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FUNDAMENTAL PROPERTIES OF SYMMETRIC SQUARE L-FUNCTIONS. I

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Dedicated with deep regards to Professor K. Ramachandra on his seventieth birthday

Abstract. We improve the existing upper bound for the mean-square of the absolute value of the Rankin-Selberg zeta-function (attached to a holomorphic cusp form) defined for the full modular group in the critical strip.

1. Introduction

A remarkable result of Selberg (see [57]) says that a positive proportion of zeros of the Riemann zeta-function are on the critical line. Similar results were obtained by Hafner for L-functions attached to cusp forms which are Hecke eigenforms (see [12]). Another important problem is studying the growth of the L-functions under consideration. In this connection, in a celebrated paper [28], Iwaniec and Sarnak proved growth estimates for eigenfunctions of certain arithmetic surfaces which break the bound that can be obtained by convexity arguments. This raises the question of proving non-trivial (in the sense of breaking the usual convexity bounds) growth estimates for general L-functions. Of course, this is closely related to the Lindelöf hypothesis. We should also point out here the important work by Iwaniec, and by Duke, Friedlander and Iwaniec (see [27], [9], and [10]), who show how one can break the convexity bounds in different aspects, namely Q and r, for certain automorphic L-functions. For an excellent exposition of these results and for further comments we refer to [56]. The problem of studying the difference between consecutive zeros on the critical line was considered by various authors; see, for example, [1], [5], [30], [31], and [54]. In [60], Shimura proved that the completed symmetric square L-functions can be continued analytically to entire functions on the whole complex plane by establishing a functional equation.

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In recent times, there has been much interest in establishing the analytic continuation and a functional equation for various symmetric power L-functions (see [59]).

We always write $s = \sigma + it, z = x + iy$. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a holomorphic cusp form of even integral weight k defined over the full modular group $SL(2,\mathbb{Z})$. We assume that a_n are eigenvalues of all Hecke operators and $a_1 = 1$. Let α_p and β_p be the complex numbers defined by the equation

$$1 - a_p p^{-s} + p^{k-1-2s} = \left(1 - \alpha_p p^{-s}\right)\left(1 - \beta_p p^{-s}\right). \tag{1.1}$$

The Hecke L-function attached to f is defined as

$$L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}. \tag{1.2}$$

It is absolutely convergent in a certain half-plane and is continuable analytically to an entire function on the whole plane. For an arbitrary primitive Dirichlet character ψ, the symmetric square L-function attached to f is defined as

$$\begin{aligned} D(s) &:= D(s,f,\psi) \\ &:= \prod_p \left(\left(1 - \psi(p)\alpha_p^2 p^{-s}\right)\left(1 - \psi(p)\beta_p^2 p^{-s}\right)\left(1 - \psi(p)p^{k-1-s}\right)\right)^{-1} \\ &:= \sum_{n=1}^{\infty} a_{n^2} n^{-s}. \end{aligned} \tag{1.3}$$

(Here a_{n^2} is just a notation and does not mean the n^2-th Fourier coefficient of f.) Following the notation in [60], throughout this paper we assume that χ is a Dirichlet character modulo M and the trivial character when $M = 1$, and that ψ is an arbitrary primitive Dirichlet character with conductor r, and the trivial character when $r = 1$.

Now, $D(s)$ converges absolutely in $\Re s > k$. The critical strip for $D(s)$ is $k - 1 \leq \sigma \leq k$, and the critical line is $\sigma = k - 1/2$. We also note that from Deligne's work (see [7] and [8]) it follows that (see the "Note added in proof" at the end of the paper)

$$|a_{n^2}| \leq (d(n))^2 n^{k-1}. \tag{1.4}$$

The following are some fundamental questions about these L-functions:

1. Is $D(k + it)$ different from zero for all $t \in \mathbb{R}$?
2. If the answer to Question 1 is yes, can we establish a reasonable zero-free region for $D(s)$?
3. Is it possible to establish mean-value theorems on certain lines?
4. If the answers to Questions 1 and 2 are yes, can we prove certain "density theorems" for the zeros of $D(s)$?

The answers to Questions 1 and 2 above are known (see Lemma 3.2 in [32] and, for example, Exercise 3.2.11 of [49]). In fact, using the Fundamental Identity (Lemma 3.1) of this paper combined with Lemma 3.2 of [32], it is not difficult to establish a reasonable zero-free region. After Shimura's work (see [60]), the answers to the above fundamental questions form the basis for any further progress. It should be mentioned here that mean values of derivatives of modular L-series had been studied earlier by Ram Murty and Kumar Murty in [50]. In this paper, we concentrate only on Question 3 above. Results related to Question 4 above will form part II of this paper, which will appear elsewhere.

The properties of Rankin-Selberg zeta-functions have been studied extensively by many authors; see, for example, [24], [25] and [34]. After normalizing the coefficients, the Rankin-Selberg zeta-function is defined by

$$Z(s) = \zeta(2s) \sum_{n=1}^{\infty} a_n^2 n^{1-k-s} = \sum_{n=1}^{\infty} c'_n n^{-s} \quad \text{(say)}. \tag{1.5}$$

This Dirichlet series is absolutely convergent in the half plane $\sigma > 1$ and can be continued as a meromorphic function to the whole complex plane with a simple pole at $s = 1$. It satisfies a nice functional equation (see [24]). For example, in [24], Ivic studied mean-value theorems for $Z(s)$ for a certain range of σ; from his work it follows that

$$\int_0^T \left| Z\left(\frac{1}{2} + it\right)\right|^2 dt \ll T^{2+\epsilon} \tag{1.6}$$

for every $\epsilon > 0$. In [34], Matsumoto proved the following result (see Theorem 2 of [34]).

Theorem A.

(i) *For* $1/2 \le \sigma \le 3/4$ *we have*

$$\int_0^T |Z(\sigma + it)|^2 dt \ll T^{4-4\sigma} (\log T)^{1+\epsilon} \tag{1.7}$$

for any $\epsilon > 0$.

(ii) *For* $3/4 < \sigma \le 1$ *we have*

$$\int_1^T |Z(\sigma + it)|^2 dt = T \sum_{n=1}^{\infty} {c'_n}^2 n^{-2\sigma} + O(T^{\theta(\sigma)+\epsilon}), \tag{1.8}$$

where

$$\theta(\sigma) = \begin{cases} \dfrac{5}{2} - 2\sigma & \text{if } \dfrac{3}{4} \le \sigma \le \dfrac{12+\sqrt{19}}{20}, \\ \dfrac{60(1-\sigma)}{29-20\sigma} & \text{if } \dfrac{12+\sqrt{19}}{20} \le \sigma \le 1. \end{cases}$$

In particular, Theorem A gives

$$\int_0^T \left|Z\left(\frac{1}{2}+it\right)\right|^2 dt \ll T^2(\log T)^{1+\epsilon}, \tag{1.9}$$

which is a slight improvement of (1.6).

The aim of this paper is twofold. Firstly, we study the analytic properties of the symmetric square L-functions. Secondly, exploring these ideas, we obtain a nice improvement of Theorem A. For example, from Theorem 4.2 below it is immediate that (unconditionally)

$$\int_0^T \left|Z\left(\frac{1}{2}+it\right)\right|^2 dt \ll T^{11/6+\epsilon} \tag{1.10}$$

for every $\epsilon > 0$. We also show (see Theorem 4.3) that, under the Lindelöf hypothesis for the Riemann zeta-function,

$$\int_0^T \left|Z\left(1/2+it\right)\right|^2 dt \ll T^{3/2+\epsilon} \tag{1.11}$$

for every $\epsilon > 0$.

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2. Notation and preliminaries

The letters C and A (with or without suffixes) denote effective positive constants unless otherwise specified. The constants need not be the same at every occurrence. Throughout the paper we assume $T \geq T_0$, where T_0 is a large positive constant. We write $f(x) \ll g(x)$, or $f(x) = O(g(x))$, to mean that $|f(x)| < C_1 g(x)$. All implied constants are effective.

We set $s = \sigma + it$ and $w = u + iv$. In any fixed strip $a \leq \sigma \leq b$ we have, as $t \to \infty$,

$$\Gamma(\sigma+it) = t^{\sigma+it-1/2}e^{-\pi/2-it+(i\pi/2)(\sigma-1/2)}\sqrt{2\pi}\left(1+O\left(\frac{1}{t}\right)\right). \tag{2.1}$$

Let

$$R(s) = \pi^{-3s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-k+2}{2}\right)D(s). \tag{2.2}$$

Then $D(s)$ satisfies the functional equation (see [60])

$$R(s) = R(2k-1-s). \tag{2.3}$$

Also we note that if

$$R_1(s) = \pi^{-(s-k+1)/2}\Gamma\left(\frac{s-k+1}{2}\right)\zeta(s-k+1), \tag{2.4}$$

then $\zeta(s-k+1)$ satisfies the functional equation

$$R_1(s) = R_1(2k-1-s). \tag{2.5}$$

Therefore if $D_1(s) = \zeta(s-k+1)D(s)$, then $D_1(s)$ satisfies the functional equation

$$R(s)R_1(s) = R(2k-1-s)R_1(2k-1-s), \tag{2.6}$$

and we see that $R(s)R_1(s)$ extends $D_1(s)$ to an analytic function in the whole plane except for a simple pole at $s = k$. We define

$$\xi(s) = -(s-k)(2k-1-s-k)R(s)R_1(s). \tag{2.7}$$

Note that

$$\xi(s) = \xi(2k-1-s). \tag{2.8}$$

We write

$$D(s) = \chi(s)D(2k-1-s), \tag{2.9}$$

where

$$\chi(s) = \pi^{\frac{-3(2k-1)}{2}+3s}\frac{\Gamma\left(\frac{2k-1-s}{2}\right)\Gamma\left(\frac{2k-s}{2}\right)\Gamma\left(\frac{k-s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-k+2}{2}\right)}. \tag{2.10}$$

From (2.1) and (2.10) it follows that, for $a \le \sigma \le b$, we have as $t \to \infty$,

$$\chi(s) = C_2(k,\sigma)t^{\frac{1}{2}(6k-6\sigma-3)}\left(\frac{t}{2\pi e}\right)^{-3it}\left(1 + O\left(\frac{1}{t}\right)\right), \tag{2.11}$$

where C_2 is a certain constant depending only on k and σ. From the maximum-modulus principle and the functional equation, we obtain

$$D(\sigma+it) \ll |t|^{\frac{3}{2}(k-\sigma)}\log|t| \tag{2.12}$$

uniformly for $k - 1/2 \le \sigma \le k$, $|t| \ge 10$.

3. Some lemmas

LEMMA 3.1 (Fundamental Identity). *We have*

$$f(s) := \sum_{n=1}^{\infty} a_n^2 n^{-s} = \frac{\zeta^2(s-k+1)}{\zeta(2s-2k+2)}\Psi(s),$$

where

$$\Psi(s) = \prod_p \left(1 + 2p^{k-1-s} - a_p^2 p^{-s} + p^{2k-2-2s}\right)^{-1}.$$

Proof. We note that a_n is multiplicative and satisfy the equations

$$a_{p^\lambda} = a_p a_{p^{\lambda-1}} - p^{k-1} a_{p^{\lambda-2}} \tag{3.1}$$

and

$$p^{k-1} a_{p^{\lambda-3}} = -a_{p^{\lambda-1}} + a_p a_{p^{\lambda-2}}. \tag{3.2}$$

From this we deduce first

$$f(s) = \prod_p \left(1 + a_p^2 p^{-s} + a_{p^2}^2 p^{-2s} + \sum_{j=3}^\infty \frac{a_{p^j}^2}{p^{js}} \right). \tag{3.3}$$

Next, computing the expression $(3.1)^2 - p^{k-1}(3.2)^2$ gives the relation

$$a_{p^\lambda}^2 - \left(a_p^2 - p^{k-1}\right) a_{p^{\lambda-1}}^2 + p^{k-1}\left(a_p^2 - p^{k-1}\right) a_{p^{\lambda-2}}^2 - p^{3(k-1)} a_{p^{\lambda-3}}^2 = 0. \tag{3.4}$$

Here

(3.5)

$$\begin{aligned}
\sum_{j=3}^\infty \frac{a_{p^j}^2}{p^{js}} &= \sum_{j=3}^\infty \frac{(a_p^2 - p^{k-1})a_{p^{j-1}}^2 - p^{k-1}(a_p^2 - p^{k-1})a_{p^{j-2}}^2 + p^{3(k-1)}a_{p^{j-3}}^2}{p^{js}} \\
&= (a_p^2 - p^{k-1})p^{-s} \left(\sum_{j=2}^\infty \frac{a_{p^j}^2}{p^{js}} \right) - p^{k-1}(a_p^2 - p^{k-1})p^{-2s} \left(\sum_{j=1}^\infty \frac{a_{p^j}^2}{p^{js}} \right) \\
&\qquad + p^{3(k-1)-3s} \left(\sum_{j=0}^\infty \frac{a_{p^j}^2}{p^{js}} \right) \\
&= \Big((a_p^2 - p^{k-1})p^{-s} - p^{k-1-2s}(a_p^2 - p^{k-1}) + p^{3(k-1)-3s} \Big) \left(\sum_{j=0}^\infty \frac{a_{p^j}^2}{p^{js}} \right) \\
&\qquad - (a_p^2 - p^{k-1})p^{-s} - a_p^2(a_p^2 - p^{k-1})p^{-2s} + (a_p^2 - p^{k-1})p^{k-1-2s}.
\end{aligned}$$

We also find that

$$\begin{aligned}
&1 + a_p^2 p^{-s} + a_{p^2}^2 p^{-2s} - (a_p^2 - p^{k-1})p^{-s} \\
&\qquad - a_p^2(a_p^2 - p^{k-1})p^{-2s} + (a_p^2 - p^{k-1})p^{k-1-2s} \\
&= 1 + p^{k-1-s}.
\end{aligned}$$

Let

$$X = 1 + a_p^2 p^{-s} + a_{p^2}^2 p^{-2s} + \sum_{j=3}^\infty \frac{a_{p^j}^2}{p^{js}}$$

and

$$Y = (a_p^2 - p^{k-1})p^{-s} - p^{k-1-2s}(a_p^2 - p^{k-1}) + p^{3(k-1)-3s}.$$

From (3.4), (3.5), and the above arguments, we observe that

$$X = XY + 1 + p^{k-1-s}. \tag{3.6}$$

Therefore we obtain

$$\begin{aligned} f(s) &= \prod_p \left(\frac{1 + p^{k-1-s}}{1 - (a_p^2 - p^{k-1})p^{-s} + p^{k-1-2s}(a_p^2 - p^{k-1}) - p^{3(k-1)-3s}} \right) \\ &= \prod_p \left(\frac{1 - p^{k-1-s}}{1 + p^{k-1-s}} (1 + 2p^{k-1-s} - a_p^2 p^{-s} + p^{2(k-1)-2s}) \right)^{-1} \\ &= \frac{\zeta^2(s-k+1)}{\zeta(2s-2k+2)} \Psi(s), \end{aligned} \tag{3.7}$$

where

$$\Psi(s) = \prod_p \left(1 + 2p^{k-1-s} - a_p^2 p^{-s} + p^{2(k-1)-2s}\right)^{-1}. \tag{3.8}$$

This proves the lemma.

Remark. This lemma is essentially due to Ramanujan. For a proof of the above lemma in the case of Dirichlet series attached to Ramanujan's τ function see, for example, [52] and [53].

Lemma 3.2 (Montgomery-Vaughan). *If h_n is an infinite sequence of complex numbers such that $\sum_{n=1}^{\infty} n|h_n|^2$ is convergent, then*

$$\int_T^{T+H} \left| \sum_{n=1}^{\infty} h_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |h_n|^2 (H + O(n)).$$

Proof. See, for example, Lemma 3.3 of [39], or [46].

4. Mean-square upper bounds on certain lines

K. Chandrasekharan and R. Narasimhan [4] showed that, whenever a Dirichlet series has a functional equation, an approximate functional equation holds, which has a nice form provided the coefficients of the Dirichlet series are positive. This result can be used to study mean value theorems. Even if the coefficients are not positive, such an application is still possible in some special cases.

K. Ramachandra observed that using only the functional equation one can prove reasonable upper bounds for mean-values on certain lines, and he has used this idea in many of his papers (see, for example, [43] and [44]). In this section we use the same idea to prove the following result.

THEOREM 4.1. *For $T \le t \le 2T$, $T \ge T_0$, we have*

$$\text{(i)} \qquad \int_T^{2T} |D(s_1)|^2 dt \ll T^{3/2} (\log T)^{17},$$

$$\text{(ii)} \qquad \int_T^{2T} |D(s_2)|^2 dt = T \sum_{n=1}^{\infty} \frac{|a_{n^2}|^2}{n^{2k-1/2}} + O\left(T^{3/4} (\log T)^{17}\right),$$

$$\text{(iii)} \qquad \int_T^{2T} |D(s_3)|^2 dt = T \sum_{n=1}^{\infty} \frac{|a_{n^2}|^2}{n^{2k+\frac{2}{\log T}}} + O\left((\log T)^{15}\right),$$

where $s_1 = k - 1/2 + it$, $s_2 = k - 1/4 + it$, $s_3 = k + (1/\log T) + it$, and the implied constants depend on the weight k.

Proof of (i). Let Y and Y_1 be two parameters satisfying $10T \le Y, Y_1 \le T^A$, to be chosen appropriately later. Let $\epsilon_1 = (\log T)^{-1}$. By Mellin's transformation, we have

$$\begin{aligned}
(4.1) \qquad S &:= \sum_{n=1}^{\infty} \frac{a_{n^2}}{n^{s_1}} e^{-n/Y} \\
&= \frac{1}{2\pi i} \int\limits_{\substack{\Re w = 1/2+\epsilon \\ |v| \le (\log T)^2}} D(s_1 + w) Y^w \Gamma(w) dw + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2}\right) \\
&= D(s_1) + \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -1+2\epsilon_1 \\ |v| \le (\log T)^2}} D(s_1 + w) Y^w \Gamma(w) dw \\
&\qquad + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2}\right) + O\left(Y^{\frac{1}{2}+\epsilon} T^C e^{-C(\log T)^2}\right) \\
&= D(s_1) + I + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2}\right) + O\left(Y^{\frac{1}{2}+\epsilon} T^C e^{-C(\log T)^2}\right),
\end{aligned}$$

say, upon moving the line of integration to $\Re w = -1 + 2\epsilon_1$. Now,

$$\begin{aligned}
(4.2) \qquad I &= \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -1+2\epsilon_1 \\ |v| \le (\log T)^2}} D(s_1 + w) Y^w \Gamma(w) dw \\
&= \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -1+2\epsilon_1 \\ |v| \le (\log T)^2}} \chi(s_1 + w) \left(Q_1 + Q_2\right) Y^w \Gamma(w) dw \\
&= I_1 + I_2,
\end{aligned}$$

where

$$Q_1 = \sum_{n \le Y_1} a_{n^2} n^{s_1 + w - 2k + 1}, \quad Q_2 = \sum_{n > Y_1} a_{n^2} n^{s_1 + w - 2k + 1}.$$

We note that in I_2, $\Re(s_1+w)=k-(3/2)+2\epsilon_1$. We have

$$\begin{aligned}(4.3)\qquad \sum_{n>Y_1} a_{n^2} n^{s_1+w-2k+1} &= \sum_{Y_1<n\le Y_1^{10}} a_{n^2} n^{s_1+w-2k+1} \\ &\quad + O\left(\sum_{n>Y_1^{10}} (d(n))^2 n^{k-1+k-\frac{3}{2}+2\epsilon_1-2k+1}\right) \\ &= \sum_{Y_1<n\le Y_1^{10}} a_{n^2} n^{s_1+w-2k+1} + O\left(Y_1^{-5+20\epsilon_1}\right),\end{aligned}$$

where we have used inequality (1.4). Using Hölder's inequality and a theorem of Montgomery and Vaughan (see [39]), we get

(4.4)

$$\begin{aligned}\int_T^{2T} |I_2|^2 dt \ll \int_T^{2T} \Bigg| \int_{\substack{\Re w=-1+2\epsilon_1\\ |v|\le(\log T)^2}} &\chi(s_1+w) \\ &\times \left(\sum_{Y_1^{10}\ge n>Y_1} a_{n^2} n^{s_1+w-2k+1}\right) Y^w \Gamma(w) dw \Bigg|^2 dt \\ &\qquad + T^{7+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\ \ll (\log T)^2 \frac{T^{6+20\epsilon_1}}{Y^2} \int_{T-(\log T)^2}^{2T+(\log T)^2} \sum_U \Bigg| &\sum_{U\le n\le 2U} a_{n^2} n^{s_1+\Re w-2k+1}\Bigg|^2 dt \\ &\qquad + T^{7+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\ \ll (\log T)^2 \frac{T^{6+20\epsilon_1}}{Y^2} \sum_U \sum_{U\le n\le 2U} &\frac{|a_{n^2}|^2}{n^{2(2k-1-(k-\frac{3}{2}+2\epsilon_1))}} n \\ &\qquad + T^{7+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\ \ll (\log T)^2 \frac{T^{6+20\epsilon_1}}{Y^2} \sum_U \sum_{U\le n\le 2U} &\frac{(d(n))^4 n^{2k-2} n}{n^{2(2k-1-(k-\frac{3}{2}+2\epsilon_1))}} \\ &\qquad + T^{7+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\ \ll (\log T)^2 \frac{T^{6+20\epsilon_1}}{Y^2} \sum_{j=0}^{C\log Y_1} &(\log Y_1)^{15} (2^j Y_1)^{-1+4\epsilon_1} \\ &\qquad + T^{7+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1} \\ \ll (\log T)^2 T^{6+20\epsilon_1} Y^{-2} Y_1^{-1+4\epsilon_1} &(\log Y_1)^{15} + T^{7+20\epsilon_1} Y^{-2} Y_1^{-10+40\epsilon_1}.\end{aligned}$$

Also we have

$$
\begin{aligned}
(4.5) \qquad I_1 &= \frac{1}{2\pi i} \int\limits_{\substack{\Re w=-1+2\epsilon_1 \\ |v|\leq(\log T)^2}} \chi(s_1+w)\left(\sum_{n\leq Y_1} a_{n^2} n^{s_1+w-2k+1}\right) Y^w \Gamma(w) dw \\
&= \frac{1}{2\pi i} \int\limits_{\substack{\Re w=-\frac{1}{2} \\ |v|\leq(\log T)^2}} \chi(s_1+w)\left(\sum_{n\leq Y_1} a_{n^2} n^{s_1+w-2k+1}\right) Y^w \Gamma(w) dw \\
&\qquad + O(T^C e^{-C\log T)^2}),
\end{aligned}
$$

upon moving the line of integration to $\Re w = -\frac{1}{2}$. Using again the Montgomery-Vaughan theorem and inequality (1.4), we obtain

$$
\begin{aligned}
(4.6) \qquad \int_T^{2T} |I_1|^2 dt &\ll (\log T)^2 \frac{T^3}{Y} \int_{T-(\log T)^2}^{2T+(\log T)^2} \left| \sum_{n\leq Y_1} a_{n^2} n^{s_1+\Re w-2k+1}\right|^2 dt \\
&\ll (\log T)^2 \frac{T^3}{Y} \sum_{n\leq Y_1} \frac{|a_{n^2}|^2 n}{n^{2(2k-1-(k-1))}} \\
&\ll (\log T)^2 \frac{T^3}{Y} \sum_{n\leq Y_1} \frac{(d(n))^4 n^{2k-1}}{n^{2(2k-1-(k-1))}} \\
&\ll (\log T)^2 \frac{T^3}{Y} (\log Y_1)^{15}.
\end{aligned}
$$

Now,

$$
S = \sum_{n=1}^{\infty} \frac{a_{n^2}}{n^{s_1}} e^{-n/Y},
$$

and so

$$
\begin{aligned}
(4.7) \qquad \int_T^{2T} |S|^2 dt &= \sum_{n=1}^{\infty} \frac{|a_{n^2}|^2}{n^{2(k-1/2)}} e^{-2n/Y} \left(T + O(n)\right) \\
&= T \left\{ \sum_{n\leq Y/2} \frac{|a_{n^2}|^2}{n^{2k-1}} \left(1 + O\left(\frac{n}{Y}\right)\right) \right. \\
&\qquad \left. + O\left(\sum_{n\geq Y/2} \frac{|a_{n^2}|^2}{n^{2k-1}} \left(\frac{Y}{n}\right)^{2\alpha+1} \right) \right\}
\end{aligned}
$$

$$+ O\left(\sum_{n \le Y/2} \frac{|a_{n^2}|^2}{n^{2k-1}} n + \sum_{n \ge Y/2} \frac{|a_{n^2}|^2}{n^{2k-1}} n \left(\frac{Y}{n}\right)^{2\alpha+2} \right)$$

$$\ll T(\log Y)^{15} + Y(\log Y)^{15}$$

with $\alpha = k$. Using inequality (1.4), we find that

$$\sum_{n \le Y/2} \frac{(d(n))^4}{n^{2k-1}} n^{2k-2} \ll (\log Y)^{15}, \tag{4.8}$$

$$\frac{1}{Y} \sum_{n \le Y/2} \frac{(d(n))^4}{n^{2k-1}} n^{2k-2} n \ll (\log Y)^{15}. \tag{4.9}$$

Choosing $\alpha = k-1$, we have

$$\begin{aligned} Y^{2k-1} \sum_{n \ge Y/2} \frac{(d(n))^4}{n^{2k-1}} n^{2k-2} \frac{1}{n^{2k-1}} & \\ \ll Y^{2k-1} \sum_{j=0}^{\infty} \sum_{U \le n < 2U,\, U = 2^j Y/2} \frac{(d(n))^4}{n^{2k}} & \\ \ll Y^{2k-1} \sum_{j=0}^{\infty} \sum_{U = 2^j Y/2} \frac{1}{U^{2k}} U (\log U)^{15} & \\ \ll (\log Y)^{15}, & \end{aligned} \tag{4.10}$$

$$\sum_{n \le Y/2} \frac{(d(n))^4}{n^{2k-1}} n^{2k-2} n \ll Y(\log Y)^{15}. \tag{4.11}$$

Similarly to (4.10), we obtain, by choosing a suitable α (say, $\alpha = k$)

$$\sum_{n \ge Y/2} \frac{(d(n))^4}{n^{2k-1}} n^{2k-1} \left(\frac{Y}{n}\right)^{2\alpha+2} \ll Y(\log Y)^{15}. \tag{4.12}$$

Part (i) of the theorem now follows if we choose $Y = Y_1 = T^{3/2}$.

Proof of (ii). Let Y and Y_1 be two parameters satisfying $10T \le Y, Y_1 \le T^A$, which will be chosen appropriately later. Let $\epsilon_1 = (\log T)^{-1}$. Since the proof of (ii) is similar to that of (i), we only give a sketch. By Mellin's transformation, we have

(4.13)
$$
\begin{aligned}
S &:= \sum_{n=1}^{\infty} \frac{a_{n^2}}{n^{s_2}} e^{-n/Y} \\
&= \frac{1}{2\pi i} \int\limits_{\substack{\Re w = 1/2+\epsilon \\ |v| \le (\log T)^2}} D(s_2+w) Y^w \Gamma(w) dw + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2} \right) \\
&= D(s_2) + \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -1+2\epsilon_1 \\ |v| \le (\log T)^2}} D(s_2+w) Y^w \Gamma(w) dw + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2} \right) \\
&\qquad + O\left(Y^{\frac{1}{2}+\epsilon} T^C e^{-C(\log T)^2} \right) \\
&= D(s_2) + I + O\left(Y^{\frac{1}{2}+\epsilon} e^{-C(\log T)^2} \right) + O\left(Y^{\frac{1}{2}+\epsilon} T^C e^{-C(\log T)^2} \right),
\end{aligned}
$$

say, upon moving the line of integration to $\Re w = -1 + 2\epsilon_1$. Now,

$$
\begin{aligned}
I &= \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -1+2\epsilon_1 \\ |v| \le (\log T)^2}} \chi(s_2+w) \left(Q_3 + Q_4 \right) Y^w \Gamma(w) dw \\
&= I_1 + I_2,
\end{aligned} \tag{4.14}
$$

where

$$
Q_3 = \sum_{n \le Y_1} a_{n^2} n^{s_2+w-2k+1}, \quad Q_4 = \sum_{n > Y_1} a_{n^2} n^{s_2+w-2k+1}.
$$

We note that in I_2, $\Re(s_2 + w) = k - (5/4) + 2\epsilon_1$. We have

$$
\sum_{n > Y_1} a_{n^2} n^{s_2+w-2k+1} = \sum_{Y_1 < n \le Y_1^{10}} a_{n^2} n^{s_2+w-2k+1} + O\left(Y_1^{-2.5+20\epsilon_1} \right), \tag{4.15}
$$

where we have used inequality (1.4). Using Hölder's inequality and a theorem of Montgomery and Vaughan (see [39]), we get as in the proof of (i)

$$
\begin{aligned}
\int_T^{2T} |I_2|^2 dt &\ll (\log T)^2 T^{4.5+20\epsilon_1} Y^{-2} Y_1^{-\frac{1}{2}+4\epsilon_1} (\log Y_1)^{15} \\
&\qquad + T^{5.5+20\epsilon_1} Y^{-2} Y_1^{-5+40\epsilon_1}.
\end{aligned} \tag{4.16}
$$

Also, we have

(4.17)

$$I_1 = \frac{1}{2\pi i} \int\limits_{\substack{\Re w = -\frac{3}{4} \\ |v| \le (\log T)^2}} \chi(s_2+w) \left(\sum_{n \le Y_1} a_{n^2} n^{s_2+w-2k+1} \right) Y^w \Gamma(w) dw \\ + O\left(T^C e^{-C \log T)^2} \right),$$

upon moving the line of integration to $\Re w = -3/4$. Using again the Montgomery-Vaughan theorem and inequality (1.4), we obtain

$$\int_T^{2T} |I_1|^2 dt \ll (\log T)^2 \frac{T^3}{Y^{3/2}} \int_{T-(\log T)^2}^{2T+(\log T)^2} \left| \sum_{n \le Y_1} a_{n^2} n^{s_2+\Re w-2k+1} \right|^2 dt \\ \ll (\log T)^2 \frac{T^3}{Y^{3/2}} (\log Y_1)^{15}. \tag{4.18}$$

Now,

$$S = \sum_{n=1}^{\infty} \frac{a_{n^2}}{n^{s_2}} e^{-n/Y},$$

and, as in the proof of (i), we obtain

$$\int_T^{2T} |S|^2 dt = T \sum_{n=1}^{\infty} \frac{|a_{n^2}|^2}{n^{2(k-1/4)}} \\ + O\left(TY^{-1/2} (\log Y)^{15} + Y^{1/2} (\log Y)^{15} \right). \tag{4.19}$$

Part (ii) of the theorem now follows by choosing $Y = Y_1 = T^{3/2}$.

Proof of (iii). Part (iii) of the theorem follows from Lemma 3.2.

We are now in a position to prove the following result.

Theorem 4.2. *Suppose that the inequality*

$$\zeta\left(\frac{1}{2}+it\right) \ll t^\kappa (\log t)$$

holds for some $\kappa > 0$ and all $t \ge 10$. Then:

(i) *For $1/2 \le \sigma \le 3/4$ we have*

$$\int_T^{2T} |Z(\sigma+it)|^2 dt \ll T^{\frac{1}{2}+(4\kappa+2)(1-\sigma)} (\log T)^{17(3-4\sigma)+2}.$$

(ii) *Let*

$$\nu = \frac{5}{2(6-4\kappa)}$$

and

$$\vartheta = \frac{29 - 10\nu + \sqrt{100\nu^2 + 420\nu - 159}}{40}.$$

For $3/4 < \sigma \leq 1$ *we have*

$$\int_T^{2T} |Z(\sigma + it)|^2 dt = T \sum_{n=1}^{\infty} {c'_n}^2 n^{-2\sigma} + O(T^{\theta(\sigma)+\epsilon}),$$

where

$$\theta(\sigma) = \begin{cases} \frac{5}{2} - 2\sigma & \text{if } \frac{3}{4} < \sigma \leq \vartheta, \\ \frac{30(1-\sigma)}{17 - 5\nu - 10\sigma} & \text{if } \vartheta \leq \sigma \leq 1. \end{cases}$$

Proof of (i). From Lemma 3.1, we have the relation

$$Z(s-k+1) = \zeta(s-k+1)D(s).$$

After normalizing the coefficients we find that

$$Z(s) = \zeta(s)D(s+k-1). \tag{4.20}$$

Let

$$J(\sigma, \lambda) = \left(\int_T^{2T} |f(\sigma + it)|^{1/\lambda} \right)^{\lambda}, \quad \lambda > 0. \tag{4.21}$$

Then Gabriel's convexity theorem (see p. 203 of [61]) asserts that for $\alpha \leq \sigma \leq \beta$

$$J(\sigma, p\lambda + q\mu) \leq J^p(\alpha, \lambda) J^q(\beta, \mu), \tag{4.22}$$

where

$$p = \frac{\beta - \sigma}{\beta - \alpha}, \quad q = \frac{\sigma - \alpha}{\beta - \alpha}. \tag{4.23}$$

We choose the parameters as follows:

$$f(s) = D(s+k-1), \quad \alpha = \frac{1}{2}, \quad \beta = \frac{3}{4}, \quad \lambda = \frac{1}{2}, \quad \mu = \frac{1}{2}.$$

Then $p + q = 1$ and $p\lambda + q\mu = 1/2$. From (i) and (ii) of Theorem 4.1, using (4.22), we obtain, for $1/2 \leq \sigma \leq 3/4$,

$$\left(\int_T^{2T} |D(\sigma + k - 1 + it)|^2 \, dt \right)^{1/2} \leq Q_5,$$

where

$$Q_5 = \left(\int_T^{2T} \left| D\left(k - \frac{1}{2} + it\right) \right|^2 dt \right)^{\frac{1}{2}(3-4\sigma)} \left(\int_T^{2T} \left| D\left(k - \frac{1}{4} + it\right) \right|^2 dt \right)^{\frac{1}{2}(4\sigma-2)}.$$

This implies that

$$\int_T^{2T} |D(\sigma + k - 1 + it)|^2 dt \ll \left(T^{3/2} (\log T)^{17} \right)^{3-4\sigma} T^{4\sigma-2} \ll T^{(5-4\sigma)/2} (\log T)^{17(3-4\sigma)}. \tag{4.24}$$

From the assumption of the theorem it follows that for $1/2 \leq \sigma \leq 1$

$$\zeta(\sigma + it) \ll t^{2\kappa(1-\sigma)} \log t. \tag{4.25}$$

Also, notice that

$$\int_T^{2T} |Z(\sigma + it)|^2 dt \ll \left(\max_{T \leq t \leq 2T} |\zeta(\sigma + it)|^2 \right) \left(\int_T^{2T} |D(\sigma + k - 1 + it)|^2 dt \right). \tag{4.26}$$

From (4.24), (4.25), and (4.26), part (i) of the theorem follows.

Proof of (ii). We only sketch the proof, since the details can be found in [34]. The function $Z(s)$ satisfies the functional equation

$$Z(s) = \Delta_1(s) Z(1 - s), \tag{4.27}$$

where

$$|\Delta_1(s)| = \left| (2\pi)^{4s-2} \frac{\Gamma(1-s)\Gamma(k-s)}{\Gamma(s)\Gamma(s+k-1)} \right| \sim \left(\frac{t}{2\pi} \right)^{2-4\sigma} \left(1 + O\left(\frac{1}{t} \right) \right) \tag{4.28}$$

for $t \geq t_0$. From the definition of ν, we have $5/12 < \nu < 1/2$. Therefore, from (4.28) and part (i) of Theorem 4.2, we find that

$$\int_T^{2T} |Z(\nu + it)|^2 dt \sim \int_T^{2T} t^{4-8\nu} |Z(1 - \nu + it)|^2 dt \ll T^{4-8\nu+\left(\frac{1}{2}+(4\kappa+2)\nu\right)+\epsilon} \ll T^{2+\epsilon}. \tag{4.29}$$

If we write

$$\sum_{n \leq x} c_n' = Cx + \Delta(x)$$

then, combining (4.29) with the method of Lemma 13.1 of [23], we obtain

$$\int_1^X |\Delta(y)|^2 dy \ll X^{1+2\nu+\epsilon}. \tag{4.30}$$

Following the arguments of [34] and using (4.30), (ii) follows.

REMARK. Using the classical value $\kappa = 1/6$, Theorem 4.2 gives

$$\int_T^{2T} \left| Z\left(\frac{1}{2} + it\right)\right|^2 dt \ll T^{11/6} (\log T)^{17},$$

which is far better than (1.9). The best known value for κ is $\kappa = \frac{89}{570} + \epsilon$ (see [21]). We also mention that part (i) of Theorem 4.2 improves upon part (i) of Theorem A only in the range $1/2 \leq \sigma \leq (3 - 8\kappa)/(4 - 8\kappa)$, since

$$\frac{1}{2} + (4\kappa + 2)(1 - \sigma) < 4 - 4\sigma$$

holds only when $\sigma \leq (3 - 8\kappa)/(4 - 8\kappa) < 3/4$.

THEOREM 4.3. *If the Lindelöf hypothesis holds, i.e., if for every positive constant* ϵ,

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{\epsilon},$$

then the inequality

$$\int_T^{2T} |Z(\sigma + it)|^2 dt \ll T^{\frac{1}{2} + 2(1-\sigma) + \epsilon}$$

holds for every positive constant ϵ *in the range* $1/2 \leq \sigma \leq 3/4 - \epsilon$.

Proof. This follows from part (i) of Theorem 4.2 upon taking $\kappa = \epsilon$.

THEOREM 4.4. *Assume that*

$$\int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right)\right|^{2j} dt \ll T^{1+\epsilon}$$

holds for some fixed integer $j \geq 1$ *and every positive constant* ϵ. *Then we have*

$$\int_T^{2T} \left| Z\left(\frac{1}{2} + it\right)\right|^2 dt \ll T^{(3/2)+(1/j)+\epsilon}.$$

for every $\epsilon > 0$.

Proof. From (4.20), on using Hölder's inequality, we obtain

(4.31)

$$\begin{aligned}
&\int_T^{2T} \left| Z\left(\frac{1}{2}+it\right)\right|^2 dt \\
&= \int_T^{2T} \left| \zeta\left(\frac{1}{2}+it\right) D\left(k-\frac{1}{2}+it\right)\right|^2 dt \\
&\le \left(\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2j} dt\right)^{1/j} \left(\int_T^{2T} \left|D\left(k-\frac{1}{2}+it\right)\right|^{2j/(j-1)} dt\right)^{(j-1)/j} \\
&\ll \left(\max_{T\le t\le 2T} \left|D\left(k-\frac{1}{2}+it\right)\right|^{2/j}\right) \left(\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2j} dt\right)^{1/j} \\
&\qquad\qquad \times \left(\int_T^{2T} \left|D\left(k-\frac{1}{2}+it\right)\right|^{2} dt\right)^{(j-1)/j}.
\end{aligned}$$

By (2.12), we have

$$D(k-\frac{1}{2}+it) \ll t^{3/4}(\log t). \tag{4.32}$$

The asserted estimate now follows from the assumption of the theorem, part (i) of Theorem 4.1, (4.31), and (4.32).

Remark. From Theorems 4.2, 4.3 and 4.4, we see that the mean-square upper bound for $Z(1/2+it)$ depends on

(1) the growth estimate of the Riemann zeta-function on the line $\sigma = \frac{1}{2}$,
(2) the higher moments of the Riemann zeta-function on the critical line $\sigma = \frac{1}{2}$,
(3) the growth estimate of the symmetric square L-function $D(s)$ on the line $\sigma = k - \frac{1}{2}$, and
(4) the higher moments of the symmetric square L-function $D(s)$ on the line $\sigma = k - \frac{1}{2}$.

In this connection we would like to point out the important work of Heath-Brown (see [15]) and of Ivic and Motohashi (see [26]). On the one hand, a general theorem of Ramachandra (see [47]) implies the mean-square lower bound

$$\int_T^{2T} \left| Z\left(\frac{1}{2}+it\right)\right|^2 dt \gg T\frac{\log T}{\log\log T}. \tag{4.33}$$

On the other hand, it is not hard to see that, under the assumption of the Lindelöf hypothesis for $D(s)$ on the line $\sigma = k - \frac{1}{2}$, i.e., the estimate

$$D\left(k - \frac{1}{2} + it\right) \ll t^{\epsilon} \tag{4.34}$$

for any small positive constant ϵ, we have

$$\int_T^{2T} \left| Z\left(\frac{1}{2} + it\right)\right|^2 dt \ll T^{1+10\epsilon}. \tag{4.35}$$

This suggests the following conjecture.

Conjecture. *We have*

$$\int_T^{2T} \left| Z\left(\frac{1}{2} + it\right)\right|^2 dt \ll T(\log T)^A \tag{4.36}$$

with some positive constant A.

Note added in proof. In fact, we have

$$D(s) := \zeta(2s - 2k + 2)\left(\sum_{n=1}^{\infty} a^*_{n^2} n^{-s}\right),$$

where

$$|a^*_{n^2}| \le d(n^2) n^{k-1}.$$

From our definition (1.3), it follows that

$$b_n := a_{n^2} = \sum_{l^2 m = n} l^{2k-2} a^*_{m^2}$$

and that b_n is a multiplicative function. It is not difficult to see that

$$\left|b_{p^j}\right| \le p^{j(k-1)} \sum_{a=0}^{j} (2a+1) \sum_{2b+a=j} 1 \le p^{j(k-1)}(j+1)^2 = p^{j(k-1)}\left(d\left(p^j\right)\right)^2$$

since for every fixed j there is at most one solution to the equation $2b + a = j$ for every fixed a. (For $j \le 4$ the inequality follows by an easy computation, and for $j \ge 5$ it can be proved by considering separately the cases when j is even and odd.) This implies (in our notation) the final inequalty of (1.4), namely

$$|a_{n^2}| \ll (d(n))^2\, n^{k-1}.$$

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