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ON COTYPE AND SUMMING PROPERTIES IN BANACH SPACES

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Abstract. Given a complex Banach space X and $2 \leq q < \infty$, we show that X has weak cotype q if and only if there is a constant $c > 0$ such that

$$\sum_k \|x_k\| \leq c n^{1-1/q} \sup_{\varepsilon_k \pm 1} \left\| \sum_k \varepsilon_k x_k \right\|$$

holds for all n-dimensional subspaces $E \subset X$ and all vectors $(x_k)_k \subset E$. Moreover, these conditions are equivalent to a decrease rate of order $k^{-1/q}$ for the sequence of eigenvalues of operators on ℓ_∞ factoring through X. This is an analog of Talagrand's theorem on the equivalence of the cotype q property and the absolutely $(q,1)$-summing property for Banach spaces in the range $q > 2$. Surprisingly, this 'weak' analog also extends to the case $q = 2$. Moreover, we show if $q > 2$ and X has weak cotype q, then the cotype q constant with n vectors can be estimated by any iterates of the function $L(x) = \max\{1, \log(x)\}$ applied to $(\log n)^{1/q}$.

1. Introduction

Unconditional and absolute convergence is a classical topic in Banach space theory; see, for example, the work of Orlicz [Or], Grothendieck [Gr], and Lindenstrauss and Pełczyński [LP]. In particular, by a well-known theorem of Dvoretzky, in infinite dimensional Banach spaces unconditional convergence does not imply absolute summability. However, Orlicz showed that in the spaces L_p, $1 \leq p \leq 2$, unconditionally convergent series are at least 2-summing. This is best possible. Indeed, the Dvoretzky-Rogers Lemma ensures that in every infinite dimensional Banach space X and for every $\delta > 0$ and every $n \in \mathbb{N}$ there exist vectors $x_1, \ldots, x_n$ such that

$$\sum_{k=1}^n \|x_k\| \geq (1-\delta)\sqrt{n} \quad \text{and} \quad \sup_{|\alpha_k| \leq 1} \left\| \sum_{k=1}^n \alpha_k x_k \right\| \leq 1.$$

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In this paper we consider Banach spaces in which the converse inequality

$$\sum_{k=1}^{n} \|x_k\| \leq c\sqrt{n} \sup_{\varepsilon_k = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_k x_k \right\|$$

holds for some constant c, all $n \in \mathbb{N}$ and all vectors $x_1, \ldots, x_n \in X$. We will prove that this condition holds if and only if X has weak cotype 2. The notion of weak cotype 2 was introduced by Milman and Pisier [MiP]; it is equivalent to the existence of a constant c' such that

$$\left(\frac{\operatorname{vol}(B_E)}{\operatorname{vol}(\mathcal{E}_{\max})} \right)^{1/n} \leq c'$$

holds for all $n \in \mathbb{N}$ and all n-dimensional subspaces $E \subset X$ with associated ellipsoid $\mathcal{E}_{\max}$ of maximal volume contained in the unit ball B_E of E. The notion of weak cotype 2, and the more general notion of weak cotype q, have become standard tools in the local theory of Banach spaces (see [Ps1]). Our results are motivated by the following result of Talagrand [Ta2].

Theorem 1 (Talagrand). *Let $2 < q < \infty$ and let X be a complex Banach space. Then the following are equivalent:*

(1) *The identity of X is absolutely $(q, 1)$-summing, i.e., there is a constant $c_1 > 0$ such that*

$$\left(\sum_{k} \|x_k\|^q \right)^{1/q} \leq c_1 \sup_{\varepsilon_k \pm 1} \left\| \sum_{k} \varepsilon_k x_k \right\|,$$

for all $(x_k)_k \subset X$.

(2) *X is of cotype q, i.e., there is a constant c_2 such that*

$$\left(\sum_{k} \|x_k\|^q \right)^{1/q} \leq c_2 \, \mathbb{E} \left\| \sum_{k} \varepsilon_k x_k \right\|,$$

for all $(x_k)_k \subset X$, where ε_k is a sequence of independent Bernoulli variables and $\mathbb{E}$ denotes the expected value.

This result does not hold for $q = 2$. Indeed, Talagrand constructed a Banach sequence space with a symmetric basis and the Orlicz property, but without cotype 2. Nevertheless, using Talagrand's techniques, we will extend his result to the context of weak cotype q spaces, and show that in this setting the result holds also for $q = 2$. Our main result is the following theorem.

THEOREM 2. *Let $2 \leq q < \infty$. For a complex Banach space X the following properties are equivalent.*

(i) *There exists a constant $c_1 > 0$ such that for all $n \in \mathbb{N}$ and all n dimensional subspaces $E \subset X$ we have*
$$\sum_k \|x_k\| \leq c_1 n^{1-1/q} \sup_{\varepsilon_k = \pm 1} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|$$
for all sequences $(x_k)_k \subset E$.

(ii) *X is of weak cotype q; equivalently, there exists a constant c_2 with $0 < c_2 < 1/(12e)$ such that for all $n \in \mathbb{N}$ and all vectors $x_1, \ldots, x_n$ satisfying*
$$\sum_{k=1}^n |\langle x_k, x^* \rangle|^2 \leq \|x^*\|^2 \text{ for } x^* \in X^*$$
and
$$\|x_k\| \geq c_2 \text{ for } k = 1, \ldots, n$$
we have
$$\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\| \geq c_2 n^{1/q}.$$

(iii) *There exists a constant c_3 such that for all operators $T : X \to X$ which factor through ℓ_∞, i.e., operators of the form $T = SR$, where $R : X \to \ell_\infty$ and $S : \ell_\infty \to X$, the sequence of eigenvalues $(\lambda_n(T))$, counted with multiplicity and arranged in non-increasing order, satisfies*
$$\sup_{n \in \mathbb{N}} n^{1/q} |\lambda_n(T)| \leq c_3 \|S\| \, \|R\| \, .$$

Moreover, if $q > 2$ and one of the above conditions is satisfied, then there is a constant C such that
$$\left(\sum_k \|x_k\|^q \right)^{1/q} \leq C^{1+l} L^{(l)}((1 + \log_2 n)^{1/q}) \, \mathbb{E} \left\| \sum_k \varepsilon_k x_k \right\|$$
holds for all $n, l \in \mathbb{N}$ and $(x_k)_k \subset E$, whenever E is an n-dimensional subspace of X. Here L is defined by $L(x) = \max\{1, \log_2\}$ and $L^{(l)}$ denotes the l-th iterate of this function.

Condition (ii) is not the standard definition of weak cotype q, and for $q > 2$ it suffices to assume $\|x_k\| \geq c_2$. For $q = 2$ equal norm cotype 2 is equivalent to cotype 2. However, the technical condition (ii) allows us to apply Talagrand's probabilistic machinery. One of our central observations is that this condition is indeed equivalent to the usual weak cotype q condition and that Talagrand's machinery works under the slightly weaker hypothesis (i). Condition (iii) does not seem to have an analog in the classical cotype q situation.

This paper is organized as follows. In Section 2 we introduce some basic notation and terminology. In Section 3 we define the notion of weak cotype and establish the equivalence of conditions (i)–(iii) of Theorem 2. In Section 4 we prove the iterated log estimate in Theorem 2. To this end we will associate with a Banach space X the symmetric sequence spaces $Y_C(X)$ and $Y_S(X)$ which encode the optimal cotype information and the optimal absolute summing information, respectively. These spaces turn out to be 'self-concave', a generalization of the well-known submultiplicativity of cotype numbers. We will use the self-concavity property to prove the iterated log estimate. This property might also be of independent interest in the general analysis of cotype properties in Banach lattices or symmetric sequence spaces.

2. Preliminaries

In what follows $c_0, c_1, \dots$ always denote universal constants. We denote by $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$ the integer part of a real number. We use standard Banach space notation as in [Pi1], [Pi2], [Ps3], [ToJ]. In particular, the classical spaces ℓ_q and ℓ_q^n, $1 \leq q \leq \infty$, $n \in \mathbb{N}$, are defined in the usual way. We will also use the Lorentz spaces ℓ_{pq}, $1 \leq p, q \leq \infty$, which consist of all sequences $\sigma \in \ell_\infty$ such that $\|\sigma\|_{pq} < \infty$, where

$$\|\sigma\|_{pq} := \left(\sum_n \left(n^{1/p}\, \sigma_n^* \right)^q n^{-1} \right)^{1/q} \; < \; \infty$$

if $q < \infty$, and

$$\|\sigma\|_{p\infty} := \sup_{n \in \mathbb{N}} \, n^{1/p}\, \sigma_n^* < \infty,$$

and where $\sigma^* = (\sigma_n^*)_{n \in \mathbb{N}}$ denotes the non-increasing rearrangement of the sequence $(|\sigma|_n)_{n \in \mathbb{N}}$. More generally, for a non-decreasing sequence $(g(n))_{n \in \mathbb{N}}$ with $g(1) = 1$ we denote by $\ell_{g,\infty}$ the space of sequences σ such that

$$\|\sigma\|_{g,\infty} := \sup_n g(n) \sigma_n^* < \infty.$$

The standard reference on operator ideals is the monograph of Pietsch [Pi1]. The ideals of linear bounded operators, finite rank operators, and integral operators are denoted by $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{I}$, respectively.

Let $1 \leq q \leq p \leq \infty$ and $n \in \mathbb{N}$. For an operator $T \in \mathcal{L}(X, Y)$ the (p, q)-summing norm of T with respect to n vectors is defined by

$$\pi_{(p,q)}^n(T) := \sup \left\{ \left(\sum_{k=1}^n \|T x_k\|^p \right)^{1/p} \;\middle|\; \sup_{\|x*\|_{X^*} \leq 1} \left(\sum_{k=1}^n |\langle x_k, x^* \rangle|^q \right)^{1/q} \leq 1 \right\}.$$

An operator T is said to be absolutely pq-summing (or simply pq-summing) if

$$\pi_{(p,q)}(T) := \sup_n \pi_{(p,q)}^n(T) < \infty.$$

We let $\Pi_{pq}(X,Y))$ denote the set of such operators T. Then $(\Pi_{pq}, \pi_{(p,q)})$ is a maximal and injective Banach ideal (in the sense of Pietsch). As usual, we set $(\Pi_q, \pi_q) := (\Pi_{qq}, \pi_{(q,q)})$ and write $\pi_{pq}^{(n)}(X)$ and $\pi_{pq}(X)$ for $\pi_{pq}^{(n)}(\mathrm{id}_X)$ and $\pi_{pq}(\mathrm{id}_X)$, respectively. We also use the standard convention that X has a certain summing property or cotype property if id_X has this property. For further information about absolutely pq-summing operators we refer to the monograph of Tomczak-Jaegermann [ToJ].

We let $(\varepsilon_k)_{k\in\mathbb{N}}$, resp. $(g_k)_{k\in\mathbb{N}}$, denote sequences of independent normalized Bernouilli, resp. Gaussian, variables. A Banach space X is of Rademacher, resp. Gaussian, cotype q if there exists a constant $c > 0$ such that for all sequences $(x_k)_{k=1}^n \subset X$ we have

$$\left(\sum_k \|x_k\|^q\right)^{1/q} \le c\mathbb{E}\left\|\sum_k \varepsilon_k x_k\right\|,$$

resp.

$$\left(\sum_k \|x_k\|^q\right)^{1/q} \le c\mathbb{E}\left\|\sum_k g_k x_k\right\|,$$

where $\mathbb{E}$ denotes the expected value. We denote the best possible constants in these inequalities by $Rc_q(X) := Rc_q(\mathrm{id}_X)$ and $c_q(X) := c_q(Id_X)$, respectively, and we let Rc_q^n and c_q^n denote the corresponding constants for sequences of n vectors (with n fixed). As usual, we will use the abbreviation

$$\ell(u) := \sup_n \left(\mathbb{E}\left\|\sum_{k=1}^n g_k u(e_k)\right\|^2\right)^{1/2}$$

for all $u \in \mathcal{L}(\ell_2, X)$. Here and in the following $(e_k)_k$ denotes the sequence of unit vectors in ℓ_2 (or in an arbitrary sequence space). From the rotation invariance of the Gaussian variables it follows that the ℓ-norm is invariant under orthogonal transformations of this basis in ℓ_2.

Finally, we recall the notions of approximation numbers and Weyl numbers. Given an operator $T \in \mathcal{L}(E,F)$ and $n \in \mathbb{N}$, the n-th *approximation number* is defined by

$$a_n(T) := \inf\{\, \|T - S\| \mid \mathrm{rank}(S) < n \,\},$$

and the n-th *Weyl number* is given by

$$x_n(T) := \sup\{\, a_n(Tu) \mid u \in \mathcal{L}(\ell_2, E),\ \|u\| \le 1 \,\}.$$

Let $s \in \{a, x\}$. By $\mathcal{L}_{pq}^{(s)}$ and $\mathcal{L}_{g,\infty}^{(s)}$ we denote the ideals of operators T such that $(s_n(T))_{n\in\mathbb{N}} \in \ell_{pq}$ and $(s_n(T))_{n\in\mathbb{N}} \in \ell_{g,\infty}$, respectively, and we define the associated quasi-norms by

$$\ell_{pq}^{(s)}(T) := \|(s_n(T))_{n\in\mathbb{N}}\|_{\ell_{pq}} \quad \text{and} \quad \ell_{g,\infty}^{(s)}(T) := \|(s_n(T))_{n\in\mathbb{N}}\|_{\ell_{g,\infty}}.$$

3. Weak cotype q

Let us recall the definition of weak cotype given in terms of approximation numbers. A linear operator $T : X \to Y$ is of *weak cotype* q, if there exists a constant $C > 0$ such that for all $u : \ell_2^n \to X$

$$\sup_k k^{1/q} a_k(Tu) \leq C\,\mathbb{E} \left\| \sum_{i=1}^{n} g_i T u(e_i) \right\| .$$

The weak cotype q-norm of T is defined as $\mathrm{wc}_q(T) := \inf C$, where the infimum is taken over all such constants C.

Remark 3.1. Mascioni [Ma] proved that for $q > 2$ an operator T is of weak cotype q if and only if T satisfies the cotype q estimate for vectors x_i with $\|Tx_i\|_Y = 1$. An operator T has 'equal norm cotype 2' if and only if T has cotype 2.

In this section, we consider a weak ℓ_q analog of absolutely q-summing operators. More precisely, an operator $T : X \to Y$ is said to be $(\ell_{q,\infty}, 1)$-*summing* if there exists a constant $C > 0$ such that for all $x_1, \dots, x_n$

$$\|(\|Tx_k\|)_k\|_{q,\infty} \leq C \sup_{\varepsilon_k = \pm 1} \left\| \sum_k \varepsilon_k x_k \right\|_X .$$

We denote the best-possible constant in this inequality by $\pi_{(q,\infty),1}(T)$; i.e., $\pi_{(q,\infty),1}(T) = \inf C$, where the infimum is taken over all such constants C. A Banach space X is said to be $(\ell_{q,\infty}, 1)$-summing if $\pi_{(q,\infty),1}(T)(\mathrm{id}_X) < \infty$.

In addition to the conditions (i), (ii) and (iii) of Theorem 2, we consider the following conditions:

(i′) $\pi_{(q,\infty),1}(\mathrm{id}_X) < \infty$.
(ii′) X has weak cotype q.
(iii′) There exists a constant $c_3 > 0$ such that every operator $T : \ell_\infty \to X$ satisfies

$$\sup_k k^{1/q} x_k(T) \leq c_3 \|T\| .$$

We note that (iii′) also makes sense for a *real* Banach space X, and we will indeed show that it is equivalent to the other conditions.

We now prove the easy implications of Theorem 2. These follow from well-known facts in the local theory of Banach spaces and are closely related to results in [Ma].

(i) $\Longrightarrow$ (i′): Indeed, if $\|x_1\| \geq \cdots \geq \|x_k\|$, then (i) implies

$$k^{1/q} \|x_k\| \leq k^{1/q-1} \sum_{j=1}^{k} \|x_j\| \leq c_1 \sup_{\varepsilon_i = \pm 1} \left\| \sum_i \varepsilon_i x_i \right\|_X .$$

Taking the supremum over all k, we deduce that id_X is $(\ell_{q,\infty}, 1)$-summing.

(i′) $\Longrightarrow$ (i): If X is $(\ell_{q,\infty}, 1)$-summing, then for any vectors $x_1, \dots, x_n$ such that $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|k_k\|$ we have

$$\begin{aligned}\sum_{j=1}^{k} \|x_j\| &\leq \sum_{j=1}^{k} j^{-1/q} \sup_i i^{1/q} \|x_i\| \\ &\leq \left(1 - \frac{1}{q}\right)^{-1} k^{1-1/q} \pi_{(q,\infty)}(\mathrm{id}_X) \sup_{\varepsilon_i = \pm 1} \left\| \sum_i \varepsilon_i x_i \right\|_X .\end{aligned}$$

(ii′) $\Longrightarrow$ (iii′): First, we note that a weak cotype q space has cotype $\tilde{q}$ for all $\tilde{q} > q$. Using Maurey's Theorem on operators on ℓ_∞ (see [ToJ]), we obtain a constant $c(X) = c(q, X)$ such that for every operator $S : \ell_\infty \to X$ and $r = 2q$

$$\pi_r(S) \leq c(X) \|S\| .$$

Now let $u : \ell_2 \to \ell_\infty$. Then we obtain from Khintchine's inequality for Gaussian random variables (see [ToJ])

$$\begin{aligned}\sup_k k^{1/q} a_k(Su) &\leq \mathrm{wc}_q(\mathrm{id}_X)\ell(Su) \\ &\leq \mathrm{wc}_q(\mathrm{id}_X) \left(\mathbb{E} \left\| \sum_i g_i Su(e_i) \right\|^r \right)^{1/r} \\ &\leq \mathrm{wc}_q(\mathrm{id}_X)\pi_r(S) \sup_{\|z\|_{\ell_\infty^*} \leq 1} \left(\mathbb{E} \left| \sum_i g_i \langle z, u(e_i) \rangle \right|^r \right)^{1/r} \\ &\leq c_0 \sqrt{r}\, \mathrm{wc}_q(\mathrm{id}_X) c(X) \|S\| \|u\| .\end{aligned}$$

Taking the supremum over all u of norm less than 1 yields the assertion.

(iii′) $\Longrightarrow$ (iii): This follows directly from the eigenvalue behaviour of the operator ideal $\mathcal{L}^{(x)}_{(q,\infty)}$; see [Pi1].

[(iii′) or (iii)] $\Longrightarrow$ (i): Let $x_1, \dots, x_n \in X$ and $x_1^*, \dots, x_n^* \in X^*$ be norm one elements such that $\langle x_j^*, x_j \rangle = 1$. Let us consider operators $R : \ell_\infty^n \to X$ and $S : X \to \ell_\infty^n$ defined by $R(e_i) = x_i$ and $S(x) = (\langle x_i^*, x \rangle)_{i=1}^n$. The classical extreme point argument shows (see [LT])

$$\|R\| \leq 2 \sup_{\varepsilon_k = \pm 1} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_X .$$

Then we obtain from (iii)

$$\begin{aligned}\sum_{k=1}^{n} \|x_k\| &= \sum_{k=1}^{n} \langle x_k, x_k^* \rangle = |\operatorname{tr}(SR)| = |\operatorname{tr}(RS)| \\ &\leq n^{1-1/q} \sup_k k^{1/q} \lambda_k(RS) \leq n^{1-1/q} c_4 \|S\| \|R\| .\end{aligned}$$

Assuming (iii′) for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, we have

$$\begin{aligned} |\operatorname{tr}(SR)| &\leq \iota_1(SR) \\ &\leq 4 \sum_{k=1}^{n} x_k(SR) \leq 4n^{1-1/q} \sup_k k^{1/q} x_k(R) \, \|S\| \\ &\leq n^{1-1/q} 4c_3 \, \|R\| \, \|S\| , \end{aligned}$$

where ι_1 is the integral norm, which can be estimated by the ℓ_1-norm of the Weyl numbers using a complexification argument.

The rest of this section is devoted to establishing the crucial implication (i′) $\Longrightarrow$ (ii) and the equivalence (ii) $\Longleftrightarrow$ (ii′). We start with the following lemma, which is essentially known (see [Ps3]), except for the values of the constants (and this is important for our applications).

Lemma 3.2. *An operator $T \in \mathcal{L}(X, Y)$ is of weak cotype q if and only if there is a number δ with $0 < \delta < 1$ and a constant $C > 0$ such that for all $n \in \mathbb{N}$ and all $u \in \mathcal{L}(\ell_2^n, X)$ there is a natural number $k \leq \max(\delta n, 1)$ such that*

$$n^{1/q} a_k(Tu) \leq C\ell(u).$$

Moreover, the best constant $C_\delta(T) = \inf C$ satisfies

$$\left(\frac{\delta}{2}\right)^{1/q} C_\delta(T) \leq \operatorname{wc}_q(T) \leq e \max(\delta^{-1/2}, (1-\delta)^{-1/2}) C_\delta(T).$$

Proof. If T has weak cotype q and k is the integer part of $\max\{\delta n, 1\}$, then we have

$$n^{1/q} a_k(Tu) \leq \left(\frac{n}{k}\right)^{1/q} \operatorname{wc}_q(T)\ell(u) \leq \left(\frac{2}{\delta}\right)^{1/q} \operatorname{wc}_q(T)\ell(u)$$

provided $\delta n \geq 1$. On the other hand, for $\delta n < 1$, we certainly have

$$n^{1/q} a_1(Tu) \leq \delta^{-1/q} \operatorname{wc}_q(T)\ell(u).$$

Thus $C_\delta(T) \leq (2/\delta)^{1/q} \operatorname{wc}_q(T)$, which is the asserted upper bound for $C_\delta(T)$.

In order to prove the corresponding lower bound, we fix $u \in \mathcal{L}(\ell_2, X)$, $v \in \Pi_2(Y, \ell_2)$, and $n \in \mathbb{N}$. By polar decomposition there is a subspace $H \subset \ell_2$ with $\dim H = n$ such that $a_n(vTu) = a_n(vTu\iota_H)$. We set $m := n - k + 1$. Using the multiplicativity of the Weyl numbers and the well-known Weyl number estimate for the 2-summing norm (see [Pi1]), we obtain

$$\begin{aligned} a_n(vTu) &= a_{n-k+1+k-1}(vTu\iota_H) \\ &\le a_k(Tu\iota_H)x_m(v) \\ &\le C_\delta(T)n^{-1/q}\ell(u)m^{-1/2}\pi_2(v) \\ &\le C_\delta(T)n^{-1/q-1/2}\sqrt{\frac{n}{m}}\ell(u)\pi_2(v). \end{aligned}$$

If $\delta n \le 1$, then $\sqrt{n/m} \le \delta^{-1/2}$. If $\delta n \ge 1$, then $k \le \delta n$, and hence $m \ge (1-\delta)n$. In either case,

$$\sqrt{\frac{n}{m}} \le \sqrt{\max(\delta^{-1}, (1-\delta)^{-1})},$$

and from [DJ1, Proposition 4.1] we deduce

$$\mathrm{wc}_q(T) \le e\max(\delta^{-1/2}, (1-\delta)^{-1/2})C_\delta(T). \qquad \square$$

The equivalence of (ii) and (ii′) follows from the following criterion.

Proposition 3.3. *Let $D \ge 16e$ be fixed. An operator T is of weak cotype q if and only if there is a constant $C > 0$ such that for all vectors $x_1, \dots, x_n \in X$ satisfying*

$$\sum_{j=1}^{n} |\langle y^*, Tx_j\rangle|^2 \le \|y^*\|^2 \quad \text{for all } y^* \in Y^*$$

and

$$\|Tx_j\| \ge \frac{1}{D} \qquad \text{for all } j = 1, \dots, n$$

we have

$$\left(\mathbb{E}\left\|\sum_{j=1}^{n} g_j x_j\right\|^2\right)^{1/2} \ge \frac{1}{C}n^{1/q}.$$

Moreover, if $C(T)$ denotes the infimum over all such constants C, then

$$\frac{1}{64D^2}C(T) \le \mathrm{wc}_q(T) \le C(T).$$

Proof. In order to prove the upper bound for $C(T)$, we fix $x_1, \dots, x_n$ as above, and let $y_1^*, \dots, y_n^*$ be unit vectors in B_{Y^*} such that

$$\frac{1}{D} \le |\langle y_j^*, Tx_j\rangle|$$

for $j = 1, \dots, n$. We consider the contraction $v : Y \to \ell_\infty^n$ defined by $v(y) = (\langle y_j^*, y\rangle)_{j=1}^n$ and note that by assumption the map $w : \ell_2^n \to Y$ defined by

$w = Tu$ is also a contraction. Let k be maximal such that $64D^2(k-1) \le n$. Using the $\ell_{2,1}^{(a)}$ estimate of the 2-summing norm (see [Pi2]), we obtain

$$\begin{aligned} \frac{\sqrt{n}}{D} &\le \left(\sum_{i=1}^{n} \|vw(e_i)\|_\infty^2 \right)^{1/2} \le \pi_2(vw) \\ &\le 2 \sum_{i=1}^{n} \frac{a_i(vw)}{\sqrt{i}} \le 2 \sum_{i=1}^{k-1} \frac{a_i(vw)}{\sqrt{i}} + a_k(Tu) 4\sqrt{n} \\ &\le 4\sqrt{k-1} + 4\sqrt{n} a_k(Tu) \le \frac{\sqrt{n}}{2D} + 4\sqrt{n} a_k(Tu). \end{aligned}$$

Hence we obtain $a_k(Tu) \ge 1/(8D)$. By the definition of weak cotype q, it follows that

$$n^{1/q} \le (64D^2)^{1/q} k^{1/q} \le 64D^2 k^{1/q} a_k(Tu) \le 64D^2 \operatorname{wc}_q(T) \ell(u),$$

which yields the desired upper bound $C(T) \le 64D^2 \operatorname{wc}_q(T)$.

We now prove the more difficult lower bound, i.e., $C(T) \ge \operatorname{wc}_q(T)$. By restricting to a finite dimensional subspace, we may assume that T is of finite rank. According to Lemma 3.2 there is a positive real number A with

$$\frac{\operatorname{wc}_q(T)}{3e} < A < C_{1/2}(T).$$

By definition there exist $m \in \mathbb{N}$ and $u \in \mathcal{L}(\ell_2^m, X)$ such that for all $k \le \max(m/2, 1)$

$$a_k(Tu) > \frac{1}{3e16^{1/q}} \quad \text{and} \quad \ell(u) \le \frac{m^{1/q}}{3e16^{1/q} A}.$$

If $m \le 4$, we set $n = 1$. If $m > 4$, we set $k = [m/2] \ge m/4$ and $n := [k/2] > m/16$. By the definition of weak cotype q, we have

$$n^{1/q} a_n(Tu) \le \operatorname{wc}_q(T) \ell(u) < 3eA \frac{m^{1/q}}{3e16^{1/q} A} \le n^{1/q}.$$

Hence there exists a subspace $H \subset \ell_2^m$ of codimension $< n$ such that

$$\|Tu|_H\| \le 1.$$

On the other hand, we deduce from the elementary properties of the approximation numbers that

$$\begin{aligned} (12e)^{-1} &\le (3e16^{1/q})^{-1} \le a_k(Tu) = a_{n+(k-n+1)-1}(Tu) \\ &\le a_{k-n+1}(Tu_{|H}) \le a_n(Tu|_H). \end{aligned}$$

By a result of Lewis (see [Ps3]) there is an orthogonal sequence $(w_j)_{j=1}^n \in H$ such that

$$\|Tu(w_j)\| \ge \frac{1}{12e} \quad \text{for all} \quad j = 1, \ldots, n.$$

We set $x_j := u(w_j)$ and $z := \sum_{j=1}^n e_j \otimes x_j : \ell_2^n \to X$. Since the vectors w_j form an orthogonal basis, we have $\|Tz\| \leq \|Tu|_H\| \leq 1$. Using the rotation invariance of the ℓ-norm, we deduce from the definition of $C(T)$

$$\frac{n^{1/q}}{C(T)} \leq \left(\mathbb{E} \left\| \sum_{j=1}^n g_j x_j \right\|^2 \right)^{1/2} = \ell(z) \leq \ell(u) \leq \frac{m^{1/q}}{3e16^{1/q}} \leq \frac{n^{1/q}}{3eA}.$$

By assumption this implies $\mathrm{wc}_q(T) \leq 3eA \leq C(T)$. □

In the following $D \geq 12e$ will be a fixed constant, corresponding to the first occurrence of c_2 in condition (ii). Talagrand's proof [Ta1] of the implication "X $(q,1)$-summing $\implies$ X has cotype q" yields the following result.

Proposition 3.4 (Talagrand). *Let X be a $(q,1)$-summing Banach space and D a constant. Then there exist a constant $c = c(q,D) > 0$ such that for all vectors $x_1, \dots, x_n$ satisfying $\|x_i\| \geq D^{-1}$ and*

$$\sum_{i=1}^n |\langle x^*, x_i \rangle|^2 \leq \|x^*\|^2 \quad \textit{for all } x^* \in X^*$$

we have

$$\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\| \geq \frac{1}{c} n^{1/q}.$$

By replacing the assumption on X in this result by the weaker assumption that X is $(\ell_{q,\infty}, 1)$-summing, we will be able to complete the proof of the remaining implication (i′) $\implies$ (ii). In the following, we will assume that X is a Banach space which is $(\ell_{q,\infty}, 1)$-summing, and we denote by $H = \pi_{(q,\infty),1}(\mathrm{id}_X)$ the corresponding constant. Under these assumptions Talagrand's proof of [Ta2, Lemma 4.3] (used in the proof of Proposition 3.4) is still valid, and we obtain the following result.

Lemma 3.5 (Starting Lemma, [Ta1, Lemma 4.3]). *Let X satisfy the same conditions as above. Let $8 \leq s \leq n$ be such that*

$$\frac{s}{\sqrt{n}} \leq \frac{1}{16HD} s^{1/q}$$

and let J be a subset of $\{1, \dots, n\}$ with $\mathrm{card}(J) \geq n/2$. Then there exists a subset $I \subset J$ with $\mathrm{card}(I) = s$ and

$$\mathbb{E} \left\| \sum_{i \in I} \varepsilon_i x_i \right\| \geq \frac{s^{1/q}}{64HD}.$$

This lemma immediately settles the case $q = 2$; see the proof of the implication (i$'$) $\Longrightarrow$ (ii) below. However, for $q > 2$, we cannot apply Lemma 3.5 for values s proportional to n. Thus we continue to follow Talagrand's iteration procedure developed in the proof of Proposition 3.4. Indeed, we will construct many subsets $I \subset \{1, \dots, n\}$ for which the expectation of the norm of $\sum_{i\in I} g_i x_i$ is large, and we will block these together to obtain a lower bound for the norm of $\sum_{i=1}^n g_i x_i$. The same blocking procedure is carried out in [Ta2, Lemma 4.2].

As mentioned above, we will only assume that X is $(\ell_{q,\infty}, 1)$-summing. Note that the proof of [Ta2, Lemma 4.2] uses the triangle inequality for ℓ_q, which is not valid in $\ell_{q,\infty}$. However, the triangle inequality does hold *up to a constant* in $\ell_{q,\infty}$, and this will be enough for the argument to carry over to our setting. Specifically, there exists a constant S and a norm $|\!|\!| \,.\, |\!|\!|$ such that

$$\frac{1}{S} \|\sigma\|_{q,\infty} \le |\!|\!|\sigma|\!|\!| \le \|\sigma\|_{q,\infty}$$

holds for all finite sequences σ. In the following, we will reserve the letter S for this constant.

Lemma 3.6 (Blocking Lemma, [Ta1, Lemma 4.2]). *There exists a constant $K > 0$ with the following property. Let $I_1, \dots, I_k$ be disjoint subsets of $\{1, \dots, n\}$ having the same cardinality, and let $I = \bigcup_{j=1}^k I_j$. Let $\alpha > 0$ be such that for all $1 \le j \le k$*

$$\mathbb{E}\left\|\sum_{i\in I_j} g_i x_i\right\| \ge \alpha \quad \textit{and} \quad k^{1/2} \le \frac{k^{1/q}\alpha}{2SKH}.$$

Then we have

$$\mathbb{E}\left\|\sum_{i\in I} g_i x_i\right\| \ge \frac{\alpha k^{1/q}}{2SH}.$$

Proof of the implication (i$'$) $\Longrightarrow$ (ii)*:* Let $D \ge 12e$ be fixed. According to Proposition 3.3 there exist $n \in \mathbb{N}$ and vectors $x_1, \dots, x_n \in X$ satisfying

$$\sum_{i=1}^n |\langle x^*, x_i\rangle|^2 \le \|x^*\|^2 \quad \text{for all } x^* \in X^*$$

and

$$\|x_i\| \ge \frac{1}{D} \quad \text{for all } i = 1, \dots, n,$$

such that

$$\mathbb{E}\left\|\sum_{i=1}^n g_i x_i\right\| \le \frac{n^{1/q}}{\mathrm{wc}_q(\mathrm{id}_X)}. \tag{3.1}$$

If $q = 2$, we choose s such that

$$s \leq \frac{n}{64H^2D^2} \leq 2s$$

and apply the Starting Lemma to obtain a subset $I \subset \{1, \dots, n\}$ with $\operatorname{card}(I) = s$ satisfying

$$\mathbb{E}\left\|\sum_{i=1}^{n} g_i x_i\right\| \geq \sqrt{\frac{2}{\pi}}\,\mathbb{E}\left\|\sum_{i\in I} \varepsilon_i x_i\right\| \geq \sqrt{\frac{2}{\pi}}\frac{\sqrt{s}}{64HD} \geq \sqrt{\frac{2}{\pi}}\frac{1}{(64HD)^2}\sqrt{n}.$$

By (3.1) it follows that

$$\mathrm{wc}_2(\mathrm{id}_X) \leq 2^{13}D^2H^2,$$

which proves the assertion in this case.

We now assume that $q > 2$. We will formulate below three conditions and first show that the result holds if these conditions are all satisfied. To complete the proof, we then consider the case when one of these conditions does not hold.

We choose a natural number $r \in \mathbb{N}$ such that

$$r \geq 4q\left(\frac{1}{2} - \frac{1}{q}\right)^2.$$

Let q' be the conjugate index to q, defined by $(1/q) + (1/q') = 1$, and set

$$s := \left[\frac{n^{q'/2}}{(16DH)^{q'}}\right].$$

Our first condition is

$$8 \leq s \leq \frac{n}{2}. \tag{3.2}$$

Under this assumption, we let $p = [n/(2s)]$. By our choice of s, we can iteratively apply the Starting Lemma (Lemma 3.5) to obtain disjoint sets $I_1, \dots, I_p$ of cardinality s such that

$$\mathbb{E}\left\|\sum_{i\in I_j} g_i x_i\right\| \quad\geq\quad \sqrt{\frac{2}{\pi}}\,\mathbb{E}\left\|\sum_{i\in I_j} \varepsilon_i x_i\right\| \geq \frac{s^{1/q}}{100HD}$$

holds for all $j = 1, \dots, p$.

Now, fix $M \in \mathbb{N} \cup \{0\}$ with

$$2^{Mr} \leq p \leq 2^{Mr+r}$$

and set $k = 2^M$.

Lemma 3.7. *Suppose k satisfies*

$$k^{1/q} \geq 2SKH \tag{3.3}$$

and

$$k^{1/2-1/q} \le \frac{s^{1/q}}{200DKSH^2}. \tag{3.4}$$

Let T be a subset of $\{1,\dots,p\}$ with $\mathrm{card}(T) = k^l$, where $l \in \mathbb{N} \cup \{0\}$, such that $k^l \le p$. Then, setting $I_T = \bigcup_{t\in T} I_t$, we have

$$\mathbb{E}\left\| \sum_{i\in I_T} g_i x_i \right\| \ge \frac{s^{1/q}}{100DH} \left(\frac{k^{1/q}}{2SH} \right)^l .$$

Proof. In the case $l = 0$ the result follows from the choice of the sets $I_1,\dots,I_p$. Proceeding by induction, we assume that $l \ge 1$ and that the assertion is true for $l-1$. A set T of cardinality k^l can be split into k sets T_j with cardinality k^{l-1}. The induction hypothesis implies that for all $j = 1,\dots,k$

$$\mathbb{E}\left\| \sum_{i\in I_{T_j}} g_i x_i \right\| \ge \frac{s^{1/q}}{100DH} \left(\frac{k^{1/q}}{2SH} \right)^{l-1} =: \alpha,$$

say. Using (3.4) and (3.3) we have

$$\sqrt{k} \;\le\; \frac{s^{1/q}}{100DH} \cdot \frac{k^{1/q}}{2KSH} \le \frac{s^{1/q}}{100DH} \left(\frac{k^{1/q}}{2SH} \right)^{l-1} \frac{k^{1/q}}{2KSH} = \alpha \frac{k^{1/q}}{2KSH}.$$

Therefore, we obtain from Lemma 3.6

$$\mathbb{E}\left\| \sum_{i\in I_T} g_i x_i \right\| \ge \frac{s^{1/q}}{100DH} \left(\frac{k^{1/q}}{2SH} \right)^l ,$$

which is the asserted bound and completes the induction. □

Now assume that conditions (3.2)–(3.4) are satisfied. Applying Lemma 3.7 with $l = r$ and $I_T = \bigcup_{j\le k^r} I_j$, we obtain

$$\begin{aligned}
\mathbb{E}\left\| \sum_{i=1}^n g_i x_i \right\| &\ge \mathbb{E}\left\| \sum_{i\in I_T} g_i x_i \right\| \\
&\ge \frac{s^{1/q}}{100DH} \left(\frac{k^{1/q}}{2SH} \right)^r \\
&\ge \left(2^r 100 D S^r H^{r+1}\right)^{-1} (sk^r)^{1/q} \\
&\ge \left(2^{r(1+1/q)} 100 D S^r H^{r+1}\right)^{-1} (sp)^{1/q} \\
&\ge \left(4^{1/q} 2^{r(1+1/q)} 100 D S^r H^{r+1}\right)^{-1} n^{1/q}.
\end{aligned}$$

The term inside the parentheses yields the desired estimate

$$\mathrm{wc}_q(\mathrm{id}_X) \le c(q,r,D) H^{r+1}.$$

We now consider the situation when one of the conditions (3.2)–(3.4) does not hold. Suppose first that (3.2) does not hold. If $s > n/2$, we apply Lemma 3.5 with $s = n/2$ and obtain from (3.1) the inequality

$$\mathrm{wc}_q(\mathrm{id}_X) \le \sqrt{\frac{\pi}{2}}2^{1/q}64DH.$$

If $s \le 8$, we have $\sqrt{n} \le 9 \cdot 16HD$ and the estimate

$$\mathrm{wc}_q(\mathrm{id}_X) \le Dn^{1/q} \le D\sqrt{n} \le 144D^2H.$$

In either case the desired estimate holds, so we can assume that (3.2) holds.

Next, suppose that (3.3) does not hold, i.e., that $k^{1/q} < 2SKH$. Then

$$\left(\frac{n}{s}\right)^{1/q} \le 2p^{1/q} \le 2^{1+r/q}k^{r/q} \le 2^{1+r/q}(2SKH)^r.$$

By the choice of I_1 and (3.1) this implies

$$\begin{aligned} n^{1/q} \le \left(\frac{n}{s}\right)^{1/q} s^{1/q} \le \left(\frac{n}{s}\right)^{1/q} 100DH\,\mathbb{E}\left\|\sum_{i\in I_1} g_i x_i\right\| \\ \le c_2(q,r,D)H^{r+1}\,\mathbb{E}\left\|\sum_{i=1}^{n} g_i x_i\right\| \\ \le c_2(q,r,D)H^{r+1}\frac{n^{1/q}}{\mathrm{wc}_q(\mathrm{id}_X)}, \end{aligned}$$

and hence

$$\mathrm{wc}_q(\mathrm{id}_X) \le c_2(q,r,D)H^{r+1}.$$

Finally, we consider condition (3.4). By our choice of r and the identity

$$\frac{1}{q'} - \frac{1}{2} = \frac{1}{2} - \frac{1}{q},$$

we have

$$\begin{aligned} k^{\frac{1}{2}-\frac{1}{q}} &\le p^{\frac{1}{r}\left(\frac{1}{2}-\frac{1}{q}\right)} \le \left(\frac{n}{2s}\right)^{\frac{1}{r}\left(\frac{1}{2}-\frac{1}{q}\right)} \le \left(\frac{(16DH)^{q'}n}{n^{\frac{q'}{2}}}\right)^{\frac{1}{r}\left(\frac{1}{2}-\frac{1}{q}\right)} \\ &= (16DH)^{\frac{q'}{r}\left(\frac{1}{2}-\frac{1}{q}\right)} n^{q'\frac{1}{r}\left(\frac{1}{2}-\frac{1}{q}\right)^2} \le (16DH)^{\frac{q'}{r}\left(\frac{1}{2}-\frac{1}{q}\right)} n^{\frac{q'}{4q}}. \end{aligned}$$

On the other hand, we have

$$\frac{s^{\frac{1}{q}}}{200DKSH^2} \quad \ge \quad \frac{n^{\frac{q'}{2q}}}{400DKSH^2(16DH)^{\frac{q'}{q}}}.$$

Thus if

$$n^{\frac{q'}{4q}} \ge 400DKSH^2(16DH)^{\frac{q'}{q}}(16DH)^{\frac{q'}{r}\left(\frac{1}{2}-\frac{1}{q}\right)},$$

then (3.4) holds. Otherwise, we have

$$\begin{aligned}\mathrm{wc}_q(\mathrm{id}_X) &\leq Dn^{\frac{1}{q}} \leq D(400DKS)^4 H^8 (16DH)^{\frac{4}{q}} (16DH)^{\frac{4}{r}\left(\frac{1}{2}-\frac{1}{q}\right)} \\ &\leq c_3(q,r,D)H^{12}.\end{aligned}$$

If $q \leq 14/5$, this estimate is far from being optimal. In this range, we choose $s = n^{1/2}$, $r = 2$, and $k = \sqrt{p} \sim n^{1/4}$ to get

$$k^{\frac{1}{2}-\frac{1}{q}} \sim n^{\frac{1}{4}\left(\frac{1}{2}-\frac{1}{q}\right)}.$$

Since $s^{1/q} \sim n^{1/(2q)}$, we see that (3.4) holds provided that

$$CH^2 \leq n^{\frac{3}{4}\frac{1}{q}-\frac{1}{8}} = n^{\frac{24-4q}{32q}}.$$

If the latter inequality does not hold, then we deduce from our assumption $q \leq 14/5$

$$\mathrm{wc}_q(\mathrm{id}_X) \leq Dn^{\frac{1}{q}} \leq D(CH^2)^{\frac{32}{24-4q}} \leq c(q,D)H^5.$$

The condition $k^{1/q} \geq CH$ yields $\mathrm{wc}_q(\mathrm{id}_X) \leq cH^4$, and the condition $n^{3/4} = s \leq n^{q'/2}(16DH)^{-q'}$ yields $\mathrm{wc}_q(\mathrm{id}_X) \leq c(q,D)H$. Thus, for $2 \leq q \leq 14/5$ we have

$$\mathrm{wc}_q(\mathrm{id}_X) \leq c(q,D)H^5.$$

This completes the proof of Theorem 2. $\square$

It is very likely that for q close to 2, we have $\mathrm{wc}_q(\mathrm{id}_X) \leq c(q,D)H^3$.

Remark 3.8. The same proof works in the following more general setting. We consider a sequence $(g(n))_{n\in\mathbb{N}}$ and assume that one of the following conditions holds:

(I) There exist numbers β and γ satisfying $0 < \gamma \leq \beta < 1$ and constants c_γ and C_β such that for $1 \leq k \leq n$,

$$c_\gamma \left(\frac{n}{k}\right)^\gamma \leq \frac{g(n)}{g(k)} \leq C_\beta \left(\frac{n}{k}\right)^\beta.$$

(II) There exists q with $2 \leq q < \infty$, a natural number $r > (q/2-1)$, and a constant M_r such that
 (a) $\lim g(n)n^{1/q} = \infty$,
 (b) $g(k^{2r}) \leq M_r g(k^r) g(k)^r$ for all $k \in \mathbb{N}$.

Using condition (I), we deduce that the space $\ell_{g,\infty}$ has an equivalent norm, with equivalence constant S, say. We assume that a Banach space satisfies the summing condition

$$\|(\|x_n\|_X)_n\|_{g,\infty} \leq H \sup_{x^*\in B_{X^*}} \sum_k |\langle x^*, x_k\rangle| \,. \tag{3.5}$$

(In the terminology introduced in the next section, this is equivalent to $Y_S \subset l_{g,\infty}$.) Proceeding as in the proof of Proposition 3.3, we obtain vectors $x_1, \dots, x_n \in X$ satisfying $D^{-1} \le \|x_i\| \le 1$ and

$$\sum_{i=1}^{n} |\langle x^*, x_i\rangle|^2 \le \|x^*\|^2$$

for all $x^* \in X^*$, and such that

$$\mathbb{E}\left\|\sum_{i=1}^{n} g_i x_i\right\| \le \frac{c_g(n)}{\mathrm{wc}_g(\mathrm{id}_X)}.$$

Here $\mathrm{wc}_g(\mathrm{id}_X)$ is the smallest constant c such that

$$\sup_k g(k) a_k(u) \le c\mathbb{E}\left\|\sum_{i=1}^{n} g_i u(e_i)\right\|,$$

and $c_g(n)$ is a constant. Furthermore, let us assume that for some $M \in \mathbb{N}$ we have

$$n = 2^{2Mr}, \quad s = 2^{Mr}, \quad k = 2^M.$$

If $8 \le s \le n$ and

$$16DH \le g(s), \quad 2KSH \le g(k), \quad k^{1/2} \le \frac{g(s)}{100DH}\frac{g(k)}{2KSH}, \tag{3.6}$$

then applying the iteration procedure of Lemma 3.7 we see that under condition (b) with $s = k^r$ we have

$$\mathbb{E}\left\|\sum_{i=1}^{n} g_i x_i\right\| \ge \frac{g(s)}{100DH}\left(\frac{g(k)}{2KSH}\right)^r \ge \frac{g(n)}{c(g)M_r H^{r+1}}.$$

Also, setting $B = 200DSH^2$, we deduce from (a) that for k, s large enough, with r defined as above,

$$g(s)g(k) \ge Bn^{\frac{1}{2q}} k^{\frac{1}{q}} \ge Bn^{\frac{1}{2r}\left(\frac{1}{2}-\frac{1}{q}\right)} n^{\frac{1}{(2rq)}} \ge Bn^{\frac{1}{4r}} = Bk^{\frac{1}{2}}.$$

Hence (3.6) holds for k and s large enough. If k or s is small, we argue as in the proof of Theorem 2 to obtain (relatively poor) estimates for the weak cotype g constant. Thus, under the conditions (I) and (II), *a Banach space is* $(\ell_{g,\infty}, 1)$*-summing (i.e., satisfies* (4)*) if and only if it has weak (Gaussian) cotype* $\ell_{g,\infty}$.

If condition (b) is not satisfied, we can still obtain some information by considering the function

$$\tilde{g}(n) := \sup\left\{ g(k^r)g(k)^r \;\middle|\; k^{2r} \le n \right\}.$$

If X satisfies the summing condition (4) with respect to g, we then obtain

$$\sup_k \tilde{g}(k) a_k(u) \leq c(H)\, \mathbb{E} \left\| \sum_{i=1}^n g_i u(e_i) \right\| .$$

An interesting special case is the function $g(n) = n^{1/q}/(1+\ln n)$. If $r > (q/2) - 1$ and X satisfies (4) with respect to g, we obtain

$$\sup_k \frac{k^{1/q}}{(1+\ln n)^{r+1}} a_k(u) \leq c(H)\, \mathbb{E} \left\| \sum_{i=1}^n g_i u(e_i) \right\| .$$

4. Optimal summing and cotype spaces

In this section we will define sequence spaces which are associated with the cotype and summing properties of a Banach space X. In this setting it is more convenient to study the Rademacher cotype. We recall that a maximal symmetric sequence space is a sequence space Y with the following properties:

(i) $\|\tau\|_\infty \leq \|\tau\|_Y \leq \|\tau\|_1$ for all sequences with finite support.
(ii) $\|\tau^*\| = \|\tau\|$, where τ^* denotes the non-increasing rearrangement of $|\tau|$.
(iii) $\|\tau\| = \sup_n \|P_n(\tau)\|$, where P_n denotes the projection onto the first n coordinates.

An operator $T \in \mathcal{L}(X,Y)$ is said to be $(Y,1)$-summing, respectively of Rademacher cotype Y, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n$ we have

$$\left\| \sum_{k=1}^n \|Tx_k\| \, e_k \right\|_Y \leq c \sup_{x^* \in B_{X^*}} \sum_{k=1}^n |\langle x^*, x_k \rangle| ,$$

respectively

$$\left\| \sum_{k=1}^n \|Tx_k\| \, e_k \right\|_Y \leq c\mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\| .$$

The corresponding norms are denoted by $\pi_{Y,1}(T)$ and $c_Y(T)$, respectively, and defined as the infimum of all constants c satisfying the respective inequalities. The properties of such spaces will be our main tool in the proof of the iterated log estimate of Theorem 2.

Given $\tau = (\tau_k)_k$, we define

$$|||\tau|||_S := \inf \left\{ \sup_{|\alpha_k| \leq 1} \left\| \sum_{k=1}^n \tau_k \alpha_k x_k \right\| \;\middle|\; (x_k)_{k=1}^n \subset X, \|x_k\| = 1 \right\}$$

and

$$|||\tau|||_C := \inf \left\{ \mathbb{E} \left\| \sum_{k=1}^n \varepsilon_k \tau_k x_k \right\| \;\middle|\; (x_k)_{k=1}^n \subset X, \|x_k\| = 1 \right\} .$$

Clearly, the expressions $\|\|\tau\|\|_S$ and $\|\|\tau\|\|_C$ are homogeneous and invariant under permutations and change of signs. In order to obtain a norm satisfying the triangle inequality, we define for $T \in \{C, S\}$

$$\|\tau\|_T^0 := \inf \left\{ \sum_1^m \|\|\tau^j\|\|_T \;\middle|\; m \in \mathbb{N}, \tau^j \text{ has finite support, } |\tau| \le \sum_{j=1}^m |\tau^j| \right\}$$

and set

$$\|\tau\|_T := \sup_n \|P_n(\tau)\|_T^0 .$$

The *normed* spaces $Y_S = Y_S(X)$ and $Y_C = Y_C(X)$ defined by this norm will be called *optimal summing space* and *optimal cotype space*, respectively. We summarize the properties of these spaces in the following lemma.

LEMMA 4.1. *Let X be a Banach space, let Y_S and Y_C denote the associated optimal summing and optimal cotype spaces, respectively, and let Z be a maximal sequence space. Then we have:*

(1) *The identity map of X is $(Y_S, 1)$-summing and of Rademacher cotype Y_C with constant* 1.
(2) *The identity map of X is $(Z, 1)$-summing (resp. of Rademacher cotype Z) if and only if*

$$Y_S \subset Z \quad (\textit{resp. } Y_C \subset Z).$$

The norm of the inclusion is $\pi_{(Z,1)}(\mathrm{id}_X)$ (resp. $C_Z(\mathrm{id}_X)$).
(3) *For $Y \in \{Y_S, Y_C\}$ and each finitely supported sequence $(\tau^k)_{k=1}^n$ we have*

$$\left\| \sum_{k=1}^n \|\tau^k\|_Y \, e_k \right\|_Y \le \left\| \sum_{k=1}^n |\tau^k| \right\|_Y .$$

Proof. (1) and (2) are obvious. In (3) we will only consider the cotype case. Let (τ^k) be a finite sequence of sequences with finite support. We denote their 'sum' by $\tau := \sum_k |\tau^k|$. Given $\delta > 0$ we can find a finite sequence $(x_i)_i \subset X$ with $\|x_i\| = 1$ such that

$$\mathbb{E} \left\| \sum_i \varepsilon_i \tau_i x_i \right\| \le (1+\delta) \|\|\tau\|\|_C .$$

For any sequence of signs ρ_k, we can find a sequence $(\gamma_i)_i$, $\gamma_i \in [-1, 1]$, such that

$$\sum_k \rho_k |\tau^k| = \gamma\tau.$$

By the sign invariance of the Bernouilli variables (ε_i) and the fact that extreme points in the unit ball of $\ell_\infty^n(\mathbb{R})$ over $\mathbb{R}$ are sequences of signs (see, e.g., [Pi1]), we have

$$\begin{aligned}\mathbb{E}_\varepsilon \left\| \sum_i \varepsilon_i \left(\sum_k \rho_k \left|\tau_i^k\right| \right) x_i \right\|_X &= \mathbb{E}_\varepsilon \left\| \sum_i \varepsilon_i \gamma_i \tau_i x_i \right\|_X \\ &\le \mathbb{E}_\varepsilon \left\| \sum_i \varepsilon_i \tau_i x_i \right\|_X \le (1+\delta) \|\!|\tau|\!\|_C .\end{aligned}$$

Taking expectations, we deduce from (1) and the triangle inequality in Y

$$\begin{aligned}(1+\delta)\|\!|\tau|\!\|_C &\ge \mathbb{E}_\varepsilon \mathbb{E}_\rho \left\| \sum_i \varepsilon_i \left(\sum_k \rho_k \left|\tau_i^k\right| \right) x_i \right\|_X \\ &\ge \mathbb{E}_\varepsilon \left\| \sum_k \left\| \sum_i \varepsilon_i \left|\tau_i^k\right| x_i \right\|_X e_k \right\|_{Y_C} \\ &\ge \left\| \sum_k \left(\mathbb{E}_\varepsilon \left\| \sum_i \varepsilon_i \left|\tau_i^k\right| x_i \right\|_X \right) e_k \right\|_{Y_C} \\ &\ge \left\| \sum_k \|\!|\tau^k|\!\|_C e_k \right\|_{Y_C} .\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\left\| \sum_k \|\!|\tau^k|\!\|_C e_k \right\|_{Y_C} \le \Big\|\!\Big| \sum_k \left|\tau^k\right| \Big|\!\Big\|_C . \tag{4.1}$$

Now let $\tau \le \sum_{j \le m} \left|\sigma^j\right|$. We define $\beta^k := \left|\tau^k\right| / \tau$ by pointwise multiplication, with the convention $0/0 = 0$. The sequences $\sigma^{kj} := \beta^k \sigma^j$ clearly satisfies

$$\sum_k \left|\sigma^{kj}\right| \le \left|\sigma^j\right| \quad \text{and} \quad \left|\tau^k\right| \le \sum_j \left|\sigma^{kj}\right| .$$

Applying (4.1) to each sequence $(\left|\sigma^{kj}\right|)_k$ $(1 \le j \le m)$, we deduce

$$\begin{aligned}\left\| \sum_k \left\|\tau^k\right\|_Y e_k \right\|_{Y_C} &\le \left\| \sum_k \left(\sum_j \|\!|\sigma^{kj}|\!\|_C \right) e_k \right\|_{Y_C} \\ &\le \sum_j \left\| \sum_k \|\!|\sigma^{kj}|\!\|_C e_k \right\|_{Y_C} \\ &\le \sum_j \Big\|\!\Big| \sum_k \left|\sigma^{kj}\right| \Big|\!\Big\|_C \le \sum_j \|\!|\sigma^j|\!\|_C .\end{aligned}$$

Taking the infimum over all $\tau \le \sum_j \sigma^j$ yields the assertion. ☐

We now relate this notion of Rademacher cotype $\ell_{q,\infty}$ to the notion of weak cotype q (see also [Ma]).

LEMMA 4.2. *Let $q > 2$. A Banach space X is of weak (Gaussian) cotype q if and only if $Y_C \subset \ell_{q,\infty}$.*

Proof. Both conditions imply finite cotype, and therefore the Gaussian and Rademacher means are comparable (see, for example, [Ps1]), i.e., we have

$$\sqrt{\frac{2}{\pi}}\,\mathbb{E}\left\|\sum_i \varepsilon_i x_i\right\| \le \mathbb{E}\left\|\sum_i g_i x_i\right\| \le c(X)\,\mathbb{E}\left\|\sum_i \varepsilon_i x_i\right\|.$$

If $Y_C \subset \ell_{q,\infty}$ with norm of the inclusion bounded by c, we obtain for any vectors x_i with $\|x_i\| \ge D^{-1}$

$$\begin{aligned}\frac{n^{1/q}}{D} &\le \|(\|x_i\|)_i\|_{q,\infty} \le c\,\|(\|x_i\|)_i\|_{Y_C}\\ &\le c\mathbb{E}\left\|\sum_i \varepsilon_i x_i\right\| \le c\sqrt{\frac{\pi}{2}}\,\mathbb{E}\left\|\sum_i g_i x_i\right\|.\end{aligned}$$

Proposition 3.3 now implies the bound $\mathrm{wc}_q(X) \le \sqrt{\pi/2}Dc$.

Conversely, assume that X is of weak cotype q. Let $x_1, \dots, x_n$ be vectors in X with $\|x_i\| \ge 1$. Using the estimate of the 2-summing norm by the $\ell_{2,1}$-norm of its approximation numbers (see [Pi2]), we obtain for the operator $u : \ell_2^n \to X$ defined by $u(e_i) := x_i$,

$$\begin{aligned}n^{1/q} &\le \left(\sum_{i=1}^n \|x_i\|^2\right)^{1/2} \le \pi_2(u) \le \sum_{j=1}^n \frac{a_j(u)}{\sqrt{j}}\\ &\le \sum_{j=1}^n j^{-1/q-1/2} \sup_k k^{1/q} a_k(u)\\ &\le \left(\frac{1}{2} - \frac{1}{q}\right)^{-1} n^{1/2-1/q}\,\mathrm{wc}_q(\mathrm{id}_X)\ell(u)\\ &\le \left(\frac{1}{2} - \frac{1}{q}\right)^{-1} n^{1/2-1/q}\,\mathrm{wc}_q(\mathrm{id}_X)C(X)\,\mathbb{E}\left\|\sum_i \varepsilon_i x_i\right\|.\end{aligned}$$

Using the same argument as in the proof of the equivalence of (i) and (i′) and the fact that $\ell_{q,\infty}$ admits an equivalent norm, we obtain the assertion. □

The proof of the iterated log estimate is based on the analysis of self-concave spaces, i.e., spaces which satisfy condition (3) in Lemma 4.1. For a maximal symmetric sequence space Y we define

$$f_Y(n) := \left\|\sum_{k=1}^n e_k\right\|_Y$$

and

$$q_Y := \inf \left\{ 0 < q < \infty \,\middle|\, \text{there exists } C \text{ with } n^{1/q} \le C f_Y(n) \right\}.$$

Obviously, for all $q > q_Y$ we have

$$Y \subset \ell_{f_Y,\infty} \subset \ell_q.$$

For self-concave spaces, the following alternative holds.

Proposition 4.3. *Let Y be a maximal symmetric sequence space which satisfies*

$$\left\| \sum_{k=1}^n \left\| \tau^k \right\|_Y e_k \right\|_Y \le \left\| \sum_{k=1}^n \left| \tau^k \right| \right\|_Y .$$

Let $1 \le p < \infty$. Then either $\ell_p \subset Y$ with inclusion norm 1, or there exists $q < p$ such that $Y \subset \ell_q \subsetneqq \ell_p$. In particular,

$$\ell_{q_Y} \subset Y.$$

Proof. Let τ be a sequence of finite support, $\tau_k = 0$ for $k \ge n$, say. For $i = 1, \dots, n$ we set

$$\sigma_i := \sum_{j=1}^n \tau_j e_{(i-1)n+j},$$

and define the product

$$\tau \otimes \tau := \sum_{i=1}^n \tau_i \sigma_i.$$

Clearly, we have $\|\tau \otimes \tau\|_p = \|\tau\|_p^2$. Our assumption on Y implies

$$\|\tau\|_Y^2 = \left\| \sum_{i=1}^n \tau_i \left\| \sigma_i \right\|_Y e_i \right\|_Y \le \|\tau \otimes \tau\|_Y . \tag{4.2}$$

In particular, f_Y is submultiplicative, i.e., we have $f_Y(n) f_Y(k) \le f_Y(nk)$.

We now consider the following two cases:

(1) There exists an $n_0 \in \mathbb{N}$ such that $f_Y(n_0) > n_0^{1/p}$.
(2) For all $n \in \mathbb{N}$ we have $f_Y(n) \le n^{1/p}$.

In the first case, we choose $q < p$ such that $f_Y(n_0) = n_0^{1/q}$. For $n \in \mathbb{N}$, let $m \in \mathbb{N}$ with $n_0^{m-1} \le n \le n_0^m$. From the submultiplicativity and the triangle inequality we deduce

$$n^{1/q} \le n_0^{m/q} = f_Y(n_0)^m \le f_Y(n_0^m) \le n_0 f_Y(n).$$

This means that $Y \subset \ell_{q,\infty}$. Hence for all r with $q < r < p$ we have the inclusion $Y \subset \ell_r \subsetneqq \ell_p$.

Now, assume that (2) holds. We will first show that $\ell_{p,1} \subset Y$. Indeed, let τ be a non-increasing positive sequence with finite support. Then we have

$$\|\tau\|_Y \leq \sum_{k=0}^{\infty} \left\| \sum_{j=2^k}^{2^{k+1}} \tau_j e_j \right\|_Y \leq \sum_{k=0}^{\infty} \tau_{2^k} f_Y(2^k)$$
$$\leq \sum_{k=0}^{\infty} \tau_{2^k} (2^k)^{1/p} \leq 5 \|\tau\|_{p,1}.$$

Setting $C_n := \|P_n : \ell_p \to Y\|$, we thus have $C_n \leq 5(1 + \ln n)$. We now use a tensor trick to complete the argument. First, we show $C_n^2 \leq C_{n^2}$. Indeed, let τ be a sequence with support contained in $\{1, \dots n\}$. From (4.2) we deduce

$$\|\tau\|_Y^2 \leq \|\tau \otimes \tau\|_Y \leq C_{n^2} \|\tau \otimes \tau\|_p \leq C_{n^2} \|\tau\|_p^2.$$

Hence, we get

$$C_n \leq \inf_k \left(C_{n^{2^k}}\right)^{1/2^k} \leq \inf_k \left(5(1 + 2^k \ln n)\right)^{1/2^k} = 1.$$

We observe that for $q < q_Y$ the inclusion $Y \subset \ell_q \subset \ell_{q,\infty}$ is impossible. Hence $\ell_{q_Y} \subset Y$ holds. $\square$

As an application, we investigate cotype properties with respect to the Lorentz space $\ell_{q,w}$.

Proposition 4.4. *Let $2 \leq q < \infty$, $1 \leq w \neq q \leq \infty$. A Banach space X is of cotype $\ell_{q,w}$ if and only if*

$$X \text{ is of cotype } \quad \begin{cases} p \text{ for some } p < q & \text{if} \quad w < q, \\ \ell_{q,\infty} & \text{if} \quad w > q. \end{cases}$$

If X is of cotype $\ell_{q,\infty}$, then there exists a constant C such that

$$c_q(\mathrm{id}_E) \leq \sqrt{\pi} C^{k+1} (\max\{1, \log_2\})^{(k)} ((1 + \log_2 n)^{1/q})$$

holds for all $k \in \mathbb{N}$ and all n-dimensional subspaces $E \subset X$. In particular,

$$c_q(\mathrm{id}_E) \leq \sqrt{\pi} 2 C^{1+k_n},$$

where k_n is the smallest integer k with $n \leq \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{k \text{ times}}$.

Proof. If $w < q$ and X is of cotype $\ell_{q,w}$, we have $Y_C \subset \ell_{q,w}$ by Proposition 4.1, but certainly not $\ell_q \subset Y_C$. By Proposition 4.3 there must exist $p < q$ such that $Y_C \subset \ell_p$. Since X is of cotype Y_C, it is also of cotype ℓ_p. Now, let us assume that X is of cotype $\ell_{q,\infty}$ with constant C, say. In particular, we then have $n^{1/q} \leq C f_{Y_C}(n)$. Let $q \leq w < \infty$ and let τ be a positive non-increasing sequence of finite support. For $k \in \mathbb{N}$, we define the disjoint elements $x_k := \tau_{2^k} \sum_{j=2^{k-1}+1}^{2^k} e_j$. Then

$$\begin{aligned} \|\tau\|_{q,w} &= \left(\sum_n (\tau_n n^{1/q})^w \frac{1}{n} \right)^{1/w} \le \left(\sum_{k=0}^{\infty} (\tau_{2^k} 2^{k/q})^w \right)^{1/w} \\ &\le \|\tau\|_\infty + 2^{1/q} C \left\| \sum_{k \in \mathbb{N}} \|x_k\|_{Y_C} e_k \right\|_w . \end{aligned}$$

If $w > q$, then $\ell_{q,\infty} \subset \ell_w$ with inclusion norm c_{qw}. Using the inequality $\sum_k x_k \le \tau$ and condition (3) in Lemma 4.1, we deduce

$$\begin{aligned} \|\tau\|_{q,w} &\le 2^{1/q} C \left(\|\tau\|_\infty + c_{qw} \left\| \sum_k \|x_k\|_{Y_C} e_k \right\|_{q,\infty} \right) \\ &\le 2^{1/q} C \left(\|\tau\|_{Y_C} + c_{qw} C \left\| \sum_k \|x_k\|_{Y_C} e_k \right\|_{Y_C} \right) \\ &\le 2^{1+1/q} c_{qw} C^2 \|\tau\|_{Y_C} . \end{aligned}$$

Hence $Y_C \subset \ell_{q,w}$ and X is of cotype $\ell_{q,w}$. In the case $q = w$, we iterate this procedure. Setting $\alpha_n := \left\| P_n : Y_C \to \ell_q^n \right\|$, with the convention $\alpha_0 = 1$, we will show

$$\alpha_n \le 2^{1+1/q} C \alpha_{[\log n]}.$$

Indeed, if the support of the given sequence τ is contained in $\{1, \ldots, n\}$, we have $x_k = 0$ whenever $2^k > n$. Therefore, we obtain from Lemma 4.1

$$\begin{aligned} \|\tau\|_q &\le \left(\|\tau\|_\infty + 2^{1/q} C \left\| \sum_{k=1}^{[\log_2 n]} \|x_k\|_{Y_C} e_k \right\|_q \right) \\ &\le \left(\|\tau\|_\infty + 2^{1/q} C \alpha_{[\log_2 n]} \left\| \sum_{k=1}^{[\log_2 n]} \|x_k\|_{Y_C} e_k \right\|_{Y_C} \right) \\ &\le (1 + 2^{1/q} C \alpha_{[\log_2 n]}) \|\tau\|_{Y_C} . \end{aligned}$$

Using the trivial estimate $\left\| \mathrm{id} : \ell_{q,\infty}^m \to \ell_q^m \right\| \le (1 + \log_2 m)^{1/q}$, we deduce by induction

$$Rc_q^n(\mathrm{id}_X) \le (2^{1+1/q} C)^{k+1} (\max\{1, \log_2\})^{(k)} ((1 + \log_2 n)^{1/q}).$$

Note that the Gaussian cotype constant of an n dimensional space E can be well estimated by the Gaussian cotype constant with n-vectors; see [ToJ] and [DJ2]. Since the Rademacher average can be estimated by the Gaussian average, we obtain

$$\begin{aligned} c_q(\mathrm{id}_E) &\le \sqrt{2} c_q^n(\mathrm{id}_E) \le \sqrt{\pi} R c_q^n(\mathrm{id}_X) \\ &\le \sqrt{\pi}(2^{1+1/q}C)^{k+1}(\max\{1,\log_2\})^{(k)}((1+\log_2 n)^{1/q}). \qquad \square \end{aligned}$$

Final Remark.

(1) The same argument can also be applied in the space Y_S, provided we have $Y_S \subset \ell_{q,\infty}$. This is of interest in the case $q = 2$. Hence in a weak cotype 2 space we have $Y_S \subset \ell_{2,\infty}$ and therefore

$$\pi_{21}^n(\mathrm{id}_X) \le C^{k+1}(\max\{1,\log_2\}^{(k)}((1+\log_2 n)^{1/2}).$$

It is an open problem whether such an estimate holds with the cotype 2 constant.

(2) For a Banach lattice of finite cotype let $Y_S \subset Z$ and let Z be p-convex for some $p > 2$. Then we can apply the generalized Maurey theorem to deduce $Y_C \subset Z_S$. This might be of particular interest in order to study the cotype properties of Orlicz spaces associated to the function

$$M(t) = \left(\frac{t}{1+|\ln t|}\right)^{1/q}.$$

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