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THE POSITIVSTELLENSATZ FOR DEFINABLE FUNCTIONS ON O-MINIMAL STRUCTURES

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ABSTRACT. In this note we prove two Positivstellensätze for definable functions of class C^r, $0 \le r < \infty$, in any o-minimal structure $\mathcal{S}$ expanding a real closed field R. Namely, we characterize the definable functions that are nonnegative (resp. strictly positive) on basic definable sets of the form $F = \{f_1 \ge 0, \dots, f_k \ge 0\}$.

1. Introduction

A classic Positivstellensatz, proved by Stengle [St], states that a polynomial g is nonnegative over the semialgebraic set $\{f_1 \ge 0, \dots, f_k \ge 0\}$ if and only if it verifies an equation of the form $p_1 g = p_2 + g^{2m}$, where p_1, p_2 are polynomials in the positive cone (or preordering) generated by $f_1, \dots, f_k$ and the sums of squares. Positivstellensätze have since then been the object of further studies, giving rise to various formulations in different contexts, as analytic germs, analytic functions on compact manifolds, etc., which in general can be seen as the geometrical counterpart of the "abstract" version in terms of the real spectra, as given in [BCR].

A significant variant is what is now called Schmüdgen's theorem, which states that if the semialgebraic set F is compact and g is strictly positive on F, then no denominators are needed; that is, p_1 can be chosen as 1 in the above expression. Moreover, Putinar [Pu, Lemma 4.1] showed that under some more restrictive hypotheses g has a representation of the form $g = v_0 + \sum_{i=1}^{k} v_i f_i$, where the v_i's are sums of squares; that is, g belongs to the module generated by the f_j's over the sums of squares, rather than the positive cone generated by them. In other words, the products of the f_j's are no longer needed.

Recently the authors showed that Schmüdgen's theorem (in fact, Putinar's result) holds also in the analytic setting without any compactness assumption; see [AAB]. In Section 2 of this paper we establish the Positivstellensätze for

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C^r functions on a differentiable manifold M, for $0 \leq r \leq \infty$. In Section 3 we show that the same results hold for definable functions of class C^r, $0 \leq r < \infty$, in any o-minimal structure $\mathcal{S}$ expanding a real closed field R. Notice that the case $r = \infty$ does not have a good behaviour in this setting and therefore is not allowed. In both cases we prove that a function g that is nonnegative on $F = \{f_1 \geq 0, \ldots, f_k \geq 0\}$ verifies an equation $p^2 g = v_0^2 + \sum v_i^2 f_i$ with $\{p = 0\} \subset \{g = 0\}$, and that if g is strictly positive over F, then we have an equation $g = v_0^2 + \sum v_i^2 f_i$. This shows, in particular, that Putinar's theorem still holds in this setting.

Finally we remark that denominators are necessary. More precisely, we prove that for any F with non-empty interior we can find definable functions that are nonnegative over F and do not belong to the precone generated by $f_1, \ldots, f_k$; i.e., the denominator p in the equation above cannot be omitted.

2. Differentiable Positivstellensatz

Let M be any differentiable manifold of class C^r, $0 \leq r \leq \infty$. We start by showing that Hilbert's 17th Problem, that is, the characterization of nonnegative functions over M, has a simple solution in this setting. In fact, we have:

Proposition 2.1. *Let $\varphi : M \longrightarrow \mathbb{R}$ be a nonnegative C^r function, $0 \leq r \leq \infty$. Then φ is the square of a quotient of C^r functions.*

Proof. First observe that the function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\xi(x) = \begin{cases} \sqrt{x}\exp(-1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is C^∞ and flat at 0. Now, given $\varphi : M \longrightarrow \mathbb{R}^+$, the composition $u = \xi \circ \varphi = \sqrt{\varphi}\exp(-1/\varphi)$ is C^r and we have $(\exp(-1/\varphi))^2 \varphi = u^2$. □

Remark 2.2. It is not true that any C^r nonnegative function is a square; see [BCR]. In the last section of this paper we give a specific example for r large enough.

To continue further we need the following elementary result:

Lemma 2.3. *Let f, φ be continuous functions on M and assume that φ is strictly positive on a closed set $\Omega \subset M$. Then there is a continuous function ε, strictly positive on M, such that $\varepsilon f < \varphi$ on Ω.*

Proof. Define $\varepsilon(x) = \varphi(x)/(1 + f(x))$ over the open set $U_0 = \{\varphi > 0\} \cap \{f \geq -1/2\}$. Set $F = \{f \geq 0\} \cap \Omega$ and $U_1 = M \setminus F$. We have $\varepsilon > 0$ on U_0 and we consider a partition of the unity $\{\varphi_0, \varphi_1\}$ subordinated to the covering $\{U_0, U_1\}$. The function $\tilde{\varepsilon} = \varphi_0 \varepsilon + \varphi_1$ is a continuous function that is strictly positive on the whole set M and coincides with ε over F. Then $\tilde{\varepsilon} f < \varphi$ on Ω, since by construction the inequality holds on the set $F = \{f \geq 0\} \cap \Omega$, and it holds trivially outside this set. $\square$

Our next lemma will be used to reduce the strict Positivstellensatz to the principal case, i.e., the case of a single inequality. Let $f_1, \dots, f_k \in C^r(M)$ and set $F = \{x \in M \mid f_1(x) \geq 0, \dots, f_k(x) \geq 0\}$. We assume that $F \neq \emptyset$. We denote by $\mathcal{M} = \mathcal{M}(f_1, \dots, f_k)$ the module over the sum of squares generated by $1, f_1, \dots, f_k$, i.e., $\mathcal{M} = \{f \in C^r(M) \mid f = s_0 + \sum s_i f_i\}$, where $s_0, \dots, s_k$ are sums of squares of C^r functions.

Lemma 2.4. *For any open set $U \supset F$ there is $h \in \mathcal{M}$ such that $F \subset \{h \geq 0\} \subset U$. Moreover, h can be taken to be of the form $h = \sum s_i f_i$ with $s_i > 0$ over M.*

Proof. Set $U_0 = U$ and $U_j = \{f_j < 0\}$, $j = 1, \dots, k$. Then $\{U_0, U_1, \dots, U_k\}$ form an open covering of M. Let $\{\varphi_0, \varphi_1, \dots, \varphi_k\}$ be a C^r partition of the unity subordinate to this covering. Then $\varphi = \sum_{j=1}^k \varphi_j f_j$ is a C^r function which verifies $\varphi < 0$ on $\Omega = M \setminus U_0$. In particular, $\{\varphi \geq 0\} \subset U_0$. We claim that we may assume that $\varphi_j > 0$ for all $j = 1, \dots, k$. Indeed, by Lemma 2.3, for each $j = 1, \dots, k$ there is $\varepsilon_j > 0$ on M such that $\varepsilon_j f_j < -\varphi/k$ on Ω. Thus on Ω we have

$$\varphi + \sum_{j=1}^k \varepsilon_j f_j < \varphi + k(-\varphi/k) = 0$$

and we can replace φ by the function

$$\varphi + \sum_{j=1}^k \varepsilon_j f_j = \sum_{j=1}^k (\varphi_j + \varepsilon_j) f_j$$

whose coefficients $\varphi_j + \varepsilon_j$ are strictly positive on M.

Thus we assume that $\varphi_j > 0$, $j = 1, \dots, k$, and we approximate these functions in the Whitney topology by C^r functions a_j such that $a_j > 0$ and $h = \sum_{j=1}^k a_j f_j$ verifies $\{h \geq 0\} \subset U_0$.

In fact, again by Lemma 2.3, for $j = 1, \dots, k$, there exist functions $\delta_j > 0$ on M such that $\delta_j |f_j| < -\varphi/2k$ on Ω. Therefore, taking a_j such that $|a_j - \varphi_j| < \min(\varphi_j/2, \delta_j)$, we have $a_j > 0$ and

$$|\varphi - h| = \left| \sum_{j=1}^k \varphi_j f_j - \sum_{j=1}^k a_j f_j \right| \leq \sum |\varphi_j - a_j||f_j| < \sum \frac{\varphi}{2k} = \frac{\varphi}{2}$$

on Ω, so that $\{h \geq 0\} \subset U_0$. This completes the proof of the lemma. $\square$

Now we are ready to prove our main result.

THEOREM 2.5 (Positivstellensatz). *Let $f_1, \dots, f_k$ be C^r-functions on M, $0 \le r \le \infty$, and assume that $\{x \in M \mid f_1 \ge 0, \dots, f_k \ge 0\} \ne \emptyset$. Let g be a C^r-function on M. Then:*

(i) $g \ge 0$ *on* $\{x \in M \mid f_1 \ge 0, \dots, f_k \ge 0\} \Longrightarrow p^2 g = v_0^2 + \sum v_i^2 f_i$,

(ii) $g > 0$ *on* $\{x \in M \mid f_1 \ge 0, \dots, f_k \ge 0\} \Longrightarrow g = v_0^2 + \sum v_i^2 f_i$,

where $p, v_1, \dots, v_k \in C^r(M) \setminus \{0\}$ *and* $\{p = 0\} \subset \{g = 0\}$.

REMARK 2.6. The condition on the zeroes of p is important in order to avoid "trivial" expressions in (i), since otherwise we could take p to have zero set $M \setminus \{g = 0\}$ and set all the $v_i's$ equal to zero.

Proof. Step 1. We start by proving the statement for principal sets, i.e., the case $k = 1$ of the result. Thus we have two differentiable functions $f, g \in C^r(M)$ such that g is positive on the nonempty set $\{f \ge 0\}$.

Let us assume first that g is strictly positive on $F = \{f \ge 0\}$. We want to find $s, t \in C^r(M)$, strictly positive on M, such that $g = s + tf$. Set $G = \{g \le 0\}$. The sets F and G are closed and we have $F \cap G = \emptyset$. We consider the function v defined by

$$v = \begin{cases} g/(f+g) & \text{over } F, \\ (f+g)/f & \text{over } G, \\ 1 & \text{elsewhere.} \end{cases}$$

Clearly v is continuous and strictly positive on M. Also, an immediate inspection shows that $g - vf > 0$ on M. Moreover, by Lemma 2.3 there exists $\varepsilon > 0$ on M such that $\varepsilon f < g - vf$ on M. Thus, taking a suitable approximation of v in the Whitney topology $t \in C^r(M)$ such that $|t - v| < \min(v/2, \varepsilon)$, we get $t > 0$ and $s = g - tf > 0$, as claimed.

Assume now that $g \ge 0$ on F and set $X = \{g = 0\}$. We repeat the above argument for the open set $M \setminus X$. Thus, we define

$$v = \begin{cases} g/(f+g) & \text{over } F \setminus X, \\ (f+g)/f & \text{over } G \setminus X, \\ 1 & \text{elsewhere,} \end{cases}$$

which is C^0 over $M \setminus X$ and satisfies $v > 0$ over this set. Also, we have $g - vf > 0$ over $M \setminus X$. We approximate v over $M \setminus X$ in the Whitney topology by a C^r function v' so that $g - v'f > 0$ over $M \setminus X$. Now, by [To, Lemma 6.1], there is a nonnegative C^∞ function $q : M \to \mathbb{R}$, that is flat over X, with $q^{-1}(0) = X$, such that qv' extends by 0 to a C^r function over R^n. In particular, we have $u_0 = qg - (qv')f \ge 0$ on M. Thus $qg = u_0 + u_1 f$ with $u_1 = qv' \ge 0$ and $\{q = 0\} \subset \{g = 0\}$.

Step 2. Consider first the case when $g > 0$ on $F = \{x \in M \mid f_1 \ge 0, \dots, f_k \ge 0\}$. By Lemma 2.4 there is a function $h = \sum t_i f_i$ such that

$F \subset \{h \geq 0\} \subset \{g > 0\}$, and by Step 1 we have

$$g = s + th = s + \sum s_i f_i.$$

This proves (i).

Assume next that $g \geq 0$ on F and let U_i be the open set $\{f_i < 0\}$, $i = 1, \dots, k$. Then the function $\varphi_i = \exp(-1/f_i^2)$ is positive over U_i and can be extended by 0 to a C^∞ function on all of M. Since the U_i's cover $M \setminus F$, the function $f = \varphi_1 f_1 + \cdots + \varphi_k f_k$ is strictly negative on $M \setminus F$ and identically zero on F. So $g \geq 0$ on $\{f \geq 0\} = F$, and by Step 1 we have

$$qg = u_0 + u_1 f = u_0 + \sum v_i f_i$$

with $\{q = 0\} \subset \{g = 0\}$, as claimed. □

3. O-minimal Positivstellensatz

In this section we show that the Positivstellensatz proved in Section 2 can be extended to the general setting of definable functions on an *o*-minimal structure. We first need to introduce some notations.

Let R be a real closed field. We consider an *o*-minimal structure $\mathcal{S}$ expanding R (see [Dr] and [Co, Definition 1.4]), and we denote by $\mathcal{D}^r$ the ring of definable functions of class C^r, $0 \leq r < \infty$ (see [DMi] for a precise definition). In particular, $\mathcal{D}^0$ stands for the ring of continuous definable functions. Finally, we let Φ^r be the set of all odd increasing $\mathcal{D}^r$ bijections of R that are r-flat at zero, i.e., all of whose derivatives or order less than or equal to r vanish at 0. The following results appear in [DMi] (see Lemma C8 and Proposition C9). In fact, there they are only stated for the case $R = \mathbb{R}$, the real numbers, because the authors work over analytic geometric structures, but the proofs carry over word for word to the case of any real closed field.

Proposition 3.1. *Let $g : A \to R, f_1, \dots, f_l : A \setminus Z(g) \to R$ be continuous and definable with A locally closed in R^m. Then there exists $\varphi \in \Phi^r$ such that $\varphi(g(x)) \cdot f_i(x) \to 0$ as $x \to y$, $x \in A \setminus Z(g)$, for each $y \in Z(g)$ and $i = 1, \dots, l$.*

Proposition 3.2. *Let $f, g : R^n \to R$ be continuous definable $\mathcal{D}^r$ functions on $R^n \setminus Z(g)$, with $Z(f) \subset Z(g)$. Then there exist $\varphi \in \Phi^r$ and $h \in \mathcal{D}^r(R^n)$ such that $\varphi \circ g = hf$.*

In other words, singularities of definable functions contained in the zero set of a definable function can be killed by multiplying by a suitable power of the latter. A consequence of these results is the existence of $\mathcal{D}^r$ *partitions of unity* subordinated to a finite definable open covering (see [Es, Theorem 4.2]). These results constitute one of the tools needed to extend the arguments of Section 2. Also, we have the following extension result:

Lemma 3.3. *Let g be a $\mathcal{D}^r$–function on R^n and set $X = g^{-1}(0)$. Let $v : R^n \setminus X \to R$ be $\mathcal{D}^r$. Then there exists $q \in \mathcal{D}^r$ with $q^{-1}(0) = X$, r-flat on X, such that qv extends to a global $\mathcal{D}^r$-function that is r-flat on X.*

Proof. By Proposition 3.1 we can find $\theta \in \Phi^r$ such that $f = \theta(g)v$ extends continuously by 0 to all of R^n. By the same result there also exists $\alpha \in \Phi^r$ such that $\alpha(g) \cdot \sigma$ extends continuously by 0 to a function over X, for each $\sigma \in \{f, \mathrm{D}^i f\}$, where $i \in \mathbb{N}^n$, $|i| \le r$. Since $\alpha(g)$ is of class C^r over R^n and vanishes exactly on X, we obtain that for k large enough $\alpha(g)^k f$ belongs to $\mathcal{D}^r$ and is r-flat on X. Thus, taking $q = \alpha(g)^k \theta(g)$ proves the result. □

The other tool we need is the approximation of definable functions. In $\mathcal{D}^r$ we consider the C^r-Whitney topology, in which a neighbourhood of a function g is given by all functions f such that $\|\mathrm{D}^\alpha f(x) - \mathrm{D}^\alpha g(x)\| < \varepsilon(x)$, where ε is a continuous strictly positive definable function on R^n and $\alpha \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n = 0, \ldots, r$ and, as usual, $\|x\|^2 = \sum x_i^2$. We will express this by saying that $|f - g| < \varepsilon$ or that f approximates g up to ε. We have the following approximation theorem, which will be used in the sequel:

Theorem 3.4 ([Es, Theorem 4.4.1]). *Let $f \in \mathcal{D}^r$ and let ε be a continuous definable strictly positive function. Then there exists an approximation $\tilde{f} \in \mathcal{D}^{r+1}$ such that $|f - \tilde{f}| < \varepsilon$.*

In particular, a repeated application of this theorem shows that $\mathcal{D}^r$ is dense in $\mathcal{D}^0$. Now, let Σ be the set of sums of squares of $\mathcal{D}^r$ and let $f_1 \ldots, f_k \in \mathcal{D}^r$. As above, we denote by $\mathcal{M}[f_1, \ldots, f_k]$ (or just $\mathcal{M}$ if no confusion is possible) the Σ–module generated by $f_1, \ldots, f_k$, that is,

$$\mathcal{M} = \Sigma + \Sigma f_1 + \cdots + \Sigma f_k.$$

The results in Section 2 extend, almost word for word, to o-minimal structures. In particular, the characterization of nonnegative definable functions is rather simple, and given by the following result:

Lemma 3.5 (Hilbert's 17th Problem). *Let $\varphi : R^n \longrightarrow R$ be a nonnegative $\mathcal{D}^r$ function. Then φ is the square of a quotient of $\mathcal{D}^r$ functions.*

Proof. Consider the definable function $\xi : R^+ \to R^+$ given by

$$\xi(x) = \begin{cases} x^k \sqrt{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

The function ξ is obviously definable, of class k, and k-flat at the origin. Now, given $\varphi : R^n \longrightarrow R^+$, the composition $u = \xi \circ \varphi = \varphi^k \sqrt{\varphi}$ is C^r for k large enough and we have $(\varphi)^{2k} \varphi = u^2$. □

Replacing partitions of unity in the proof of Lemma 2.3 by definable partitions of unity we get the following analogue of this result:

LEMMA 3.6. *Let f, φ be continuous definable functions on R^n and assume that φ is strictly positive on a closed set $\Omega \subset R^n$. Then there is a continuous definable function ε, strictly positive on R^n, such that $\varepsilon f < \varphi$ on Ω.*

We also have the following definable versions of the Positivstellensätze:

THEOREM 3.7 (Definable Positivstellensatz). *Let $f_1, \ldots, f_k$ be $\mathcal{D}^r$-functions on R^n, $0 \leq r < \infty$, and assume that the set $F = \{x \in R^n \mid f_1 \geq 0, \ldots, f_k \geq 0\}$ is not empty. Let g be a $\mathcal{D}^r$-function on R^n. Then:*

(i) $g \geq 0$ *on* $\{x \in R^n \mid f_1 \geq 0, \ldots, f_k \geq 0\} \Longrightarrow p^2 g = v_0^2 + \sum v_i^2 f_i$,
(ii) $g > 0$ *on* $\{x \in R^n \mid f_1 \geq 0, \ldots, f_k \geq 0\} \Longrightarrow g = v_0^2 + \sum v_i^2 f_i$,

where $p, v_0, \ldots, v_k \in \mathcal{D}^r(R^n)$ and $\{p = 0\} \subset \{g = 0\}$

Proof. The proof is identical to that of Theorem 2.5, with the necessary auxiliary results replaced by their definable analogues and Tougeron's lemma replaced by Proposition 3.3. □

Recall that $\mathcal{D}^r$ denotes the ring of definable function class C^r. Let $\tilde{R}_{\mathrm{def}}$ be the real spectrum of $\mathcal{D}^r$ (see [BCR]). Notice that for any point $x \in R^n$ there exists a unique point $\tilde{x} \in \tilde{R}_{\mathrm{def}}$, namely the prime cone of all functions $f \in \mathcal{D}^r$ such that $f(x) \geq 0$. Thus we have a canonical inclusion $R^n \subset \tilde{R}_{\mathrm{def}}$. Now, given any set $\tilde{S} = \{\alpha \in \tilde{R}_{\mathrm{def}} \mid g(\alpha) \neq 0, f_1(\alpha) \geq 0, \ldots, f_r(\alpha) \geq 0\}$, we can consider its restriction $S = \tilde{S} \cap R^n$, which is the definable subset of R^n defined by the same inequalities. A natural question (usually referred to as the Artin-Lang property) is to decide whether $\tilde{S} \neq \emptyset$ implies $S \neq \emptyset$ (the converse being immediate). Using the Positivstellensatz above we show:

COROLLARY 3.8 (Artin-Lang for $\mathcal{D}^r$ functions).

(a) $\tilde{S} = \{\alpha \in \tilde{R}_{\mathrm{def}} \mid g(\alpha) > 0, f_1(\alpha) \geq 0, \ldots, f_r(\alpha) \geq 0\}$ *is not empty if and only if S is not empty.*
(b) $\tilde{S} = \{\alpha \in \tilde{R}_{\mathrm{def}} \mid g(\alpha) \geq 0, f_1(\alpha) \geq 0, \ldots, f_r(\alpha) \geq 0\}$ *is not empty if and only if S is not empty.*

Proof. Assume that $\tilde{S}$ is empty. Then by the abstract Positivstellensatz (see [BCR]) we have an equation $p_1(-g^2) = p_2$ for some p_1, p_2 in the positive cone generated by $f_1, \ldots, f_r$; that is, $p_i = a_{oi} + \sum a_\varepsilon f_1^{\varepsilon_1} \cdots f_r^{\varepsilon_r}$, where the coefficients are sums of squares in $\mathcal{D}^r$ and $\{p_1 = 0\} \cap \{x \in R^n \mid f_1 \geq 0, \ldots, f_r \geq 0\} \subset \{g = 0\}$. Assume now that there exists some point $x \in S$. Evaluating the above expression at x we see that the left hand side of the equality is strictly negative, while the right hand side is greater than or equal to zero, a contradiction. Thus, S must be empty.

Conversely, assume that S is empty. Then by the Positivstellensatz (Theorem 3.6) we have an equation $p^2(-g^2) = q$ with $\{p = 0\} \subset \{g = 0\}$ and

$q \in \mathcal{M}(f_1, \dots, f_r)$. By the abstract Positivstellensatz this means that $\tilde{S} = \emptyset$. This proves (a); part (b) is shown analogously. □

Remark 3.9 (About denominators). It is well known that denominators are necessary in Hilbert's 17th Problem in the algebraic case (see [BCR]). One of the first explicit examples was given by Motzkin [Mo] with the polynomial $f(x, y) = 1 + x^2y^2(x^2 + y^2 - 3)$. The set of common zeroes of all possible denominators is called the *bad points set* of f (see [Br], [De]) and has been an object of study since then. The homogenization of Motzkin's polynomial shows also that denominators are necessary for expressing it as a sum of squares of analytic germs at the origin, as well as a formal power series, and therefore also as a sum of squares of C^∞ functions in any neighbourhood of the origin.

In fact, any homogeneous nonnegative polynomial f with a bad point at 0 cannot be written as a sum of squares of smooth functions in any neighbourhood of 0, since otherwise, by considering the homogeneous initial part of the Taylor series of such a representation, we would get a representation of f as a sum of squares of polynomials, contradicting the assumption that 0 is a bad point for f.

Moreover, for any definable set of the form $F = \{f_1 \geq 0, \dots, f_k \geq 0\} \subset R^n$, $n \geq 3$, with nonempty interior and $f_i \in \mathcal{D}^r$ with r suitably large, there are definable functions f which are nonnegative over F and do not belong to the cone generated by $f_1, \dots, f_k$ and the sum of squares. Therefore such a function requires a "true" denominator in the Positivstellensatz (Theorem 3.7). Indeed, consider the following example, which uses an idea similar to that of C. Scheiderer [Sche].

Assume that the origin is an inner point of F and take any polynomial f which has a bad point at the origin, for instance, Motzkin's homogeneous polynomial

$$f(x, y, z) = z^6 + x^2y^2(x^2 + y^2 - z^2).$$

Now, if f belongs to the preordering generated by the functions $f_1, \dots, f_k$, we would have a representation

$$g = \sum \alpha_I f_1^{i_1} \dots f_k^{i_k},$$

where $I = (i_1, \dots, i_k) \in \{0, 1\}^k$. Since $f_j(0) > 0$, these functions are squares of C^r functions in a neighbourhood of the origin, and considering the Taylor polynomials at 0 on both sides of the equation we would get a representation of Motzkin's polynomial as a sum of squares of polynomials, which is a contradiction.

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