = \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,\nabla''A_j,\dots,A_k)\ , \tag{A.32}$$

and if $\nabla'_H = D'_H + \Theta^*_H$, then

$$\partial\phi(A_1,\dots,A_k) = \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,\nabla'_H A_j,\dots,A_k)\ . \tag{A.33}$$

Proof. If we apply Lemma A.11 to the Chern connection $\overline{\partial}_E + D'_H$, and decompose both side of (A.30) according to holomorphic type, we get

$$\overline{\partial}\phi(A_1,\dots,A_k) = \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,\overline{\partial}_E A_j,\dots,A_k), \tag{A.34}$$

$$\partial\phi(A_1,\dots,A_k) = \sum_j (-1)^{p_1+\cdots+p_{j-1}}\phi(A_1,\dots,D'_H A_j,\dots,A_k). \tag{A.35}$$

But $\nabla'' A_j = \overline{\partial}_E A_j + (-1)^{p_j}[A_j, \Theta]$. Equation (A.34) thus yields

$$\overline{\partial}\phi(A_1, \dots, A_k) = \sum_j (-1)^{p_1+\cdots+p_{j-1}} \phi(A_1, \dots, \nabla'' A_j, \dots, A_k) - \sum_j (-1)^{p_1+\cdots+p_{j-1}+p_j} \phi(A_1, \dots, [A_j, \Theta], \dots, A_k) \ . \tag{A.36}$$

The last summation in (A.36) vanishes by (A.7) in Lemma A.4, i.e., by the invariance of ϕ. Equation (A.33) follows similarly from (A.35), using the invariance of ϕ and $\nabla'_H A_j = D'_H A_j + (-1)^{p_j}[A_j, \Theta^*_H]$. □

Using (A.32) and (A.33) of Corollary A.12, the Bianchi identities (4.16), and Lemma A.4, the terms on the right hand side of (A.29) thus become

$$\sum_{j=2}^{k} \phi(u, F_H^\nabla, \dots, F_H^\nabla, \nabla''\nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla) = -\sum_{j=2}^{k} \phi(\nabla''(u), F_H^\nabla, \dots, F_H^\nabla, \nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla) - \overline{\partial}\alpha(u,v) \tag{A.37}$$

and

$$\sum_{j=2}^{k} \phi(v, F_H^\nabla, \dots, F_H^\nabla, \nabla'_H\nabla''(u), F_H^\nabla, \dots, F_H^\nabla) = -\sum_{j=2}^{k} \phi(\nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla, \nabla''(u), F_H^\nabla, \dots, F_H^\nabla) - \partial\beta(u,v) \ . \tag{A.38}$$

The forms α and β are forms on X, given by

$$-\alpha(u,v) = \phi(u, F_H^\nabla, \dots, F_H^\nabla, \nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla) \tag{A.39}$$

and

$$-\beta(u,v) = \phi(v, F_H^\nabla, \dots, F_H^\nabla, \nabla''(u), F_H^\nabla, \dots, F_H^\nabla) \ . \tag{A.40}$$

Furthermore, since $\nabla'_H(v)$ and $\nabla''(u)$ are 1-forms and F_H^∇ is a 2-form, it follows by the invariance of ϕ (cf. the remark after Definition A.1) that

$$\phi(\nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla, \nabla''(u), F_H^\nabla, \dots, F_H^\nabla) + \phi(\nabla''(u), F_H^\nabla, \dots, F_H^\nabla, \nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla) = 0 \ . \tag{A.41}$$

Equation (A.29) thus reduces to

$$d\theta_H(U,V) = \overline{\partial}\alpha(u,v) + \partial\beta(u,v) \ . \tag{A.42}$$

LEMMA A.13. *The expression $\overline{\partial}\alpha(u,v) + \partial\beta(u,v)$ defines a 2-form on* $\mathrm{Met}(E)$ *with values in* $\mathrm{Im}\,\overline{\partial} + \mathrm{Im}\,\partial$

Proof. Applying (A.33) in Corollary A.12 to $\phi(u, F_H^\nabla, \dots, F_H^\nabla, v, F_H^\nabla, \dots, F_H^\nabla)$ gives

$$\begin{aligned}\phi(u, F_H^\nabla, \dots, F_H^\nabla, \nabla'_H(v), F_H^\nabla, \dots, F_H^\nabla) \\ = -\phi(\nabla'_H(u), F_H^\nabla, \dots, F_H^\nabla, v, F_H^\nabla, \dots, F_H^\nabla) \\ + \partial\phi(u, F_H^\nabla, \dots, F_H^\nabla, v, F_H^\nabla, \dots, F_H^\nabla)\ ,\end{aligned} \tag{A.43}$$

and hence

$$\begin{aligned}\overline{\partial}\alpha(u,v) = \overline{\partial}\phi(\nabla'_H(u), F_H^\nabla, \dots, F_H^\nabla, v, F_H^\nabla, \dots, F_H^\nabla) \\ + \overline{\partial}\partial\phi(u, F_H^\nabla, \dots, F_H^\nabla, v, F_H^\nabla, \dots, F_H^\nabla)\ .\end{aligned} \tag{A.44}$$

Similarly, applying (A.32) to $\phi(v, F_H^\nabla, \dots, F_H^\nabla, u, F_H^\nabla, \dots, F_H^\nabla)$ gives

$$\begin{aligned}\partial\beta(u,v) = \partial\phi(\nabla''(v), F_H^\nabla, \dots, F_H^\nabla, u, F_H^\nabla, \dots, F_H^\nabla) \\ + \partial\overline{\partial}\phi(v, F_H^\nabla, \dots, F_H^\nabla, u, F_H^\nabla, \dots, F_H^\nabla)\ .\end{aligned} \tag{A.45}$$

Notice that in each occurrence of ϕ in (A.44) and (A.45) the arguments include at most one form of odd degree. By the remark after Definition A.1 the expressions are thus symmetric functions of their arguments. Recall also that $\overline{\partial}\partial + \partial\overline{\partial} = 0$. Combining (A.44) and (A.45) thus yields

$$\begin{aligned}\overline{\partial}\alpha(u,v) + \partial\beta(u,v) &= \overline{\partial}\phi(v,, F_H^\nabla, \dots, F_H^\nabla, \nabla'_H(u), F_H^\nabla, \dots, F_H^\nabla) \\ &\quad + \partial\phi(u, F_H^\nabla, \dots, F_H^\nabla, \nabla''(v), F_H^\nabla, \dots, F_H^\nabla) \\ &= -(\overline{\partial}\alpha(v,u) + \partial\beta(v,u))\ .\end{aligned} \tag{A.46}$$

□

We can now prove:

PROPOSITION A.14. *Up to terms in* $\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}$*,* $R_{\mathrm{Higgs}}(H,K)$ *is independent of the path* $H(t)$ *used to compute it in Definition A.19. Thus the map*

$$H \longmapsto R_{\mathrm{Higgs}}(H,K) \tag{A.47}$$

gives a well defined map from $\mathrm{Met}(E)$ *(the space of metrics) to the space* $\Omega^k(X, \mathbf{C})/\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}$*.*

Proof. Let γ_1, γ_2 be any two paths from H to K in Met. Then $\gamma_1 - \gamma_2$ bounds a disk, say Γ, and Stokes' Theorem implies

$$\int_{\gamma_1} \theta - \int_{\gamma_2} \theta = \int_\Gamma d\theta = \int_\Gamma (\overline{\partial}\alpha + \partial\beta)\ . \tag{A.48}$$

□

The rest of Theorem A.3 now follows from the definition of R_{Higgs}.

REMARK A.15. It follows from the definition of R_{Higgs} that if $H(t)$ is a smooth 1-parameter family of metrics, then

$$\frac{d}{dt} R_{\text{Higgs}}(H(t), K) = -ik\phi(L_t, F_t, \dots, F_t) \ , \tag{A.49}$$

where L_t is as in (A.8) and F_t is the curvature of the Higgs connection corresponding to $H(t)$.

A.4. Two special cases.

Case 1. If $k = 1$ and $\phi(A) = \text{Tr}(A)$, then

$$\phi'(F_t, L_t) = \phi(L_t) = \text{Tr}(\dot{H}(t)H(t)^{-1}) \ . \tag{A.50}$$

Thus, denoting the corresponding function R_{Higgs} by $R^{(1)}_{\text{Higgs}}$, we get

$$R^{(1)}_{\text{Higgs}}(H, K) = -i \int_0^1 \text{Tr}(\dot{H}(t)H(t)^{-1}) dt \ . \tag{A.51}$$

Notice that this is the same as the corresponding Bott-Chern form defined on a holomorphic bundle. In both cases (i.e., with or without the extra Higgs bundle structure) we get

$$R^{(1)}_{\text{Higgs}}(H, K) = -i \ln HK^{-1} \ , \tag{A.52}$$

which is manifestly independent of the path from H to K.

Case 2. If $k = 2$ and $\phi(A_1, A_2) = -\frac{1}{2}\text{Tr}(A_1A_2 + A_2A_1)$, then

$$\phi'(F_t, L_t) = \phi(F_t, L_t) = -\text{Tr}(F_tL_t) \ , \tag{A.53}$$

$$R^{(2)}_{\text{Higgs}}(H, K) = i \int_0^1 \text{Tr}(F_tL_t) dt \ .$$

The functional defined by Simpson in [S3] is

$$M_S(H, K) = \int_X R^{(2)}_{\text{Higgs}}(H, K) \wedge \omega^{d-1} \ . \tag{A.54}$$

This is the Higgs analog of the function defined by Donaldson in [Do1], which is given by the same formula, but with the Bott-Chern form $R^{(2)}(H, K)$ in place of the Higgs Bott-Chern form $R^{(2)}_{\text{Higgs}}(H, K)$.

PROPOSITION A.16. *Take $H(t) = Ke^ts$, with $s = s^{*_K}$. Then*

$$\frac{d}{dt} M_S(H(t), K) = -2i \int_X \phi'(F_t, s) \wedge \omega^{d-1} = 2i \int_X \text{Tr}(F_ts) \wedge \omega^{d-1}, \tag{A.55}$$

$$\frac{d^2}{dt^2} M_S(H(t), K)|_{t=0} = |\nabla''(s)|^2_K. \tag{A.56}$$

Proof. The formulae for $\frac{d}{dt}M_S$ follow directly from (A.49). Using this result, plus the fact that (cf. (A.26)) $dF_t/dt = \nabla''\nabla'_t(s)$, we get

$$\begin{aligned}\frac{d^2}{dt^2}M_S(H(t),K)|_{t=0} &= 2i\int_X \mathrm{Tr}(\nabla''\nabla'_K(s)s)\wedge\omega^{d-1} \\ &= -2i\int_X \mathrm{Tr}(\nabla''(s)\wedge\nabla'_K(s))\wedge\omega^{d-1} \\ &= 2\int_X |\nabla''(s)|^2_K\wedge\omega^{d-1}\ .\end{aligned}\tag{A.57}$$

The second equality follows by (A.37). The third follows by Lemma 3.1(b) in [S3]. □

References

[BNR] A. Beauville, M.S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor,* J. Reine Angew. Math. **398** (1989), 169–179.

[BC] R. Bott and S.S. Chern, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections,* Acta. Math. **114** (1965), 71–112.

[B1] S. Bradlow, *Special metrics and stability for holomorphic bundles with global sections,* J. Differential Geom. **33** (1991), 169–213.

[BGP] S. Bradlow and O. Garcia-Prada, *Higher cohomology triples and holomorphic extensions,* Comm. Anal. Geom. **3** (1995), 421–463.

[BDGW] S. Bradlow, G.D. Daskalopoulos, O. Garcia-Prada, and R. Wentworth, *Stable augmented bundles over Riemann surfaces,* Vector bundles in algebraic geometry (Durham, 1993), London Math. Soc. Lecture Notes Ser., vol. 208, Cambridge Univ. Press, Cambridge, 1995, pp. 15–67.

[DUW] G. Daskalopoulos, K. Uhlenbeck, and R. Wentworth, *Moduli of extensions of holomorphic bundles on Kähler manifolds,* Comm. Anal. Geom. **3** (1995), 479–522.

[Do1] S. Donaldson, *Anti-self-dual Yang-Mills connections on a complex algebraic surface and stable vector bundles,* Proc. London Math. Soc. **50** (1985), 1–26.

[Do2] ______, *Infinite determinants, stable bundles and curvature,* Duke Math. J. **54** (1987), 231–247.

[H] N. Hitchin, *The self-duality equations on a Riemann surface,* Proc. London Math. Soc. **55** (1987), 59–126.

[HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves,* Aspects of Mathematics, E31, Vieweg, Braunschweig, 1997.

[Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton Univ. Press, Princeton, 1987.

[S1] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47–129.

[S2] ______, *Moduli of representations of the fundamental group of a smooth projective variety II,* Publ. Math. I.H.E.S. **80** (1995), 5–79.

[S3] ______, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization,* J. Amer. Math. Soc. **1** (1988), 867–918.

[S4] ______, *Higgs bundles and local systems*, Publ. Math. I.H.E.S. **75** (1992), 5–95.

[UY] K. Uhlenbeck and S.T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles,* Frontiers of the mathematical sciences: 1985 (New York, 1985), Comm. Pure and Appl. Math. **39** (1986), S257–S293.

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