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ALMOST SURE CONVERGENCE OF WEIGHTED SERIES OF CONTRACTIONS

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ABSTRACT. In this paper we consider the almost sure convergence of a series of contractions (of an arbitrary Hilbert space) with random weights. The paper is a continuation of a previous work [PSW], in which only convergence in operator norm was investigated. We obtain conditions ensuring the existence of universal sets on which these series are converging almost everywhere, for any contraction. The paper is also a continuation of the paper [SW], in which an analogous problem concerning ergodic averages was considered, as well as the paper [S], which deals with a variant of the problem. The proofs of our results rely on uniform estimates of random polynomials which were established in a recent paper by the second author and proved by means of metric entropy methods.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a probability space. The purpose of the paper is to establish conditions for the convergence almost everywhere of the randomly weighted series of contractions

$$\sum_{k=1}^{\infty} W_k(\omega) T^{p_k}, \tag{1.1}$$

where $\{W_k\}_{k\geq 1}$ is a sequence of independent, mean zero, square integrable random variables, defined on some probability space $(\Omega, \mathcal{B}, \mathbf{P})$, T is a linear contraction in the Hilbert space $H = L^2(\mu)$, $\{p_k\}_{k\geq 1}$ is a non-decreasing sequence of non-negative integers, and $\omega \in \Omega$. Our main goal is to find sufficient conditions for the convergence almost everywhere of the series in (1.1) which are valid for all Hilbert spaces $H = L^2(\mu)$ and all contractions T in H. The paper is in this sense a continuation of the paper [PSW], where we obtained sufficient conditions for the convergence in norm of the series in (1.1). More precisely, we proved the following theorem:

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Theorem 1.1 ([PSW, Theorm 3.1, p. 272, and Remark 3.2]). *Let $\{W_k\}_{k\ge1}$ be a sequence of independent, mean zero, random variables, defined on some probability space $(\Omega, \mathcal{B}, \mathbf{P})$, and let $\{p_k\}_{k\ge1}$ be a non-decreasing sequence of non-negative integers, with $p_1 > 1$. Suppose that there exist integers $0 := N_0 < N_1 < N_2 < \cdots$ such that the following condition is satisfied:*

$$\sum_{i=0}^{\infty} \sqrt{\log(p_{N_{i+1}})}\mathbf{E}\left[\left(\sum_{k=N_i+1}^{N_{i+1}} |W_k|^2\right)^{1/2}\right] \quad \textit{converges.} \tag{1.2}$$

Then there exists a (universal) sequence of $\mathbf{P}$-integrable random variables $M = \{M_J\}_{J\ge1}$ defined on $(\Omega, \mathcal{B}, \mathbf{P})$ which converges to zero $\mathbf{P}$-a.s. and in $\mathbf{P}$-mean, such that for any Hilbert space H and any contraction T in H we have

$$\sup_{R>N_J} \left\| \sum_{k=N_J+1}^{R} W_k(\omega)T^{p_k} \right\| \le M_J(\omega) \tag{1.3}$$

for all $\omega \in \Omega$ and all $N \ge 1$. In particular, there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that the series

$$\sum_{k=1}^{\infty} W_k(\omega)T^{p_k} \tag{1.4}$$

converges in operator norm for all $\omega \in \Omega\backslash N^*$, whenever H is a Hilbert space and T is a contraction in H.

We will show (see Theorem 3.1) that Condition (1.2) is, in fact, already enough to imply that there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega\backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, if we define

$$\forall \omega \in \Omega,\ \forall x \in X,\ \forall n \ge 1, \qquad S_n(\omega, x) = \sum_{k=1}^{n} W_k(\omega)T^{p_k} f(x), \tag{1.5}$$

then the sequence $S_{N_k}(\omega, \cdot)$ converges μ-almost surely.

If, in addition to Condition (1.2), we have

$$\sum_k \min\left[\log^2(N_{k+1} - N_k)\log p_{N_{k+1}}\left(\sum_{j=N_k+1}^{N_{k+1}} \mathbf{E}(W_j^2)\right), \right. \\ \left. \mathbf{E}\left(\sum_{j=N_k+1}^{N_{k+1}} |W_j|\right)^2\right] < \infty, \tag{1.6}$$

then there also exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega\backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, the sequence S_n, $n = 1, 2, \ldots$, converges μ-almost surely.

The method of proof is essentially based on an improvement of a classical result of Salem and Zygmund (see Lemma 2.2) providing a global uniform estimate for random polynomials. For the proof of our main result, we will also invoke a classical uniform estimate arising in the theory of stochastic processes (see Theorem 2.3).

2. Preliminary results

We begin by recalling a useful tool, the spectral inequality, which reduces the problem of evaluating norms to Fourier analysis questions. Let T be a contraction in a Hilbert H, that is, a linear operator such that $\|T(f)\| \le \|f\|$ for each f in H. Let $f \in H$, and put

$$P_n(f) = \langle T^n(f), f\rangle \quad \text{for} \quad n \ge 0 \quad \text{and} \quad P_n(f) = \overline{P_{-n}(f)} \quad \text{for} \quad n \le 0.$$

The sequence $(P_n(f))_{n\in\mathbf{Z}}$ is non-negative definite, and thus, by Herglotz' Theorem, there exists a finite positive measure μ_f on $\mathcal{B}(]-\pi,\pi])$ (called the spectral measure of f) such that for all $n \ge 0$

$$\langle T^n(f), f\rangle = \int_{-\pi}^{\pi} e^{in\lambda}\mu_f(d\lambda).$$

From this fact and the Dilation Theorem of Sz-Nagy (see Theorem 1 in [N]) one deduces:

Lemma 2.1 (Spectral Inequality). *If T is a contraction in a Hilbert space H and f is an element from this space with spectral measure μ_f, then we have*

$$\|P(T)f\| \le \left(\int_{-\pi}^{\pi} \left|P(e^{i\lambda})\right|^2 \mu_f(d\lambda)\right)^{1/2}, \tag{2.1}$$

whenever $P(z) = \sum_{k=0}^{N} a_k z^k$ is a complex polynomial of degree $N \ge 0$.

We next state a stronger form of the classical Salem-Zygmund bound for random polynomials, on which our results in the next section are based. Consider the Young function $G(t) = \exp(t^2) - 1$, where t is real, together with the associated Orlicz space $L^G(\mathbf{P})$, that is, the set of $\mathcal{B}$-measurable functions $f\colon \Omega \to \mathbf{R}$ such that $\mathbf{E}G(af) < \infty$ for some real $0 < a < \infty$. We recall that $L^G(\mathbf{P})$ is endowed with the norm

$$\forall f \in L^G(\mathbf{P}), \qquad \|f\|_G = \inf\{c > 0 : \mathbf{E}G(f/c) \le 1\},$$

and that $(L^G(\mathbf{P}), \|\cdot\|_G)$ is a Banach space.

LEMMA 2.2 ([W, Theorem 7]). *Let $\mathcal{W}=(W_k)_{k=1}^{\infty}$ be a sequence of independent, symmetric real random variables. Then*

$$\left\| \sup_{N<M} \sup_{0\le t\le 1} \frac{\left|\sum_{k=N+1}^{M} W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{1/2}} \right\|_G \le C, \tag{2.2}$$

where C is a universal constant.

It is easily seen, by means of the Cauchy-Schwarz inequality, that Lemma 2.2 is only interesting when the sequence $(p_m)_{m\ge 1}$ grows at most geometrically.

We define the sequence of random polynomials

$$U_N(t) = \sum_{k=1}^{N} a_k \left\{ e^{2i\pi t(p_k+P_k)} - \mathbf{E} e^{2i\pi t(p_k+P_k)} \right\}, \qquad N=1,2,\dots, \tag{2.3}$$

where $\{a_k, k\ge 1\}$ is a sequence of reals and $\mathcal{P}\colon=\{P_1,P_2,\dots\}$ a sequence of $\mathbf{Z}$-valued, independent random variables defined on a probability space $(\Omega,\mathcal{B},\mathbf{P})$, and satisfying

$$\mathbf{P}\left\{p_i+P_i\ge 0\right\}=1, \qquad i=1,2,\dots. \tag{2.4}$$

Let us assume that the following assumption, in which $\Phi\colon \mathbf{N}\to\mathbf{N}$ is an increasing map, is satisfied:

$$C(\mathcal{P},\Phi) = \mathbf{E} \sup_{M=1}^{\infty} \frac{\left[\log_+(p_M+P_M)\right]^{1/2}}{\Phi(M)} < \infty. \tag{2.5}$$

The following lemma will be used for proving Theorem 3.7, in which we examine a variant of the problem considered in [S].

LEMMA 2.3. *There exists a universal constant C such that*

$$\mathbf{E} \sup_{N<M} \sup_{0\le t\le 1} \frac{|U_M(t)-U_N(t)|}{\left(\sum_{k=N+1}^{M} a_k^2\right)^{1/2} \Phi(M)} \le C\cdot C(\mathcal{P},\Phi). \tag{2.6}$$

Proof. The proof is a modification of that of Theorem 9 in [W]. We consider the symmetrized sequence

$$V_N(t) = \sum_{k=1}^{N} a_k \varepsilon_k e^{2i\pi t(p_k+P_k)}, \qquad N=1,2,\dots,$$

where $\varepsilon_1,\varepsilon_2,\dots$ is a Rademacher sequence defined on another probability space $(\Omega_\varepsilon,\mathcal{B}_\varepsilon,\mathbf{P}_\varepsilon)$. We denote by $\mathbf{E}_\varepsilon$ the corresponding symbol of integration. Since conditionally to the sequence $(P_k)_k$ these polynomials are of the same type as those in Lemma 2.2, this lemma can be applied to estimate their extrema, and we obtain

$$\left\| \sup_{N<M} \sup_{0\le t\le 1} \frac{|V_M(t)-V_N(t)|}{\left\{\left(\sum_{k=N+1}^M a_k^2\right)\log_+(p_M+P_M)\right\}^{1/2}} \right\|_{G,\mathbf{P}_\varepsilon} \le C,$$

where C is a universal constant. Hence

$$\mathbf{E}_\varepsilon \sup_{N<M} \sup_{0\le t\le 1} \frac{|V_M(t)-V_N(t)|}{\left\{\left(\sum_{k=N+1}^M a_k^2\right)\log_+(p_M+P_M)\right\}^{1/2}} \le C.$$

Thus

$$\begin{aligned}
&\mathbf{E}\ \mathbf{E}_\varepsilon \sup_{N<M} \sup_{0\le t\le 1} \frac{|V_M(t)-V_N(t)|}{\left(\sum_{k=N+1}^M a_k^2\right)^{1/2}\Phi(M)} \\
&\quad \le \mathbf{E}\ \mathbf{E}_\varepsilon \sup_{N<M} \sup_{0\le t\le 1} \frac{|V_M(t)-V_N(t)|}{\left[\left(\sum_{k=N+1}^M a_k^2\right)\log_+(p_M+P_M)\right]^{1/2}} \\
&\qquad\qquad \times \sup_M \frac{\left[\log_+(p_M+P_M)\right]^{1/2}}{\Phi(M)} \\
&\quad \le C\mathbf{E}\sup_M \left[\log_+(p_M+P_M)\right]^{1/2}\Phi(M) \le C\ C(\mathcal{P},\Phi).
\end{aligned}$$

To conclude, observe that, by means of the usual symmetrization procedure,

$$\begin{aligned}
&\mathbf{E}\ \mathbf{E}_\varepsilon \sup_{N<M} \sup_{0\le t\le 1} \frac{|U_M(t)-U_N(t)|}{\left(\sum_{k=N+1}^M a_k^2\right)^{1/2}\Phi(M)} \\
&\quad = \mathbf{E} \sup_{N<M} \sup_{0\le t\le 1} \frac{\left|\sum_{k=N+1}^M e^{2i\pi t(p_k+P_k)} - \mathbf{E}' e^{2i\pi t(p_k+P_k')}\right|}{\left(\sum_{k=N+1}^M a_k^2\right)^{1/2}\Phi(M)} \\
&\quad \le \mathbf{E}\mathbf{E}' \sup_{N<M} \sup_{0\le t\le 1} \frac{\left|\sum_{k=N+1}^M e^{2i\pi t(p_k+P_k)} - e^{2i\pi t(p_k+P_k')}\right|}{\left(\sum_{k=N+1}^M a_k^2\right)^{1/2}\Phi(M)} \\
&\quad \le 2\mathbf{E} \sup_{N<M} \sup_{0\le t\le 1} \frac{\left|\sum_{k=N+1}^M \varepsilon_k e^{2i\pi t(p_k+P_k)}\right|}{\left(\sum_{k=N+1}^M a_k^2\right)^{1/2}\Phi(M)} \\
&\quad = \mathbf{E}\ \mathbf{E}_\varepsilon \sup_{N<M} \sup_{0\le t\le 1} \frac{|V_M(t)-V_N(t)|}{\left(\sum_{k=N+1}^M a_k^2\right)^{1/2}\Phi(M)} \le C\cdot C(\mathcal{P},\Phi),
\end{aligned}$$

where $P_1^{'}, P_2^{'}, \dots$ is an independent copy of the sequence $P_1, P_2, \dots,$ defined on another probability space $(\Omega^{'}, \mathcal{B}^{'}, \mathbf{P}^{'})$, with $\mathbf{E}^{'}$ as the corresponding symbol of integration. □

We conclude this section by recalling a classical tool from the theory of stochastic processes given in the following statement:

THEOREM 2.4 ([P, Theorem 2.1]). *Let (E, d) be a compact space endowed with a continuous pseudo-metric d. Assume that*

$$\int_0^1 N_d(E, \epsilon)^{1/2}\, d\epsilon < \infty, \tag{2.7}$$

where $N_d(E, \epsilon)$ is the smallest number of d-open balls of radius ϵ needed to cover E. Then any stochastic process $X = \{X_t, t \in E\}$ satisfying

$$\forall s, t \in E, \quad \|X_s - X_t\|_2 \le d(s, t) \tag{2.8}$$

possesses a modification with continuous sample paths on E. Moreover,

$$\left[\mathbf{E} \sup_{E \times E} |X_t - X_s|^2\right]^{1/2} \le K \int_0^{K_1 D} N_d(E, \epsilon)^{1/2}\, d\epsilon, \tag{2.9}$$

where $D = \sup_{E\times E} d(s, t)$ and K_1 is an absolute constant.

3. Almost sure convergence

3.1. In this section, we present the main result of the paper.

THEOREM 3.1. *Let $(W_k)_{k\in\mathbf{N}}$ be a sequence of independent, symmetric, square integrable random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$, and let $(p_n)_{n\in\mathbf{N}}$ be an increasing sequence of positive integers. Assume that condition* (1.2) *is satisfied for some arbitrary, but given increasing sequence $(N_k)_k$ of positive integers. Then there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega\backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, if we define*

$$\forall \omega \in \Omega,\ \forall x \in X,\ \forall n \ge 1, \qquad S_n(\omega, x) = \sum_{k=1}^{n} W_k(\omega) T^{p_k} f(x), \tag{3.1}$$

the sequence $(S_{N_k}(\omega, \cdot))_{k\ge1}$ converges μ-almost surely.

If, in addition to condition (1.2)*, we assume*

$$\sum_k \min\left[\log^2(N_{k+1} - N_k) \log p_{N_{k+1}} \left(\sum_{j=N_k+1}^{N_{k+1}} \mathbf{E}(W_j^2)\right), \right.$$
$$\left. \mathbf{E}\left(\sum_{j=N_k+1}^{N_{k+1}} |W_j|\right)^2\right] < \infty, \tag{3.2}$$

then there exists a (universal) $\mathbf{P}$*-null set* $N^* \in \mathcal{B}$ *such that for each* $\omega \in \Omega\backslash N^*$*, for any probability space* $(X, \mathcal{F}, \mu)$*, any contraction* T *on* $L^2(\mu)$*, and any* $f \in L^2(\mu)$*, the sequence* $(S_n)_{n\geq 1}$ *converges* μ*-almost surely.*

Proof. Let $f \in L^2(\mu)$ and assume that $\|f\|_{2,\mu} = 1$. Define

$$\forall \omega \in \Omega,\ \forall k \geq 1, \qquad \psi_k(\omega) = \sum_{j=N_k+1}^{N_{k+1}} W_j(\omega) T^{p_j} f.$$

Then, by Lemmas 2.1 and 2.2, there exists a $\mathbf{P}$-integrable random variable C such that

$$\forall \omega \in \Omega,\ \forall k \geq 1, \quad \|\psi_k(\omega)\|_{2,\mu} \leq C(\omega) \left(\log p_{N_{k+1}} \sum_{N_k+1}^{N_{k+1}} W_j^2(\omega) \right)^{1/2}.$$

Since

$$S_{N_m} = \sum_{j=1}^{N_m} W_j(\omega) T^{p_j} f = \sum_{k=0}^{m-1} \psi_k(\omega)$$

and

$$\sum_{k=0}^{\infty} \|\psi_k(\omega)\|_{1,\mu} \leq \sum_{k=0}^{\infty} \|\psi_k\|_{2,\mu} \leq C(\omega) \sum_{k=0}^{\infty} \left(\log p_{N_{k+1}} \sum_{N_k+1}^{N_{k+1}} W_j^2(\omega) \right)^{1/2},$$

we deduce from assumption (1.2) that there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that, if $\omega \in \Omega\backslash N^*$, then

$$\sum_{k=0}^{\infty} \|\psi_k(\omega)\|_{1,\mu} < \infty.$$

Consequently, the series $\sum_{k=0}^{m} |\psi_k(\omega)|$ converges μ-almost surely for each $\omega \in \Omega\backslash N^*$. Therefore, for each $\omega \in \Omega\backslash N^*$, the sequence of partial sums

$$\sum_{j=1}^{N_{k+1}} W_j(\omega) T^{p_j} f(x), \qquad k = 1, 2, \ldots,$$

converges μ-almost surely. The general case follows from this by considering, for an arbitrary function $g \in L^2(\mu)$ with $g \neq 0$, the function $f = g/\|g\|_{2,\mu}$. This is the first assertion of Theorem 3.1.

To prove the second assertion, we proceed as before, by considering $f \in L^2(\mu)$ with $\|f\|_{2,\mu} = 1$. Set $I_k =]N_k, N_{k+1}]$, $k = 1, 2, \ldots$. Let $s, t \in I_k$ with $s < t$ and estimate the increments $\|S_t - S_s\|$. By Lemmas 2.1 and 2.2, there exists a $\mathbf{P}$-integrable random variable C such that

$$\forall \omega \in \Omega,\ \forall k \geq 1,\ \forall s, t \in I_k, \qquad \|S_t - S_s\|_{2,\mu}^2 \leq C(\omega) \log p_{N_{k+1}} \sum_{j=s+1}^{t} W_j^2(\omega).$$

Since T is a contraction, we also have

$$\|S_t - S_s\|_{2,\mu} \le \sum_{j=s+1}^{t} |W_j(\omega)|.$$

Replacing, if necessary, C by $\max(C,1)$, we deduce

$$\begin{aligned}
&\forall \omega \in \Omega,\ \forall k \ge 1,\ \forall s,t \in I_k,\\
&\qquad d(s,t) := \|S_t - S_s\|_{2,\mu}\\
&\qquad\qquad \le C(\omega) \min\left[\left(\log p_{N_{k+1}} \sum_{j=s+1}^{t} W_j^2(\omega)\right)^{1/2}, \sum_{j=s+1}^{t} |W_j(\omega)|\right].
\end{aligned}$$

We will control the oscillation $\max_{s,t\in I_k} |S_t - S_s|$ by means of Theorem 2.4. In our case, the set E will be the interval of integers I_k (where, here and in what follows, we fix k), endowed with the pseudo-metric d.

We next consider the entropy number $N(E,d,\epsilon)$. Put

$$A = \left(\log p_{N_{k+1}} \sum_{j=N_k+1}^{N_{k+1}} W_j^2(\omega)\right)^{1/2}, \qquad B = \sum_{j=N_k+1}^{N_{k+1}} |W_j(\omega)|.$$

Then $\max_{s,t\in E} d(s,t) \le \min(A,B)$. For $0 < \varepsilon \le \min(A,B)$, we now estimate the entropy number $N(E,d,\varepsilon)$. Define for each integer $l \ge 0$

$$S_1^l = \left\{ t : l\varepsilon^2 C(\omega)^{-2} \le \log p_{N_{k+1}} \sum_{j=N_k}^{t} W_j^2(\omega) \le C(\omega)^{-2}(l+1)\varepsilon^2 \right\}.$$

Then $s,t \in S_1^l$ implies $d(s,t) \le \varepsilon$. Moreover,

$$\bigcup_{l=0}^{L} S_1^l = \left\{ t : \log p_{N_{k+1}} \sum_{j=N_k}^{t} W_j^2(\omega) \le C(\omega)^{-2}(L+1)\varepsilon^2 \right\}.$$

Let $L = L_1$ be the smallest integer such that

$$C(\omega)^{-2}(L_1+1)\varepsilon^2 \ge \log p_{N_{k+1}} \sum_{j=N_k}^{N_{k+1}} W_j^2(\omega).$$

Then $N(E,d,\varepsilon) \le L_1 + 1$. Similarly, define

$$S_2^l = \left\{ t : l\varepsilon C(\omega)^{-1} \le \sum_{j=N_k}^{t} |W_j| \le C(\omega)^{-1}(l+1)\varepsilon \right\}.$$

Then $s, t \in S_2^l$ implies $d(s,t) \le \varepsilon$, and

$$\bigcup_{l=0}^{L} S_2^l = \left\{ t : \sum_{j=N_k}^{t} |W_j| \le C(\omega)^{-1}(L+1)\varepsilon \right\}.$$

Let $L = L_2$ be the smallest integer such that

$$C(\omega)^{-1}(L_2+1)\varepsilon \ge \sum_{j=N_k}^{N_{k+1}} |W_j|.$$

Then $N(E,d,\varepsilon) \le L_2 + 1$. We have thus

$$N(E,d,\varepsilon) \le K \min\left[\frac{A^2 C(\omega)^2}{\varepsilon^2}, \frac{BC(\omega)}{\varepsilon}, N_{k+1} - N_k\right],$$

where K is an absolute constant. Replacing, if necessary, C by $\max(C,1)$, we estimate $I := \int_0^{\min(A,B)} N(E,d,\varepsilon)^{1/2}\, d\varepsilon$ by means of the crude inequality

$$I \le K \min\Bigg(C \int_0^A \min\left[\frac{A^2}{\varepsilon^2}, N_{k+1} - N_k\right]^{1/2} d\varepsilon, \qquad \sqrt{C}\int_0^B \min\left[\frac{B}{\varepsilon}, N_{k+1} - N_k\right]^{1/2} d\varepsilon \Bigg).$$

On the one hand, we have

$$\begin{aligned}
&\int_0^A \min\left[\frac{A^2}{\varepsilon^2}, N_{k+1} - N_k\right]^{1/2} d\varepsilon \\
&\quad = \left(\int_0^{\frac{A}{\sqrt{N_{k+1}-N_k}}} + \int_{\frac{A}{\sqrt{N_{k+1}-N_k}}}^A \right) \min\left[\frac{A^2}{\varepsilon^2}, N_{k+1} - N_k\right]^{1/2} d\varepsilon \\
&\quad \le A + A\int_{\frac{A}{\sqrt{N_{k+1}-N_k}}}^A \frac{d\varepsilon}{\varepsilon} = A + \frac{A}{2}\log[N_{k+1} - N_k].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^B \min\left[\frac{B}{\varepsilon}, N_{k+1} - N_k\right]^{1/2} d\varepsilon \\
&\quad = \left(\int_0^{\frac{B}{N_{k+1}-N_k}} + \int_{\frac{B}{N_{k+1}-N_k}}^B \right) \min\left[\frac{B}{\varepsilon}, N_{k+1} - N_k\right]^{1/2} d\varepsilon \\
&\quad \le \frac{B}{\sqrt{N_{k+1}-N_k}} + 2\sqrt{B}\left[\sqrt{B} - \sqrt{BN_{k+1} - N_k}\right] \le 2B.
\end{aligned}$$

Therefore

$$\begin{aligned}I &\le C(\omega)\min\left(A\log[N_{k+1}-N_k], B\right)\\ &= C(\omega)\min\left(\log[N_{k+1}-N_k]\left(\log p_{N_{k+1}}\sum_{j=N_k+1}^{N_{k+1}} W_j^2(\omega)\right)^{1/2},\right.\\ &\qquad\qquad\left.\sum_{j=N_k+1}^{N_{k+1}}|W_j|(\omega)\right).\end{aligned}$$

We deduce from these computations and Theorem 2.4

$$\begin{aligned}&\left\|\max_{s,t\in I_k}|S_t-S_s|\right\|_{2,\mu}^2\\ &\qquad\le C(\omega)\min\left[\log^2\left(N_{k+1}-N_k\right)\log p_{N_{k+1}}\sum_{j=N_k+1}^{N_{k+1}} W_j^2(\omega),\right.\\ &\qquad\qquad\left.\left(\sum_{j=N_k+1}^{N_{k+1}}|W_j(\omega)|\right)^2\right].\end{aligned}$$

Put, for $k=1,2,\dots,$

$$\begin{aligned}X_k = \min&\left[\log(N_{k+1}-N_k)\left(\log p_{N_{k+1}}\sum_{j=N_k+1}^{N_{k+1}} W_j^2(\omega)\right),\right.\\ &\qquad\left.\left(\sum_{j=N_k+1}^{N_{k+1}}|W_j(\omega)|\right)^2\right].\end{aligned}$$

Now note that the variables X_k form a sequence of non-negative, independent random variables, so if Condition (3.2) is fulfilled, then there exists a $\mathbf{P}$-null set $N^*\in\mathcal{B}$ such that, for each $\omega\in\Omega\backslash N^*$,

$$\sum_{k=1}^{\infty}\left\|\max_{s,t\in I_k}|S_t-S_s|\right\|_{2,\mu}^2 \le \sum_{k=1}^{\infty}X_k<\infty.$$

Combining this result with the one obtained in the first step of the proof easily leads to the conclusion. □

Remark 3.2. Suppose that Conditions (1.2) and (3.2) are fulfilled for given sequences of integers $(N_k)_{k\ge1}$, $(p_k)_{k\ge1}$, and a sequence of independent, symmetric, square integrable random variables $(W_k)_{k\geq 1}$, defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$. Consider a sequence of contractions $(T_k)_{k\geq 1}$ defined on $L^2(X, \mathcal{F}, \mu)$. Then the conclusions of Theorem 3.1 are also true for the series

$$S_n^{'}(f) = \sum_{j=1}^{n} W_j(\omega) U_j^{p_j}(f) \quad \text{for } n \geq 1,\ \omega \in \Omega,\ f \in L^2(\mu),$$

where $U_j = T_k$ if $N_k < j \leq N_{k+1}$, $k \geq 1$. To see this, observe that for all $k \geq 1$, $s, t \in I_k$ with $s < t$, and $f \in L^2(\mu)$ with $\|f\|_{2,\mu} \leq 1$, one has

$$\|S_t^{'} - S_s^{'}\|_{2,\mu}^2 \leq C(\omega) \log p_{N_{k+1}} \sum_{j=N_k+1}^{N_{k+1}} W_j^2(\omega)$$

and

$$\|S_t^{'} - S_s^{'}\|_{2,\mu} \leq \sum_{j=N_k+1}^{N_{k+1}} |W_j(\omega)|.$$

Hence, the assertion of Theorem 3.1 remains true.

3.2. We now apply Theorem 3.1 to improve some previous results on randomly weighted means of contractions (see [A], [SW], and [R]).

Corollary 3.3. *Let $(Z_k)_{k\in\mathbf{N}}$ be a sequence of independent, symmetric, square integrable, identically distributed random variables, defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$. Let $\alpha > 1/2$, $\beta > 2$. Then there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega \backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, the sequences $(S_n(\omega, \cdot))_{n\geq 1}$, $(R_n(\omega, \cdot))_{n\geq 1}$ defined by*

$$\forall x \in X,\ \forall n \geq 1, \qquad S_n(\omega, x) = \sum_{k=1}^{n} \frac{Z_k(\omega)}{k^{\alpha}} T^k f(x) \tag{3.3}$$

and

$$\forall x \in X,\ \forall n \geq 1, \qquad R_n(\omega, x) = \sum_{k=1}^{n} \frac{Z_k(\omega)}{\sqrt{k} \log^{\beta} k} T^k f(x) \tag{3.4}$$

converge μ-almost surely.

Proof. Set $N_k = 2^k$ for all $k \in \mathbf{N}$. By Theorem 3.1 it is enough to verify Conditions (1.2) and (3.2).

If $\alpha > 1/2$, then for the series (3.3), Condition (1.2) becomes

$$\sum_{i=0}^{\infty} \sqrt{\log(2^{i+1})} \left[\sum_{k=2^i+1}^{2^{i+1}} \frac{\mathbf{E}(|Z_k|^2)}{k^{2\alpha}} \right]^{1/2} \leq K(\mathbf{E}|Z_1|^2)^{1/2} \sum_{i=0}^{\infty} \frac{\sqrt{i+1}}{2^{(\alpha-1/2)i}} < \infty,$$

while Condition (3.2) becomes

$$\sum_{k=1}^{\infty} \min \left[\log^2 (2^{k+1} - 2^k) \log 2^{k+1} \left(\sum_{j=2^k+1}^{2^{k+1}} \frac{\mathbf{E}\,|Z_j|^2}{j^{2\alpha}} \right), \right.$$
$$\left. \mathbf{E} \left(\sum_{j=2^k+1}^{2^{k+1}} \frac{|Z_j|}{j^{\alpha}} \right)^2 \right] \leq K\mathbf{E}(|Z_1|^2) \sum_{k=1}^{\infty} \frac{k^3}{2^{(2\alpha-1)k}} < \infty.$$

If $\beta > 2$, then for the series (3.4), Condition (1.2) becomes

$$\sum_{i=0}^{\infty} \sqrt{\log(2^{i+1})} \left[\sum_{k=2^i+1}^{2^{i+1}} \frac{\mathbf{E}(|Z_k|^2)}{k \log^{2\beta} k} \right]^{1/2} \leq K(\mathbf{E}|Z_1|^2)^{1/2} \sum_{i=1}^{\infty} \frac{1}{i^{(\beta-1/2)}} < \infty,$$

while Condition (3.2) becomes

$$\sum_{k=1}^{\infty} \min \left[\log^2 (2^{k+1} - 2^k) \log 2^{k+1} \left(\sum_{j=2^k+1}^{2^{k+1}} \frac{\mathbf{E}(|Z_j|^2)}{j \log^{2\beta} j} \right), \right.$$
$$\left. \mathbf{E} \left(\sum_{j=2^k+1}^{2^{k+1}} \frac{|Z_j|}{j \log^{2\beta} j} \right)^2 \right] \leq K\mathbf{E}(|Z_1|^2) \sum_{k=1}^{\infty} \frac{1}{k^{2\beta-3}} < \infty.$$

Hence, Conditions (1.2) and (3.2) are fulfilled for the series (3.3) and (3.4). This completes the proof of Corollary 3.3. □

REMARK 3.4. If $\alpha \leq 1/2$ and $\mathbf{P}\{|Z_1| > 0\} > 0$, then the series (3.3) does not converge. To see this, it is enough to take $T = I$ and $\alpha = 1/2$ in (3.3). Then we have

$$\forall \omega \in \Omega,\ \forall x \in X, \qquad \sum_{k=1}^{\infty} \frac{Z_k(\omega)}{\sqrt{k}} T^k f(x) = f(x) \sum_{k=1}^{\infty} \frac{Z_k(\omega)}{\sqrt{k}}.$$

But by the 0–1 law and the Central Limit Theorem, the series on the right-hand side diverges almost surely. The case $\alpha < 1/2$ is treated in exactly the same manner, completing the proof of our claim. Corollary 3.3 strengthens a previous result of J. Rosenblatt [R], who obtained this result with a factor n instead of $\sqrt{n}\log^{\beta} n$ and the Rademacher sequence instead of general sequence of independent, symmetric, identically distributed random variables.

By combining Corollary 3.3 with Kronecker's Lemma, we get:

THEOREM 3.5. *If $(Z_k)_{k\in\mathbf{N}}$ is a sequence of independent, symmetric, square integrable, identically distributed random variables on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$ and if $\beta > 2$, then there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$*

such that for each $\omega \in \Omega \backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, the sequence

$$A_n(\omega, x) = \frac{1}{\sqrt{n}\log^\beta n} \sum_{k=1}^n Z_k(\omega) T^k f(x), \qquad x \in X, n \geq 1 \tag{3.5}$$

converges to zero μ-almost surely.

REMARK 3.6. The almost sure convergence of the weighted means

$$\frac{1}{n} \sum_{k=1}^n Z_k(\omega) T^k f \tag{3.6}$$

has been studied in [A], [R], and [SW]. Indeed, in [A] the almost sure convergence to zero of these means is established when $(Z_k)_{k\in\mathbf{N}}$ is an i.i.d sequence of symmetric random variables such that $E(|Z_1|^p) < \infty$ for some $1 < p < \infty$, and T is the transformation induced by a measure preserving transformation, whereas in [R] these means are studied when T is a contraction on $L^p(\mu)$, $1 < p < \infty$, and $(Z_k)_{k\in\mathbf{N}}$ is a Rademacher sequence. In [SW] a Gaussian randomization technique is used to prove the almost sure convergence of the means (3.6), and a similar result is obtained when the sequence $(Z_k)_{k\geq 1}$ is positive.

3.3. Now let $(Z_k)_{k\in\mathbf{N}}$ be as in Theorem 3.4, let $(X, \mathcal{F}, \mu)$ be a probability space, and T a contraction on $L^1(\mu)$, which is also assumed to be a contraction on any $L^p(\mu)$, $p \geq 1$. Consider the series

$$\forall \omega \in \Omega,\ \forall x \in X,\ \forall n \geq 1, \qquad S_n(\omega, x) = \sum_{k=1}^n \frac{Z_k(\omega)}{k} T^k f(x). \tag{3.7}$$

By using the above results and a complex interpolation method, we will prove the almost sure convergence of this series, for all $f \in L^p(\mu)$, $p > 1$.

THEOREM 3.7. *Let $(Z_k)_{k\in\mathbf{N}}$ be a sequence of independent, symmetric, square integrable, identically distributed random variables on some probability space $(\Omega, \mathcal{B}, \mathbf{P})$. Then there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$, such that for each $\omega \in \Omega \backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^1(\mu)$ which is also a contraction on every $L^p(\mu)$, and for any $f \in L^p(\mu)$, $p > 1$, the series $S_n(\omega, \cdot)$ defined in (3.7) converges μ-almost surely. Furthermore, if we define*

$$\forall \omega \in \Omega \backslash N^*,\ \forall p > 1,\ \forall f \in L^p(\mu), \qquad S^*(f) = \sup_{n\geq 1} \left| \sum_{k=1}^n \frac{Z_k(\omega)}{k} T^k f \right|, \tag{3.8}$$

then we have the strong maximal inequality

$$\forall \omega \in \Omega \backslash N^*,\ \forall p > 1,\ \forall f \in L^p(\mu), \qquad \|S^*(f)\|_p \leq C(p, \omega)\|f\|_p. \tag{3.9}$$

Proof. Let $\alpha > 1/2$, $z \in \mathcal{C}$ with $0 \le \Re(z) \le 1$, and $N_j = 2^j$ for $j = 1, 2, \ldots$. Let $\nu\colon X \to \mathbf{N}^*$ be a measurable map and define for $\omega \in \Omega$ and $p \ge 1$ the following operators on $L^p(\mu)$:

$$\forall f \in L^p(\mu), \qquad S_z^\nu(f) = \sum_{j=1}^{\nu} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^k(f) \tag{3.10}$$

and

$$S_z^*(f) = \sup_{\nu \ge 1} |S_z^\nu(f)|. \tag{3.11}$$

We first establish a useful estimate for $\|S_z^\nu(f)\|_2$ when $f \in L^2(\mu)$. Let $x \in X$. Then there exists a positive integer $k_0 = k_0(\nu)$, such that $2^{k_0} < \nu(x) \le 2^{k_0+1}$. Thus

$$\begin{aligned}
|S_z^\nu(f)|(x) &= \left|\sum_{j=1}^{\nu(x)} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f(x)\right| \\
&\le \left|\sum_{j=1}^{2^{k_0}} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f(x)\right| + \max_{2^{k_0} < n \le 2^{k_0+1}} \left|\sum_{j=2^{k_0}+1}^{n} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f(x)\right| \\
&\le |Z_1(\omega) T(f)(x)| + \sum_{k=0}^{\infty} \left|\sum_{j=2^k+1}^{2^{k+1}} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f(x)\right| \\
&\qquad + \sum_{k=0}^{\infty} \max_{2^k < n \le 2^{k+1}} \left|\sum_{j=2^k+1}^{n} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f(x)\right| \\
&\le |Z_1(\omega) T(f)(x)| + 2\sum_{k=0}^{\infty} \max_{2^k < n \le 2^{k+1}} \left|\sum_{j=2^k+1}^{n} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f(x)\right|.
\end{aligned}$$

But, by the second step in the proof of Theorem 3.1, we have

$$\sum_{k=0}^{\infty} \left\| \max_{2^k < n \le 2^{k+1}} \left| \sum_{2^k+1}^{n} \frac{Z_j(\omega)}{j^{(\alpha+z/2)}} T^j f \right| \right\|_2 \le C_2(\omega) \|f\|_2,$$

where

$$C_2(\omega) = |Z_1(\omega)| + C(\omega) \sum_{k=1}^{\infty} \left(k^3 \sum_{j=2^k+1}^{2^{k+1}} \frac{|Z_j(\omega)|^2}{j^{2\alpha}} \right)^{1/2}$$

is a $\mathbf{P}$-almost surely finite random variable. Hence,

$$\forall f \in L^2(\mu), \qquad \|S_z^\nu(f)\|_2 \le C_2(\omega) \|f\|_2. \tag{3.12}$$

Now let f_1, f_2 be two simple functions and consider the map $\Phi\colon z \to \int S_z^\nu(f_1) f_2 \, d\mu$. Then Φ is analytic on $\{z \mid 0 < \Re(z) < 1\}$ and uniformly bounded for $0 \le \Re(z) \le 1$. To see this, it is enough to consider the case when f_1 and f_2 are characteristic functions. Indeed, let $A_1, A_2 \in \mathcal{B}$, and denote by $\mathbf{1}_{A_1}$ and $\mathbf{1}_{A_2}$ their respective characteristic functions. We have

$$|\Phi(z)| = \left| \int_{A_2} \sum_{n=1}^{\nu(x)} \frac{Z_n(\omega)}{n^{(\alpha+z/2)}} T^n(\mathbf{1}_{A_1})(x)\, d\mu(x) \right| \\ \le \sqrt{\mu(A_2)} \left\| \sum_{n=1}^{\nu} \frac{Z_n(\omega)}{n^{(\alpha+z/2)}} T^n(\mathbf{1}_{A_1}) \right\|_2 .$$

Hence, by using inequality (3.12), we obtain

$$|\Phi(z)| \le C(\omega)\sqrt{\mu(A_1)\mu(A_2)}.$$

This proves the uniform boundedness of $\Phi(z)$ when $0 \le \Re(z) \le 1$ and f_1, f_2 are the characteristic functions of measurable sets. By the same method, we obtain the result for the case when f_1, f_2 are simple functions.

Let $f \in L^1(\mu)$, $N \ge 1$, and set $z = 1 + iy, y \in \mathbf{R}$. Then

$$\left\| \sup_{n \le N} \left| \sum_{j=1}^{n} \frac{Z_j(\omega)}{j^{(\alpha+(1+iy)/2)}} T^k(f) \right| \right\|_1 \le \sum_{j=1}^{\infty} \frac{|Z_j(\omega)|}{j^{(\alpha+1/2)}} \|f\|_1.$$

But

$$\mathbf{E}\left(\sum_{j=1}^{\infty} \frac{|Z_j|}{j^{(\alpha+1/2)}} \right) < \infty.$$

Hence

$$\forall f \in L^1(\mu), \qquad \|S^{\nu}_{1+iy}(f)\|_1 \le C_1(\omega)\|f\|_1, \tag{3.13}$$

where

$$C_1(\omega) = \left(\sum_{j=1}^{\infty} \frac{|Z_j(\omega)|}{j^{(\alpha+1/2)}} \right).$$

Further, if we choose $z = iy$, $y \in \mathbf{R}$, in inequality (3.12), we have

$$\forall f \in L^2(\mu), \qquad \|S^{\nu}_{iy}(f)\|_2 \le C_2(\omega)\|f\|_2. \tag{3.14}$$

By inequalities (3.13), (3.14), and the Interpolation Theorem of Stein (see Theorem 1.39 in [Z]), we have

$$\forall\, 1 \le p \le 2,\ \forall f \in L^p(\mu), \qquad \|S^{\nu}_r(f)\|_p \le C_0(p,\omega)\|f\|_p,$$

where $0 \le r \le 1$, and $1/p = (1-r)/2 + r$. But, if $1 < p \le 2$, we can choose $\alpha > 1/2$ such that $\alpha + r/2 = 1$. Hence, for $1 < p \le 2$ and $f \in L^p(\mu)$ we have

$$\left\| \sum_{j=1}^{\nu} \frac{Z_j(\omega)}{j} T^j f \right\|_p \le C_p(\omega)\|f\|_p.$$

Now if we put, for $1 < p \le 2$, $f \in L^p(\mu)$, and $N \ge 1$,

$$\begin{aligned}\nu(x) &= \nu(\omega, f, N, x)\\ &= \min\left\{\nu \le N : \max_{n\le N}\left|\sum_{j=1}^{n}\frac{Z_j(\omega)}{j}T^k(f)(x)\right| = \left|\sum_{j=1}^{\nu}\frac{Z_j(\omega)}{j}T^k(f)(x)\right|\right\},\end{aligned}$$

we have, for $1 < p \le 2$, $f \in L^p(\mu)$, and $N \ge 1$,

$$\left\|\max_{n\le N}\left|\sum_{j=1}^{n}\frac{Z_j(\omega)}{j}T^j f\right|\right\|_p \le C_0(p,\omega)\|f\|_p.$$

But $C_0(p,\omega)$ is independent of N, so taking the limit in the above inequality gives

$$\forall\, 1 < p \le 2,\ \forall f \in L^p(\mu), \qquad \|S^*(f)\|_p \le C_0(p,\omega)\|f\|_p. \tag{3.15}$$

Using the same argument, we get a maximal inequality for $p \ge 2$. Indeed, let $2 \le p \le \infty$, $f \in L^\infty(\mu)$, $N \ge 1$, and set $z = 1 + iy$, $y \in \mathbf{R}$. Then

$$\left\|\sum_{j=1}^{\nu}\frac{Z_j(\omega)}{j^{(\alpha+(1+iy)/2)}}T^k(f)\right\|_\infty \le \left(\sum_{j=1}^{\infty}\frac{|Z_j(\omega)|}{j^{(\alpha+1/2)}}\right)\|f\|_\infty.$$

Thus,

$$\forall f \in L^\infty(\mu), \qquad \|S^\nu_{1+iy}(f)\|_\infty \le C_1(\omega)\ \|f\|_\infty. \tag{3.16}$$

Hence, by (3.14), (3.16), and Stein's Interpolation Theorem, there exists a $\mathbf{P}$-null set $N^* \in \mathcal{B}$, such that, for each $\omega \in \Omega\backslash N^*$ and for $2 \le p \le \infty$,

$$\forall f \in L^p(\mu), \qquad \|S^\nu_r(f)\|_p \le C_3(p,\omega)\|f\|_p,$$

where $0 \le r \le 1$, $2 \le p \le \infty$, and $1/p = (1-r)/2$. But if $2 \le p < \infty$, we can choose α such that $\alpha + r/2 = 1$. Hence, as above we obtain

$$\forall\, 2 < p < \infty,\ \forall f \in L^p(\mu), \qquad \|S^*(f)\|_p \le C_3(p,\omega)\|f\|_p.$$

Consequently, for $1 < p < \infty$ there exists a $\mathbf{P}$-null set $N^* \in \mathcal{B}$, and a $\mathbf{P}$-almost surely finite random variable $C(p,\omega)$, such that, for each $\omega \in \Omega\backslash N^*$,

$$\forall f \in L^p(\mu), \qquad \|S^*(f)\|_p \le C(p,\omega)\|f\|_p. \qquad \square$$

Remark 3.8. Inequality (3.15) and the fact that the series (3.7) converges μ-almost surely when $f \in L^2$, imply that the series (3.7) converges μ-almost surely, for any $f \in L^p(\mu)$, $1 < p < \infty$. This observation implies, by Kronecker's Lemma,

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}Z_j(\omega)T^j f = 0$$

μ-almost surely, for each $\omega \in \Omega\backslash N^*$ and $f \in L^p(\mu)$, $p > 1$.

3.4. We now focus on a series of operators perturbed by random variables. By invoking Lemma 2.3 this time, we will establish conditions for the convergence μ-almost everywhere of these series and show how to deduce a result of Schneider (Theorem 1.1. in [S]).

THEOREM 3.9. *Let $(a_k)_{k\geq 1}$ be a sequence of real numbers, $(p_k)_{k\in\mathbf{N}}$ a sequence of positive integers, and $(\theta_k)_{k\geq 1}$ a sequence of independent, identically distributed, $\mathbf{N}$-valued random variables with positive moments, defined on a probability space $(\Omega, \mathcal{B}, \boldsymbol{P})$. Assume that there exist integers $0 := N_0 < N_1 < N_2 < \cdots$ such that the following conditions are satisfied:*

$$\sum_{i=0}^{\infty} \sqrt{\log p_{N_{i+1}}} \left(\sum_{k=N_i+1}^{N_{i+1}} a_k^2 \right)^{1/2} < \infty \tag{3.17}$$

and

$$\sum_k \min \left[\log^2 (N_{k+1} - N_k) \log p_{N_{k+1}} \left(\sum_{j=N_k+1}^{N_{k+1}} a_j^2 \right), \left(\sum_{j=N_k+1}^{N_{k+1}} a_j \right)^2 \right] < \infty. \tag{3.18}$$

Then there exists a (universal) $\boldsymbol{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega \backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, the sequence $(R_n(\omega, \cdot))_{n\geq 1}$ defined by

$$\forall \omega \in \Omega,\ \forall x \in X,\ \forall n \geq 1, \quad R_n(\omega, x) = \sum_{k=1}^{n} a_k \left[T^{p_k + \theta_k(\omega)}(f) - \mathbf{E}\left(T^{p_k+\theta_1}(f)\right)\right] \tag{3.19}$$

converges μ-almost surely.

Proof. Let $f \in L^2(\mu)$ and assume that $\|f\|_{2,\mu} = 1$. Define

$$\forall \omega \in \Omega,\ \forall k \geq 1, \qquad \psi_k(\omega) = \sum_{j=N_k+1}^{N_{k+1}} a_j \left[T^{p_j+\theta_j(\omega)}(f) - \mathbf{E}\left(T^{p_j+\theta_1}(f)\right)\right].$$

By Lemma 2.3 in [S], we have

$$A = \mathbf{E}\left(\sup_{m\geq 1} \sqrt{\frac{\log (p_m + \theta_1)}{\log p_m}} \right) < \infty.$$

By means of the spectral inequality we may write

$$\begin{aligned}\|\psi_k(\omega)\|_{2,\mu} &\le \sup_{0\le\lambda\le1}\left|\sum_{j=N_k+1}^{N_{k+1}} a_j\left[e^{2i\pi\lambda(p_j+\theta_j(\omega))}-\mathbf{E}\left(e^{2i\pi\lambda(p_j+\theta_1)}\right)\right]\right|\\ &\le \sup_{N<M}\sup_{0\le\lambda\le1}\frac{\left|\sum_{j=N}^{M} a_j\left[e^{2i\pi\lambda(p_j+\theta_j(\omega))}-\mathbf{E}\left(e^{2i\pi\lambda(p_j+\theta_1)}\right)\right]\right|}{\left\{(\log p_M)\sum_{j=N}^{M} a_j^2\right\}}\\ &\qquad\qquad\times\left\{(\log p_{N_{k+1}})\sum_{j=N_k+1}^{N_{k+1}} a_j^2\right\}^{1/2}.\end{aligned}$$

By Lemma 2.3 we have

$$\mathbf{E}\left(\sup_{N<M}\sup_{0\le\lambda\le1}\frac{\left|\sum_{j=N+1}^{M} a_j\left[e^{2i\pi\lambda(p_j+\theta_j)}-\mathbf{E}\left(e^{2i\pi\lambda(p_j+\theta_1)}\right)\right]\right|}{\left\{(\log p_M)\sum_{j=N}^{M} a_j^2\right\}^{1/2}}\right)\le CA.$$

There thus exists a $\mathbf{P}$-almost surely finite random variable $C(\omega)$, such that

$$\|\psi_k(\omega)\|_{2,\mu}\le C(\omega)\left\{\log p_{N_{k+1}}\sum_{j=N_k+1}^{N_{k+1}} a_j^2\right\}^{1/2}.$$

Since

$$R_{N_m}=\sum_{j=1}^{N_m} a_j\left[T^{p_j+\theta_j(\omega)}(f)-\mathbf{E}\left(T^{p_j+\theta_1}(f)\right)\right]=\sum_{k=0}^{m-1}\psi_k(\omega)$$

and

$$\sum_{k=0}^{\infty}\|\psi_k(\omega)\|_{1,\mu}\le\sum_{k=0}^{\infty}\|\psi_k\|_{2,\mu}\le C(\omega)\sum_{k=0}^{\infty}\left(\log p_{N_{k+1}}\sum_{N_k+1}^{N_{k+1}} a_j^2\right)^{1/2},$$

there exists a (universal) $\mathbf{P}$-null set $N^*\in\mathcal{B}$, such that, for each $\omega\in\Omega\backslash N^*$,

$$\sum_{k=0}^{\infty}\|\psi_k(\omega)\|_{1,\mu}<\infty.$$

Consequently, the series $\sum_{k=0}^{m}|\psi_k(\omega)|$, converges μ-almost surely for each $\omega\in\Omega\backslash N^*$. Therefore, for each $\omega\in\Omega\backslash N^*$, the sequence of partial sums

$$\sum_{j=1}^{N_{k+1}} a_j\left[T^{p_j+\theta_j(\omega)}(f)-\mathbf{E}\left(T^{p_j+\theta_1}(f)\right)\right],\quad k=1,2,\dots,$$

converges μ-almost surely. The general case follows from this by considering, for an arbitrary function $g\in L^2(\mu)$ with $g\ne0$, the function $f=g/\|g\|_{2,\mu}$. As in the second half of the proof of Theorem 3.1, and by means of Theorem 2.4, we control the oscillations of the sequence R_n on the interval $I_k =]N_k, N_{k+1}]$ by using the following inequalities, which result from the Spectral Lemma and Lemma 2.3:

$$\|R_t - R_s\|_{2,\mu}^2 \leq C(\omega) \log p_{N_{k+1}} \sum_{j=s+1}^{t} a_j^2$$

and

$$\|R_t - R_s\|_{2,\mu} \leq 2 \sum_{j=s+1}^{t} |a_j|,$$

for each $\omega \in \Omega$, $k \geq 1$ and $s, t \in I_k$. Hence we obtain

$$\sum_{k=1}^{\infty} \left\| \max_{s,t \in I_k} |R_t - R_s| \right\|_{2,\mu}^2$$
$$\leq C(\omega) \sum_k \min \left[\log^2 (N_{k+1} - N_k) \log p_{N_{k+1}} \left(\sum_{j=N_k+1}^{N_{k+1}} a_j^2 \right), \left(\sum_{j=N_k+1}^{N_{k+1}} a_j \right)^2 \right] < \infty.$$

This completes the proof of Theorem 3.9. □

Now take $a_k = 1/k$ and $N_k = 2^k$ in Theorem 3.9 and suppose that the sequence $(p_k)_{k \in \mathbf{N}}$ satisfies

(a1) $$p_k = O(2^{k^\delta}) \quad \text{for some } \delta \in]0, 1[.$$

Define also a contraction Δ on $L^2(\mu)$ by

(a2) $$\Delta \colon f \in L^2(\mu) \to \Delta(f) = \mathbf{E}\left(T^{\theta_1}(f)\right).$$

In this case, Conditions (3.17) and (3.18) are fulfilled, so there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega \backslash N^*$ the series

$$\sum_{k=1}^{n} \frac{1}{k} \left[T^{p_k + \theta_k(\omega)}(f) - T^{p_k}(\Delta(f)) \right], \quad n = 1, 2, \ldots,$$

converges μ-almost surely. This gives immediately the following result:

Corollary 3.10. *Let $(p_k)_{k \in \mathbf{N}}$ be a sequence of positive integers which satisfies* (a1) *and let $(\theta_k)_{k \geq 1}$ be a sequence of independent, identically distributed, $\mathbf{N}$-valued random variables with positive moments, defined on a probability space $(\Omega, \mathcal{B}, \boldsymbol{P})$. There exists a (universal) $\boldsymbol{P}$-null set $N^* \in \mathcal{B}$ such that* *for each $\omega \in \Omega \backslash N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, and any $f \in L^2(\mu)$, if Δ is the contraction defined in* (a2)*, one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T^{p_k+\theta_k(\omega)}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T^{p_k}(\Delta(f)), \quad \mu\text{-almost surely.}$$

Corollary 3.10 states that if $(p_k)_{k\in\mathbf{N}}$ is a sequence of positive integers which satisfies (a1) and which is good for the pointwise ergodic theorem, i.e., $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} T^{p_k}(f)$ exists μ-almost surely, there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega\backslash N^*$, the perturbed sequence $\{p_k + \theta_k(\omega),\ k \in \mathbf{N}\}$ is also good for the pointwise ergodic theorem. Thus we have:

(1) If T is the operator induced by a measure preserving transformation and $(p_k)_{k\in\mathbf{N}}$ is the sequence $\{k^d, \quad k \geq 1\}$, $d \geq 1$, or the sequence of prime numbers, then the perturbed sequence $\{p_k + \theta_k(\omega), \quad k \in \mathbf{N}\}$ is good for the pointwise ergodic theorem. Furthermore, if $d = 1$ (resp. $d \geq 2$) and T is ergodic (resp. T^n is ergodic for each $n \in \mathbf{N}$), then for any $\omega \in \Omega\backslash N^*$ one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T^{p_k+\theta_k(\omega)}(f) = \int_X \Delta(f)\, d\mu = \int_X f\, d\mu$$

μ-almost surely. Hence, in this case the limit is independent of the sequence $(\theta_k)_{k\geq 1}$.

(2) If T is a positive contraction, then the sequence $\{k + \theta_k(\omega), \quad k \in \mathbf{N}\}$ is also good for the pointwise ergodic theorem.

On the other hand, we can deduce from Corollary 3.10 that if $(p_k)_{k\in\mathbf{N}}$ is a sequence of positive integers which satisfies (a1) and which is bad for the ergodic theorem (i.e., there exist an $f \in L^2(\mu)$ and $X_f \in \mathcal{F}$ with $\mu(X_f) > 0$ such that, for each $x \in X_f$, $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} T^{p_k}(f)(x)$ does not exist), then there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega\backslash N^*$ the sequence $\{p_k+\theta_k(\omega),\ k \in \mathbf{N}\}$ is also bad for the pointwise ergodic theorem. This was observed in [S], where Corollary 3.10 was proved using Gaussian randomization techniques.

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