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FØLNER NUMBERS AND FØLNER TYPE CONDITIONS FOR AMENABLE SEMIGROUPS

BY

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As well-known, the Følner condition for semigroups does not imply the left amenability. In 1964, Namioka gave two sufficient conditions of Følner type for a semigroup to be left amenable. He asked the question whether they are necessary. In this paper we show that these conditions are not necessary for left amenability by studying in details the Følner number of a semigroup. We also prove that for a semidirect product of two semigroups satisfying the strong Følner condition, the two Namioka-Følner conditions are equivalent to the strong Følner condition. We answer in this paper a problem of Klawe on the homomorphic images of a semigroup with the strong Følner condition. Some general properties of Følner numbers and Følner type conditions are also studied.

1. Introduction

Let S be a semigroup, and let $m(S)$ be the Banach space of bounded real-valued functions on S with the supremum norm. A linear functional $\mu \in m(S)^*$ is called a mean if μ is positive and $\|\mu\| = 1$. A mean μ is left invariant if $\mu(f) = \mu(l_s f)$ for all $f \in m(S)$ and $s \in S$, where $l_s f \in m(S)$ is defined by $(l_s f)(s_1) = f(ss_1)$, $s_1 \in S$. A semigroup is left amenable if it has a left invariant mean. For general properties of left amenable semigroups, see Day [3] and [4].

For subsets A, B of S and $s \in S$, we define

$$A \cdot B = \{uv \mid u \in A \text{ and } v \in B\},$$
$$sA = \{su \mid u \in A\} \text{ and } As = \{us \mid u \in A\}.$$

We denote by $A^2 = A \cdot A$, and so on. χ_A is used to denote the characteristic function of A. And if A is finite, then $|A|$ will be its cardinality.

Consider the following Følner type conditions on S:

(A) There exists a number k, $0 < k < 1$, such that for any elements $s_1, \ldots, s_n$ of S (not necessarily distinct), there is a finite subset A of S

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satisfying

$$\frac{1}{n}\sum_{i+1}^{n}|A\setminus s_iA| \le k|A|.$$

(B) Given any finite subset F of S, and any number $\varepsilon > 0$, there exists a finite subset A of S, such that for each $s \in F$, $|A \setminus sA| \le \varepsilon |A|$.

We call condition (A) the weak Følner condition (WFC) and condition (B), as in [1] and [8], the strong Følner condition (SFC). When S is a group, Følner [5] proved that both WFC and SFC are equivalent to the amenability of S. Frey [6] introduced the condition FC, which is equivalent to SFC when S is left cancellative (see [1]):

(FC) Given any finite subset F of S, and any number $\varepsilon > 0$, there exists a finite subset A of S, such that for each $s \in F$, $|sA \setminus A| \le \varepsilon |A|$.

He proved that if S is left amenable, the FC holds, but the converse is not true (see Namioka [9] for an elegant proof of this fact). In general, SFC is sufficient for the left amenability (LA) of S (cf. [1], also [9]). However, it is not necessary (see Klawe [8] for an example). Also WFC is not sufficient for LA (see Namioka [9] and also see our Theorem 2.3). In 1964, Namioka gave two sufficient conditions stronger than WFC. We will refer to them as the weak and strong Namioka-Følner conditions.

(WNFC) There exists a number k, $0 < k < 1$, such that for any elements $s_1, \dots, s_n; s_1', \dots, s_n'$ of S, there is a finite subset A of S satisfying

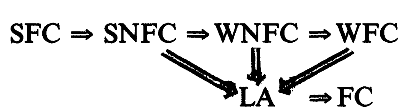
$$\frac{1}{n}\sum_{i+1}^{n}|s_iA \cap s_i'A| \ge k|A|.$$

(SNFC) There exists a number k, $0 < k < 1/2$, such that for any elements $s_1, \dots, s_n$ of S, there is a finite subset A of S satisfying

$$\frac{1}{n}\sum_{i=1}^{n}|A\setminus s_iA| \le k|A|.$$

Namioka [9] proved that SNFC implies WNFC and WNFC implies LA. In fact he showed that if SNFC holds for k then WNFC holds for $1 - 2k$. Also it is easy to see that if WNFC holds for k, then SNFC(WFC) holds for $1 - k$. Namioka [9, p. 26] posted the problem whether those conditions are necessary; i.e., whether LA implies WNFC or SNFC.

The following diagram summarizes the known implications among the various Følner-type conditions for a semigroup mentioned above.



We prove in §4 that LA $\not\Rightarrow$ WFC. This answers negatively Namioka's problem (same as what he conjectured). Notice that this also shows that WFC is not weaker than FC.

Motivated by WFC and SNFC, James Wong [14] defined the Følner number $\varphi(S)$ for an arbitrary semigroup S to be the infimum of all numbers $k \leq 1$ such that WFC holds. Clearly $\varphi(S) < 1$ and $\varphi(S) < 1/2$ correspond to WFC and SNFC, respectively. Furthermore, $\varphi(S) = 0$ is equivalent to SFC as we will show in Proposition 2.1.

In Section 2 of this paper, we investigate some general properties of $\varphi(S)$ and competely determine $\varphi(S)$ for all finite semigroups and cancellative semigroups. In Section 3 we obtain, by some combinatorial computations, two inequalities for $\varphi(S)$ related to the cancellation behavior of S. One of them is the main tool to solve Namioka's problem.

In [8], Klawe studied amenability of the semidirect product of two semigroups and she was able to construct a left amenable semigroup not satisfying SFC. In Section 4, we show that Klawe's semigroup has Følner number 1, which answers Namioka's problem. Then we give some necessary and sufficient conditions for a semidirect product to be left amenable. We also answer negatively a problem of Klawe [8, p. 102]: whether a homomorphic image of a semigroup satisfying SFC also satisfies SFC.

The last section of this paper is devoted to the Følner number of a semidirect product. We prove by the inequality we obtain in Theorem 3.3 and 3.9 that there is a large collection of semidirect products which are left amenable and have Følner number 1. We also show that for those semigroups the two Namioka-Følner conditions are in fact equivalent to SFC. But the problem of whether they are always equivalent is still open.

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2. Følner numbers

In this section we give a formula for Følner numbers of finite semigroups related to the numbers of minimal right ideals. Then we show that the Følner number of a cancellative semigroup S is 0 or 1 according as S is left amenable or not.

We follow Wong [14] for the definition of Følner number of a semigroup. Let S be a semigroup and $0 < k \leq 1$. We say that S has property (F_k) if for any $s_1, \dots, s_n \in S$ (not necessarily distinct), there is a finite (nonempty) subset A of S such that

$$\frac{1}{n} \sum_{i=1}^{n} |A \setminus s_i A| \leq k|A|.$$

The Følner number of S is defined by

$$\varphi(S) = \inf\{k \mid 0 < k \leq 1 \text{ and } S \text{ has property } (F_k)\}.$$

$\varphi(S)$ is well-defined since every semigroup has property (F_1).

By the definition we can see that WFC $\Leftrightarrow$ $\varphi(S) < 1$ and SNFC $\Leftrightarrow$ $\varphi(S) < 1/2$. Also it is easy to see that SFC implies $\varphi(S) = 0$. Our first result is about the converse (compare with Wong [14], Theorem 2.2(1)).

PROPOSITION 2.1. *Let S be a semigroup. If $\varphi(S) = 0$, then S satisfies* SFC.

Proof. Let $F = \{s_1, \dots, s_n\}$ be any finite subset of S, and $\varepsilon > 0$. Since $\varphi(S) = 0$, there exists a finite subset A of S, such that

$$\frac{1}{n}\sum_{i=1}^{n} |A \setminus s_i A| \leq \frac{\varepsilon}{n}|A|.$$

Therefore $|A \setminus s_i A| \leq \varepsilon |A|$ for all i, $1 \leq i \leq n$. □

PROPOSITION 2.2. *Let S be a semigroup. If there are n disjoint right ideals $I_1, \dots, I_n$ in S, then $\varphi(S) \geq (n-1)/n$.*

Proof. Pick $s_i \in I_i$ for $i = 1, \dots, n$. For any finite subset A of S, the sets $s_i A$ are mutually disjoint. So $\sum_{i=1}^{n} |A \cap s_i A| \leq A|$. This implies that

$$\frac{1}{n}\sum_{i=1}^{n} |A \setminus s_i A| \geq \frac{n-1}{n}|A| \qquad \square$$

THEOREM 2.3. *If S is a finite semigroup, then $\varphi(S) = 1 - 1/n$, where n is the number of minial right ideals of S.*

Proof. By Proposition 2.2, $\varphi(S) \geq 1 - 1/n$. On the other hand, let $I_1, \dots, I_n$ be the n minimal right ideals of S, and $A = \bigcup_{i=1}^{n} I_i$. Since any two minimal right ideals in a finite semigroup have the same cardinality, we have $|A| = n|I_1|$. For any $s \in S$, sA is a right ideal, so it contains a minimal right ideal I_i. Thus $|A \cap sA| \geq |I_i| = n^{-1}|A|$, and

$$|A \setminus sA| \leq (1 - 1/n)|A|. \qquad \square$$

COROLLARY 2.4. *For a finite semigroup S, the following are equivalent*:

(1) *S is left amenable*;

(2) $\varphi(S) = 0$ (*S satisfies* SFC):

(3) $\varphi(S) < 1/2$ (*S satisfies* SNFC).

Proof. A finite semigroup is left amenable if and only if it contains a unique minimal right ideal (see [11]). □

COROLLARY 2.5([9]). *There are semigroups which satisfy* WFC *but are not left amenable.*

COROLLARY 2.6. *Let S be a semigroup, h a homomorphism of S onto a finite semigroup. Then* $\varphi(S) \geq \varphi(h(S))$.

Proof. If $h(S)$ has n minimal right ideals, then S admits at least n disjoint right ideals. By Proposition 2.2, $\varphi(S) \geq 1 - 1/n = \varphi(h(S))$. □

It is well known (see [3]) that a homomorphic image of a left amenable semigroup is also left amenable. It would be desirable if Corollary 2.6 holds for arbitrary h. Unfortunately, this is not true in general. An example that $\varphi(S) = 0$ but $\varphi(h(S)) = 1$ is given in §4.

If G is a group, then $\varphi(G) = 0$ or 1 according to G is amenable or not [14, Theorem 2.2(3)]. This is also true for cancellative semigroups. In other words, the Følner number of a cancellative semigroup never takes values other than 0 and 1.

THEOREM 2.7. *If S is a cancellative semigroup, then $\varphi(S) = 0$ or 1 according as S is left amenable or not.*

Proof. If S is left amenable, then $\varphi(S) = 0$ since now SFC is equivalent to FC (see [1]).

Suppose S is not left amenable.

Case (i). S has two disjoint right ideals I_1 and I_2. Take $s_1 \in I_1$ and $s_2 \in I_2$. $s_1 I_1$ and $s_1 I_2$ are disjoint right ideals contained in I_1. Also $s_2 I_1$ and $s_2 I_2$ are disjoint right ideals contained in I_2. Thus we have got four disjoint right ideals. Inductively, for any positive integer n, we can find 2^n disjoint right ideals in S. By Proposition 2.2, $\varphi(S) = 1$.

Case (ii). Any two right ideals of S have nonempty intersection. By Dubreil's theorem [2, p. 36], S can be embedded into a group G, such that

$$G = \{xy^{-1} | x, y \in S\}.$$

By a theorem of Frey (See Pier [10, Prop. 23.32]), G is not amenable. Hence $\varphi(G) = 1$. Suppose $\varphi(S) < k < 1$, and take $x_1 y_1^{-1}, x_2 y_2^{-1}, \dots, x_n y_n^{-1} \in G$, where x_i, $y_i \in S$. We prove first that there exists an element $s \in S$ such that $x_1 y_1^{-1} s, \dots, x_n y_n^{-1} s$ are all in S. By induction, suppose that there exists $s' \in S$, such that $x_1 y_1^{-1} s', \dots, x_{n-1} y_{n-1}^{-1} s' \in S$. By the structure of G, $y_n^{-1} s'$ can be written as ab^{-1}, where $a, b \in S$. Let $s = s'b$. Then

$$x_i y_i^{-1} s = (x_i y_i^{-1} s')b \in S \quad \text{for } i \leq n - 1,$$

and $x_n y_n^{-1} s = x_n a \in S$.

Write $s_i = x_i y_i^{-1} s$. Notice that $s_i s^{-1} = x_i y_i^{-1}$ for $1 \le i \le n$. By the assumption $\varphi(S) < k < 1$, there is a finite subset of A of S, such that

$$\frac{1}{n} \sum_{i=1}^{n} |A \setminus s_i A| \le k|A|.$$

It follows that

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{n} |(A \cup sA) \setminus s_i s^{-1}(A \cup sA)| \\
&\quad = |A \cup sA| - \frac{1}{n} \sum_{i=1}^{n} |(A \cup sA) \cap s_i s^{-1}(A \cup sA)| \\
&\quad \le |A \cup sA| - \frac{1}{n} \sum_{i=1}^{n} |A \cap s_i A| \\
&\quad = |A \cup sA| - |A| + \frac{1}{n} \sum_{i=1}^{n} |A \setminus s_i A| \\
&\quad \le |A \cup sA| - (1 - k)|A| \\
&\quad \le |A \cup sA| - \frac{1-k}{2}|A \cup sA| \\
&\quad = \frac{1+k}{2}|A \cup sA|.
\end{aligned}$$

This means $\varphi(G) \le (1 + k)/2 < 1$, which contradicts the fact that $\varphi(G) = 1$. □

COROLLARY 2.8. *For a cancellative semigroup S, the following are equivalent*:

(i) *S is left amenable*;
(ii) $\varphi(S) = 0$ (*S satisfies* SFC);
(iii) $\varphi(S) < 1$ (*S satisfies* WFC).

Let S be a semigroup having the finite intersection property for right ideals; i.e., any two right ideals of S have nonempty intersection (e.g. any left amenable semigroup has this property). We can define an equivalence relation R on S by

$$sRt \Leftrightarrow \exists x \in S, sx = tx.$$

The set $S/(R)$ of the R-equivalence classes forms a right cancellative semigroup—the right cancellative quotient of S. We refer the readers to [7] for more details about the semigroup $S/(R)$. Whenever $S/(R)$ exists, S is left amenable if and only if $S/(R)$ is left amenable [13], and $\varphi(S) = 0$ if and only if $\varphi(S/(R)) = 0$ ([1] and [8]).

THEOREM 2.9. *Let S be a semigroup with the finite intersection property for right ideals. Then $\varphi(S) \leq \varphi(S/(R))$.*

Proof. This follows from the proof of Theorem 4 in Argabright and Wilde [1]. □

We are unable to prove the equality in Theorem 2.9. This gives rise to the open problem of whether the strict inequality can hold.

3. Følner number and left cancellation

For a right cancellative semigroup $S, \varphi(S) = 0$ if and only if S is left amenable and left cancellative ([1] and [8]). In this section we will see that $\varphi(S)$ really depends on the left cancellativity of S. The first result is how $\varphi(S)$ relates to the size of left cancellative classes.

THEOREM 3.1. *Let S be a right cancellative semigroup. If there exist distinct elements $s_1, s_2, \ldots, s_{2n}$ of S, and $r \in S$, such that $rs_1 = rs_2 = \ldots rs_{2n}$, then $\varphi(S) \geq 1/3 - 1/6n$.*

Proof. Suppose S has property (F_k) for some $k \in (0, 1]$ (see the beginning of §2). We are going to prove $k \geq 1/3 - 1/6n$. By (F_k) we know that there exists a finite subset A of S such that

$$\frac{1}{3n}\left(n|A \setminus rA| + \sum_{i=1}^{2n} |A \setminus s_i A|\right) \leq k|A|. \tag{3.1}$$

Define $f\colon S \to \mathbf{Z}^+$ by $f = \sum_{i=1}^{2n} \chi_{s_i^{-1}A}$, where $s_i^{-1}A \subset S$ is the set of all $x \in S$ such that $s_i x \in A$. Let $W_j = \{a \in A | f(a) = j\}$ for $0 \leq j \leq 2n$. Let $T_0 = A$ and

$$T_j = \left\{ y \in A | y = s_i a \text{ for some } a \in \bigcup_{m=j}^{2n} W_m \text{ and } i \in \{1, \ldots, 2n\} \right\},$$

for $j = 1, \ldots, 2n$. Finally, let $S_j = T_j \setminus T_{j+1}$, $j = 0, 1, \ldots, 2n-1$ and $S_{2n} = T_{2n}$.

Since $S_j \subset (\bigcup_{i=1}^{2n} s_i w_j) \cap A$, it is not difficult to see that

$$|W_j| \geq j^{-1}|S_j| \text{ for } j \geq 1,$$

by the definition of f. Thus we have

$$
\begin{aligned}
(3.2)\qquad \sum_{i=1}^{2n} |A \setminus s_i A| &\geq \sum_{i=1}^{2n} |A \setminus s_i^{-1} A| \\
&= 2n|A| - \sum_{i=1}^{2n} |A \cap s_i^{-1} A| \\
&= 2n|A| - \sum_{a \in A} f(a) \\
&= 2n \sum_{j=0}^{2n} |W_j| - \sum_{j=1}^{2n} j|W_j| \\
&\geq \sum_{j=1}^{2n} (2n-j)|W_j| \geq \sum_{j=1}^{2n} \frac{2n-j}{j}|S_j|.
\end{aligned}
$$

Also since $T_1 = A \cap \bigcup_{i=1}^{2n} s_i A$, $S_0 \subset A \setminus s_i A$ for all $i = 1, \dots, 2n$. Thus we have the inequality

$$\sum_{i=1}^{2n} |A \setminus s_i A| \geq 2n|S_0|,$$

and hence by (3.2),

$$(3.3)\qquad \sum_{i=1}^{2n} |A \setminus s_i A| \geq n|S_0| + \frac{1}{2} \sum_{j=1}^{2n} \frac{2n-j}{j}|S_j|.$$

Now consider $|A \setminus rA|$. We claim that for $j \geq 1$,

$$(3.4)\qquad \left|rS_j \setminus \bigcup_{m=j+1}^{2n} rS_m\right| \leq \frac{1}{j}|S_j|.$$

Suppose $x \in rS_j \setminus \bigcup_{m=j+1}^{2n} rS_m$. Then there is $s \in S_j$ with $x = rs$, where $s = s_{i_0} a$ for some i_0 and $a \in A$ with $f(a) = j$. Here the equality holds since $s \notin T_{j+1}$. Thus there are j distinct s_i such that $s_i a \in A$. Also those $s_i a$ are distinct by right cancellation of S. Since

$$rs_i a = rs_{i_0} a = x \notin \bigcup_{m=j+1}^{2n} rS_m = rT_{j+1},$$

those $s_i a$ are all in S_j. Now we have proved that for any $x \in rS_j \setminus \bigcup_{m=j+1}^{2n} rS_n$, there are at least j elements $s \in S_j$, such that $rs = x$. This gives (3.4).

Summing up for $j = 0, 1, \ldots, 2n$, we obtain

$$|rA| \le |rS_0| + \sum_{j=1}^{2n} \frac{1}{j}|S_j| \le |S_0| + \sum_{j=1}^{2n} \frac{1}{j}|S_j|,$$

and

$$|A \setminus rA| \ge |A| - |rA| \ge \sum_{j=1}^{2n} \left(1 - \frac{1}{j}\right)|S_j|. \tag{3.5}$$

Finally, from (3.1), (3.3) and (3.5),

$$\begin{aligned} k|A| &\ge \frac{1}{3n}\left(n|A \setminus rA| + \sum_{i=1}^{2n} |A \setminus s_i A|\right) \\ &\ge \frac{1}{3n}\left(n \sum_{j=1}^{2n} \left(1 - \frac{1}{j}\right)|S_j| + n|S_0| + \frac{1}{2}\sum_{j=1}^{2n} \frac{2n-j}{j}|S_j|\right) \\ &= \frac{1}{3n}\left(n|S_0| + \sum_{j=1}^{2n}\left(n - \frac{1}{2}\right)|S_j|\right) \\ &\ge \frac{1}{3n}\sum_{j=0}^{2n}\left(n - \frac{1}{2}\right)|S_j| = \left(\frac{1}{3} + \frac{1}{6n}\right)|A|; \end{aligned}$$

i.e., $k \ge 1/3 - 1/6n$. □

It can be seen in the proof that for an arbitrary semigroup S, the same result also holds with the additional condition that $s_1, \ldots, s_{2n}$ belong to different right cancellative classes. In other words, for any $a \in S$, $i \ne j$ implies $s_i a \ne s_j a$.

COROLLARY 3.2. *For any semigroup S, if $\varphi(S) \ne 0$, then $\varphi(S) \ge 1/6$.*

Proof. We may assume that S is left amenable. By Lemma 2.1 in [8], there exist $r, s, t \in S$ with $rs = rt$ but $sx \ne tx$ for any $x \in S$. Now our theorem applies with $n = 1$. □

If there is a subset in S having a sort of "uniform cancellation property", we can get a much sharper inequality for $\varphi(S)$ which will be used to solve Namioka's problem.

THEOREM 3.3. *Suppose S is a right cancellative semigroup. If there exists a finite subset F in S such that*
(i) $|F| = n \ge 2$,
(ii) $\forall r, s, t \in F$, $rs = rt$,
(iii) $\forall r_1, r_2 \in F, \forall s, t \in S$, $r_1 s = r_1 t \Leftrightarrow r_2 s = r_2 t$,
then $\varphi(S) \ge 1 - 1/n$.

We divide the proof into some lemmas.

LEMMA 3.4. *For any positive integer $m \geq 2$, the set F^m also has properties* (i)–(iii).

Proof. (i) Take $r \in F$. Then $F^m = Fr^{m-1}$ by (ii). But $|Fr^{m-1}| = n$ since S is right cancellative.

(ii) This follows from the fact that $r_1 \dots r_m r_1' \dots r_m' = r_1^{2m}$ for $r_1, \dots, r_m, r_1', \dots, r_m' \in F$.

(iii) For $r_1 \dots r_m$ and $r_1' \dots r_m' \in F^m$, and $s, t \in S$, if

$$r_1 \dots r_m s = r_1 \dots r_m t,$$

then

$$r_1' \dots r_m' s = r_1' r_1^{m-1} s = r_1' r_1^{m-1} t = r_1' \dots r_m' t,$$

by (iii), since $r_1 r_1^{m-1} s = r_1 \dots r_m s = r_1 \dots r_m t = r_1 r_1^{m-1} t$. □

Now let A be a finite subset of S. Given a positive integer m, we define an equivalence relation $\sim_m$ on A by

$$s \sim_m t \Leftrightarrow \exists r \in F^m, rs = rt.$$

By (iii) it defines an equivalence relation. An equivalence class for the relations $\sim_m$ is called a class of level m. Denote by N_m the total number of classes of level m in A. Since $s \sim_m t \Rightarrow s \sim_{m+1} t$, each class of level $m+1$ is the disjoint union of some classes of level m, and $|A| \geq N_1 \geq N_2 \geq \dots$. Let

$$k_m = \frac{1}{n} \sum_{r \in F^m} \frac{|A \setminus rA|}{|A|}.$$

LEMMA 3.5. *For any (nonempty) finite subset A of S and any real number $\delta > 1$, if $k_m < 1$, then*

$$N_m - N_{2m} > \frac{\delta - 1}{\delta^2}(1 - k_m)^2 \left[1 - \frac{\delta}{(1 - k_m)n}\right]|A|.$$

Proof. Define a function $f\colon S \to \mathbf{Z}^+$ by $f = \sum_{r \in F^m} \chi_{rA}$. We have $0 \leq f(s) \leq n$, and the average of f on A is given by

$$\begin{aligned}\frac{1}{|A|} \sum_{s \in A} f(s) &= \frac{1}{|A|} \sum_{r \in F^m} |rA \cap A| \\ &= \frac{1}{|A|} \sum_{r \in F^m} (|A| - |A \setminus rA|) \\ &= n - k_m n = (1 - k_m) n.\end{aligned}$$

Let $A_1 = \{s \in A | f(s) > (1 - k_m)n/\delta\}$, and $A_2 = A \setminus A_1$. Then

$$\begin{aligned}(1 - k_m)n|A| &= \sum_{s \in A} f(s) \\ &= \sum_{s \in A_1} f(s) + \sum_{s \in A_2} f(s) \\ &\le n|A_1| + \frac{(1 - k_m)n}{\delta}|A|.\end{aligned}$$

So

$$|A_1| \ge \left(1 - \frac{1}{\delta}\right)(1 - k_m)|A|. \tag{3.6}$$

Let C be a class of level m. Then for any $r \in F^m$, $|rC| = 1$. Furthermore, if $s \sim_m t$ and $r_1, r_2 \in F^m$, then $r_1 s \sim_m r_2 t$, by (ii). Thus $(F^m \cdot C) \cap A$ is contained in a single class of level m (maybe empty).

Suppose that there exists $s \in C$ with $f(s) > 0$. Then $s \in r_i A$ for distinct $r_1, r_2, \ldots, r_{f(s)} \in F^m$. In other words, there exist $f(s)$ classes $C_1, C_2, \ldots, C_{f(s)}$ of level m with $r_i C_i = \{s\}$. It is easy to see that those C_i are disjoint. By (ii), those classes are contained in the same class $\overline{C}$ of level $2m$. For a class C' of level m such that $(F^m \cdot C') \cap A \neq \varnothing$, $C' \subset \overline{C}$ if and only if $(F^m \cdot C') \cap A \subset C$. For let $t_1 \in C'$ and $r \in F^m$ be such that $rt_1 \in A$, and $t_2 \in C_1 \subset \overline{C}$. Then

$$\begin{aligned}(F^m \cdot C') \cap A \subset C \Leftrightarrow rt_1 \in C \Leftrightarrow rt_1 \sim_m rt_2 \Leftrightarrow r^2 t_1 \\ = r^2 t_2 \Leftrightarrow t_1 \sim_{2m} t_2 \Leftrightarrow C' \subset \overline{C}.\end{aligned}$$

This means that the map $C \to \overline{C}$ is independent of a choice of s and it is 1-1.

For every class C of level m for which $\overline{C}$ is defined, let $V(\overline{C})$ be the number of classes of level m contained in $\overline{C}$. Then for any $r \in F^m$, $|r\overline{C}| = V(\overline{C})$. So $\sum_{s \in C} f(s) \le n \cdot V(\overline{C})$ by the definition of f, and

$$|C \cap A_1| < n \cdot V(\overline{C}) \frac{(1 - k_m)n}{\delta} = \frac{\delta \cdot V(\overline{C})}{1 - k_m}. \tag{3.7}$$

If $C \cap A_1 \neq \varnothing$, then $\exists s \in C$ with $f(s) > (1 - k_m)n/\delta$. So $V(\overline{C}) \ge f(s) > (1 - k_m)n/\delta$. Thus by (3.7),

$$\begin{aligned}\frac{V(\overline{C}) - 1}{|C \cap A_1|} &> \frac{V(\overline{C}) - 1}{\frac{\delta \cdot V(\overline{C})}{1 - k_m}} \\ &= \frac{1 - k_m}{\delta}\left(1 - \frac{1}{V(\overline{C})}\right) \\ &> \frac{1 - k_m}{\delta}\left[1 - \frac{\delta}{(1 - k_m)n}\right].\end{aligned} \tag{3.8}$$

And then from (3.6) and (3.8),

$$\begin{aligned} N_m - N_{2m} &\geq \sum_{\overline{C}} (V(\overline{C}) - 1) \\ &\geq \sum \{ V(\overline{C}) - 1 | C \cap A_1 \neq \varnothing \} \\ &> \sum_{C} |C \cap A_1| \frac{1 - k_m}{\delta} \left[1 - \frac{\delta}{(1 - k_m)n} \right] \\ &= \frac{1 - k_m}{\delta} \left[1 - \frac{\delta}{(1 - k_m)n} \right] |A_1| \\ &\geq \frac{\delta - 1}{\delta^2} (1 - k_m)^2 \left[1 - \frac{\delta}{(1 - k_m)n} \right] |A|. \qquad \square \end{aligned}$$

Proof of Theorem 3.3. Suppose $\varphi(S) < 1 - 1/n$. Then WFC holds for some $k < 1 - 1/n$. Choose δ so that $1 < \delta < \sqrt{n(1 - k)}$, and an integer l so that

$$\text{(3.9)} \qquad l > \frac{n\delta^5}{(\delta - 1)^2 (1 - k)^2 [(1 - k)n - \delta^2]} + \frac{k\delta}{(1 - k)(\delta - 1)}.$$

By WFC, there exists a finite subset A of S such that

$$\frac{1}{(l + 1)n} \sum_{i=0}^{l} \sum_{r \in F^{2^i}} |A \setminus rA| \leq k|A|.$$

Adapting the above notations, we have

$$\frac{1}{l + 1} \sum_{i=0}^{l} k_{2^i} \leq k, \quad \text{or} \quad \frac{1}{l + 1} \sum_{i=0}^{l} (1 - k_{2^i}) \geq 1 - k.$$

Let $K = \{2^i | i = 0, 1, \ldots, l - 1;\ 1 - k_{2^i} > (1 - k)/\delta\}$. Then

$$\begin{aligned} 1 - k &\leq \frac{1}{l + 1} \sum_{i=0}^{l} (1 - k_{2^i}) \\ &= \frac{1}{l + 1} \left[(1 - k_{2^l}) + \sum_{i=0}^{l-1} (1 - k_{2_i}) \right] \\ &\leq \frac{1}{l + 1} \left[1 + l \frac{1 - k}{\delta} + \sum_{m \in K} (1 - k_m) \right] \\ &\leq \frac{1}{l + 1} \left[1 + l \frac{1 - k}{\delta} + |K| \right]. \end{aligned}$$

This and (3.9) imply

$$|K| \geq l(1-k)\frac{\delta-1}{\delta} - k > \frac{n\delta^4}{(\delta-1)(1-k)[(1-k)n-\delta^2]}.$$

Finally, by Lemma 3.5,

$$\begin{aligned}
|A| &\geq \sum_{i=0}^{l-1} (N_{2^i} - N_{2^{i+1}}) \\
&\geq \sum \{ N_m - N_{2m} | m \in K \} \\
&> \sum_{m \in K} \frac{\delta-1}{\delta^2}(1-k_m)^2\left[1 - \frac{\delta}{(1-k_m)n}\right]|A| \\
&> \frac{\delta-1}{\delta^2}\left(\frac{1-k}{\delta}\right)^2\left[1 - \frac{\delta^2}{(1-k)n}\right]|K|\,|A| \\
&> |A|,
\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 3.3. □

COROLLARY 3.6. *Let S be a right cancellative semigroup. If there exists a finite subset F of S satisfying conditions* (i)–(iii) *of Theorem* 3.3, *then S does not satisfy* SNFC.

COROLLARY 3.7. *Let S be a right cancellative semigroup. If there exists a sequence $\{F_n\}$ of finite subsets of S satisfying conditions* (ii) *and* (iii) *of Theorem* 3.3, *and $|F_n| \to \infty$, then S does not satisfy* WFC.

REMARK 3.8. The conclusion $\varphi(S) \geq 1 - 1/n$ is the best possible. For consider the semigroup $\{a_1, \ldots, a_n\}$ with the operation $a_i a_j = a_i$. It is easy to check that this semigroup, with F equal to itself, satisfies all the conditions of Theorem 3.3, and $\varphi(S) = 1 - 1/n$ by Theorem 2.3.

For later application we need a slightly different form of Theorem 3.3.

THEOREM 3.9. *Let S be a semigroup with the finite intersection property for right ideals. Suppose S has a finite subset F with the following properties*:

(i) $|F| = n \geq 2$.

(ii) $\forall r, s, t \in F$, $rs = rt$.

(iii)′ $\forall r_1, r_2 \in F, \forall s, t \in S$, $r_1 s R r_1 t \Leftrightarrow r_2 s R r_2 t$.

(iv) *Different elements of F belong to different right cancellative classes*; *i.e.*, $\forall r_1, r_2 \in F$, $r_1 R r_2 \Rightarrow r_1 = r_2$.

Then $\varphi(S) \geq 1 - 1/n$. (See the last part of §2 *for the relation R.)*

To prove Theorem 3.9, we need to change the equivalence relation $\sim_m$ into $\sim'_m$ defined by

$$s \sim'_m t \Leftrightarrow \exists r \in F^m, rsRrt$$

in the proof of Theorem 3.3. All the rest works with little modification.

4. Semidirect products and left amenability

For a semigroup U, we denote by $\mathrm{End}(U)$ the semigroup of all endomorphisms of U. Similarly, $\mathrm{Inj}(U)$ and $\mathrm{Sur}(U)$ will be the semigroups of all injective or surjective endomorphisms of U, respectively. And $\mathrm{Aut}(U) = \mathrm{Inj}(U) \cap \mathrm{Sur}(U)$.

Let U and T be two semigroups, ρ a homomorphism of T into $\mathrm{End}(U)$. The semidirect product of U by T (with respect to ρ) is the set $U \times T$ associated with the multiplication $\langle u, a\rangle\langle v, b\rangle = \langle u\rho_a(v), ab\rangle$, denoted by $U \times_\rho T$. It is also a semigroup.

Maria Klawe [8] initiated the study of semidirect products for amenable semigroups. For convenience, we collect some of her results here (Propositions 4.1–4.5).

PROPOSITION 4.1. *If U and T are right cancellative, so is $S = U \times_\rho T$. If U and T are left cancellative, then S is left cancellative iff $\rho(T) \subset Inj(U)$.*

PROPOSITION 4.2. *If U and T are left amenable and $\rho(T) \subset \mathrm{Sur}(U)$, then $S = U \times_\rho T$ is also left amenable.*

PROPOSITION 4.3. *If $S = U \times_\rho T$ is left amenable, then U and T are left amenable.*

PROPOSITION 4.4. *If U and T satisfy* SFC *and $\rho(T) \subset \mathrm{Aut}(T)$, then $S = U \times_\rho T$ also satisfies* SFC.

PROPOSITION 4.5. *If $S = U \times_\rho T$ satisfies* SFC, *then U and T also satisfy* SFC.

From those results we see that if U and T are two left amenable cancellative semigroups, ρ: $T \to \mathrm{Sur}(U)$ a homomorphism such that $\rho(T) \not\subset \mathrm{Inj}(U)$, then $S = U \times_\rho T$ is left amenable, right cancellative, but not left cancellative. So it does not satisfy SFC (see [8] or our Theorem 3.1). The following example is given by Klawe.

Example 4.6 [8]. Let U be the free abelian semigroup generated by the elements $\{u_i | i = 0, 1, 2, \dots\}$, and T the infinite cyclic semigroup with generator a. Define ρ: $T \to \mathrm{Sur}(U)$ by $\rho_a(u_i) = u_{i-1}$ if $i \geq 1$ and $\rho_a(u_0) = u_0$. Since $\rho_a \notin \mathrm{Inj}(U)$, the semidirect product $S = U \times_\rho T$ is left amenable but does not satisfy SFC.

In the rest part of this section, we will use Klawe's example 4.6 to solve Namioka's problem and Klawe's problem on the homomorphic image of a semigroup with SFC. Then we will give some necessary and sufficient conditions for a semidirect product to be left amenable.

PROPOSITION 4.7. *There exist left amenable semigroups with Følner number equal to* 1. *So none of* SNFC, WNFC *or* WFC *is necessary for a semigroup to be left amenable.*

Proof. Klawe's example S is left amenable and right cancellative. Let

$$F_n = \{\langle u_0^{j-1} u_1^{n-j}, a\rangle \mid j = 1, \dots, n\},$$

where $u^0 u^n$ is understood to be u^n. Then F_n satisfies conditions (i)–(iii) of Theorem 3.3 with $|F_n| = n$. So $\varphi(S) = 1$. (This also can be obtained directly from Theorem. 5.1). □

Klawe [8] asked whether homomorphic images of semigroups satisfying SFC also satisfy SFC. We now show that Klawe's example is a homomorphic image of some semigroup having SFC.

PROPOSITION 4.8. *There exists a semigroup X and a homomorphism h from X such that $\varphi(X) = 0$ and $\varphi(h(X)) = 1$.*

Proof. Let Y be the free abelian semigroup generated by $\{u_i \mid i \in \mathbf{Z}\}$, U, T and ρ as in 4.6. Define τ: $T \to \mathrm{Aut}(Y)$ by $\tau_a(u_i) = u_{i-1}$, for $i \in \mathbf{Z}$. Let $X = Y \times_\tau T$. Then $\varphi(X) = 0$ by Proposition 4.4. Define a homomorphism h': $Y \to U$ by

$$h'(u_i) = \begin{cases} u_i, & i \geq 1; \\ u_0, & i \leq 0. \end{cases}$$

Note that $h' \circ \tau_a = \rho_a \circ h'$. Now define h: $X \to S = U \times_\rho T$ by

$$h(\langle x, a^n\rangle) = \langle h'(x), a^n\rangle.$$

Then

$$\begin{aligned} h(\langle x, a^n\rangle\langle y, a^m\rangle) &= h(\langle x\tau_{a^n}(y), a^{n+m}\rangle) \\ &= \langle h'(x)h'(\tau_{a^n}(y)), a^{n+m}\rangle \\ &= \langle h'(x)\rho_{a^n}(h'(y)), a^{n+m}\rangle \\ &= \langle h'(x), a^n\rangle\langle h'(y), a^m\rangle \\ &= h(\langle x, a^n\rangle)h(\langle y, a^m\rangle). \end{aligned}$$

So h is a homomorphism of X onto S. By Proposition 4.7, $\varphi(S) = 1$. □

Among other properties of S, we point out that any left amenable subsemigroup of S has Følner number either 0 or 1, and any finite generated left amenable subsemigroup of S is abelian. The proofs are omitted.

Now we give two necessary and sufficient conditions for a semidirect product to be left amenable. In the next section we will give necessary and sufficient conditions for a semidirect product to satisfy SFC.

THEOREM 4.9. *Let U and T be two left amenable semigroups, $\rho: T \to \mathrm{End}(U)$ a homomorphism. Then the following are equivalent:*

(i) $S = U \times_\rho T$ *is left amenable;*

(ii) $S = U \times_\rho T$ *has the finite intersection property for rights ideals;*

(iii) $\forall u \in U, \forall a \in T, u\rho_a(U) \cap \rho_a(U) \neq \varnothing$.

Proof. (i) $\Rightarrow$ (ii). This is a well-known fact (see [7]).

(ii) $\Rightarrow$ (iii). Take $u \in U$, $a \in T$. By (ii), $\langle u, a\rangle S \cap \langle \rho_a(u), a\rangle S \neq \varnothing$. This implies that $u\rho_a(U) \cap \rho_a(u)\rho_a(U) = u\rho_a(U) \cap \rho_a(uU) \neq \varnothing$.

(iii) $\Rightarrow$ (i). For each $a \in T$, define a linear operator P_a on $m(U)$ by $P_a g(u) = g(\rho_a(u))$ for $g \in m(U)$ and $u \in U$. Each P_a induces a dual operator P_a^* on $m(U)^*$ given by $P_a^*\psi(g) = \psi(P_a g)$ for $\psi \in m(U)^*$ and $g \in m(U)$. Obviously when ψ is a mean on $m(U)$, $P_a^*\psi$ is also a mean on $m(U)$. Suppose ψ is a left invariant mean in $m(U)$, $v \in U$. By (iii), there are $x, y \in U$, such that $v\rho_a(x) = \rho_a(y)$. We have

$$\begin{aligned} P_a^*\psi(l_v g) &= \psi(P_a(l_v g)) = \psi(l_x P_a(l_v g)) = \psi\left(P_a\left(l_{v\rho_a(x)} g\right)\right) \\ &= \psi\left(P_a\left(l_{\rho_a(y)} g\right)\right) = \psi(l_y(P_a(g)) = \psi(P_a g) = P_a^*\psi(g). \end{aligned}$$

Thus $P_a^*\psi$ is also a left invariant mean. As in the proof of [8, Lemma 3.3 and Prop. 3.4], the map $a \to P_a^*$ is a representation of T in the set of linear mappings on the set $ML(U)$ of all left invariant means on $m(U)$. Since $ML(U)$ is w^*-compact and convex, by the fixed point theorem [4, Theorem 6.1] there exists $\psi \in ML(U)$ with $P_a^*\psi = \psi$ for each $a \in T$. For each $f \in m(S)$ define $\bar{f} \in m(T)$ by $\bar{f}(a) = \psi(f_a)$, where $f_a \in m(U)$ is defined as $f_a(u) = f(u, a)$. Choose $\nu \in ML(T)$ and define $\mu \in m(S)^*$ by $\mu(F) = \nu(\bar{f})$. It follows by routine computation that μ is a left invariant mean on S (see [8, Prop. 3.4]). So S is left amenable. □

COROLLARY 4.10. *Let U and T be two left amenable semigroups, $\rho: T \to \mathrm{End}(U)$ a homomorphism. If for any $a \in T$, $\rho_a(U)$ contains a right ideal of U, then $S = U \times_\rho T$ is left amenable.*

Proof. Take $u \in U$ and $a \in T$. Since $\rho_a(U)$ contains a right ideal, $u\rho_a(U)$ also contains a right ideal. U as a left amenable semigroup has the finite intersection property for right ideals. Therefore $u\rho_a(U) \cap \rho_a(U) \neq \varnothing$. □

Examples 4.11. We give some applications of Theorem 4.9 and Corollary 4.10.

(i) Let $U = \{q \in \mathbf{Q} | q \geq 1\}$ with the usual addition. $T = \{r \in \mathbf{Q} | r \geq 1\}$ with the usual multiplication. T acts on U in the way that $\rho_r(q) = rq$, $r \in T$, $q \in U$. Since for any $r \in T$, $\rho_r(U) = \{q \in \mathrm{U} | q \geq r\}$ is an ideal in U, by Corollary 4.10, $S = U \times_\rho T$ is left amenable.

(ii) Let $\mathbf{Q}^+$ be the set of nonnegative rationals, $\mathbf{Z}^+$ that of integers, with the usual addition. Let $U = \mathbf{Q}^+ \oplus \mathbf{Z}^+$, T the infinite cyclic semigroup generated by a. Define $\rho_a(\langle r, n\rangle) = \langle r + n, n\rangle$. Then $\rho_a(U)$ does not contain any ideal of U. But by Theorem 4.9, $S = U \times_\rho T$ is still left amenable.

5. Semidirect products and Følner type conditions

For left cancellative semigroups, finite semigroups, and abelian semigroups, SFC, SNFC and WNFC are all equivalent (to the left amenability). It is natural to ask whether these conditions are equivalent in general. In this section we will prove that for a semidirect product of two semigroups satisfying SFC, they are equivalent (to LA + WFC).

If a semigroup S has the finite intersection property for right ideals and its right cancellative quotient semigroup $S/(R)$ is left cancellative, we say S satisfies Sorenson's condition. See [12] for Sorenson's conjecture. It is known that S satisfies SFC if and only if S is left amenable and satisfies Sorenson's condition (cf. [1] and [8]).

Let U be a semigroup with the finite intersection property for right ideals, and $h \in \mathrm{End}(U)$. Since sRt implies $h(s)Rh(t)$, h can be reduced to

$$\bar{h} \in \mathrm{End}(U/(R)),$$

defined by $\bar{h}(\bar{s}) = \overline{h(s)}$. And for $h_1, h_2 \in \mathrm{End}(U)$, $\bar{h}_1 \circ \bar{h}_2 = \overline{h_1 \circ h_2}$. If ρ: $T \to \mathrm{End}(U)$ is a homomorphism from another semigroup T into $\mathrm{End}(U)$, then we can define $\bar{\rho}$: $T \to \mathrm{End}(U/(R))$ by $\bar{\rho}_a = \overline{\rho_a}$. $\bar{\rho}$ is also a homomorphism.

THEOREM 5.1. *Let U and T be two semigroups where U satisfies Sorenson's condition. Suppose ρ: $T \to \mathrm{End}(U)$ is a homomorphism such that*

$$\bar{\rho}(T) \not\subset \mathrm{Inj}(U/(R)).$$

Then the semidirect product $S = U \times_\rho T$ is either not left amenable or $\varphi(S) = 1$. In both cases S does not satisfy WNFC.

Proof. For convenience we write $\sim$ for the right cancellative relation R on U. Sorenson's condition implies that $\forall u, v, w \in U$, $wu = wv \Rightarrow u \sim v$.

Assume that S is left amenable and $\bar{\rho}(T) \not\subset \text{Inj}(U/(R))$. Then there exists $a \in T$ and $u, v \in U$ such that $u \nsim v$ but $\rho_a(u) \sim \rho_a(v)$.

We claim that for any positive integer n, there are two elements $u_n, v_n \in U$ such that $\rho_{a^{n-1}}(u_n) \nsim P_{a^{n-1}}(v_n)$ but $\rho_{a^n}(u_n) = \rho_{a^n}(v_n)$.

Take $w \in U$ with $\rho_a(u)w = \rho_a(v)w$. Since S is left amenable, by Theorem 4.9, $wU \cap \rho_a(U) \neq \varnothing$. Choose $w' \in U$ with $\rho_a(w') \in wU$. Then $\rho_a(uw') = \rho_a(vw')$, and $uw' \nsim vw'$, since $u \nsim v$. Let $u_1 = uw'$ and $v_1 = vw'$.

Suppose $n \geq 2$. Again since S is left amenable,

$$\langle u_1, a^{n-1}\rangle S \cap \langle v_1, a^{n-1}\rangle S \neq \varnothing .$$

Therefore $u_1\rho_{a^{n-1}}(U) \cap v_1\rho_{a^{n-1}}(U) \neq \varnothing$. Choose $w', w'' \in U$ so that

$$u_1\rho_{a^{n-1}}(w') = v_1\rho_{a^{n-1}}(w''). \tag{5.1}$$

Since $u_1 \nsim v_1$, $\rho_{a^{n-1}}(w') \nsim \rho_{a^{n-1}}(w'')$. Applying ρ_a to both sides of (5.1), we get $\rho_a(u_1)\rho_{a^n}(w') = \rho_a(v_1)\rho_{a^n}(w'') = \rho_a(u_1)\rho_{a^n}(w'')$. Sorenson's condition on U gives that $\rho_{a^n}(w') \sim \rho_{a^n}(w'')$. By the same argument as in the previous paragraph, we can find $w \in U$, with $\rho_{a^n}(w'w) = \rho_{a^n}(w''w)$, and also $\rho_{a^{n-1}}(w'w) \nsim \rho_{a^{n-1}}(w''w)$. Let $u_n = w'w$ and $v_n = w''w$.

Let

$$F_n = \{\langle w_1w_2 \dots w_n, a^n\rangle \in S \mid w_i = u_i \text{ or } v_i\}.$$

Then F_n satisfies conditions (i), (ii), (iii)′ and (iv) in Theorem 3.9 with $|F_n| = 2^n$, as we will show.

(i) and (iv). We prove by induction that any two different words $w_1w_2 \dots w_n$ are not in the same right cancellative class of U. This implies (i) and (iv).

Suppose this is true for $n = k - 1 \geq 1$. Let

$$F_k' = \{w_1w_2 \dots w_{k-1}u_k \mid w_i = u_i \text{ or } v_i\},$$

and

$$F_k'' = \{w_1w_2 \dots w_{k-1}v_k \mid w_i = u_i \text{ or } v_i\}.$$

By the induction hypothesis and the fact that $ac \sim bc \Rightarrow a \sim b$, each set F_k' or F_k'' satisfies our requirement. Let $w_1 \dots w_{k-1}u_k \in F_k'$ and $w_1' \dots w_{k-1}'v_k \in F_k''$. If they are in the same right cancellative class, then

$$\begin{aligned}
&\rho_{a^{k-1}}(u_1)\rho_{a^{k-1}}(u_2) \dots \rho_{a^{k-1}}(u_{k-1})\rho_{a^{k-1}}(u_k) \\
&\quad = \rho_{a^{k-1}}(w_1w_2 \dots w_{k-1}u_k) \\
&\quad \sim \rho_{a^{k-1}}(w_1'w_2' \dots w_{k-1}'v_k) \\
&\quad = \rho_{a^{k-1}}(u_1)\rho_{a^{k-1}}(u_2) \dots \rho_{a^{k-1}}(u_{k-1})\rho_{a^{k-1}}(v_k).
\end{aligned}$$

Since U satisfies Sorenson's condition, we have

$$\rho_{a^{k-1}}(u_k) \sim \rho_{a^{k-1}}(v_k).$$

This contradicts to our choice for u_k and v_k.

(ii) This follows from the fact that

$$\rho_{a^n}(w_1 w_2 \dots w_n) = \rho_{a^n}(u_1)\rho_{a^n}(u_2)\dots\rho_{a^n}(u_n).$$

(iii)′ For $s \in S$, write $s = \langle P_1(s), P_2(s)\rangle$. Suppose $r_1, r_2 \in F_n$ and $s, t \in S$ are such that there exists $x \in S$, $r_2 sx = r_1 tx$. Equivalently we have

$$P_1(r_1)\rho_{a^n}(P_1(sx)) = P_1(r_1)\rho_{a^n}(P_1(tx)), \tag{5.2}$$

and

$$a^n P_2(sx) = a^n P_2(tx), \tag{5.3}$$

by the definition of semidirect products. By Sorenson's condition, there exists $w \in U$ such that $\rho_{a^n}(P_1(sx))w = \rho_{a^n}(P_1(tx))w$. Theorem 4.9 gives

$$wU \cap \rho_{a^n P_2(sx)}(U) \neq \varnothing.$$

Thus there exists $w' \in U$ such that

$$\rho_{a^n}(P_1(sx))\rho_{a^n P_2(sx)}(w') = \rho_{a^n}(P_1(tx))\rho_{a^n P_2(tx)}(w'), \tag{5.4}$$

since $a^n P_2(sx) = a^n P_2(tx)$. Let $y = x\langle w', a\rangle$. Then it is easy to check that

$$\rho_{a^n}(P_1(sy)) = \rho_{a^n}(P_1(ty)) \quad \text{and} \quad a^n P_2(sy) = a^n P_2(ty),$$

by (5.4) and (5.3). It follows that $r_2 sy = r_2 ty$.

S as a left amenable semigroup has the finite intersection property for right ideals. So by Theorem 3.9, $\varphi(S) = 1$. □

COROLLARY 5.2. *Let U and T be two semigroups where U satisfies* SFC *and T is left amenable. Suppose $\rho\colon T \to \operatorname{End}(U)$ is a homomorphism satisfying condition* (iii) *in Theorem* 4.9 *and such that $\bar{\rho}(T) \not\subset \operatorname{Inj}(U/(R))$. Then the semidirect product $S = U \times_\rho T$ is left amenable and $\varphi(S) = 1$; i.e., S does not satisfy* WFC.

Proof. By Theorem 4.9, S is left amenable. □

This corollary gives a large class of counterexamples for Namioka's problem.

Now we consider the conditions under which S satisfies SFC.

Let U and T be two semigroups satisfying SFC, and $\rho\colon T \to \operatorname{End}(U)$ a homomorphism. Suppose $S = U \times_\rho T$ is left amenable, and $\bar{\rho}(T) \subset \operatorname{Inj}(U/(R))$. Note that those conditions are necessary for S to satisfy SFC by Proposition 4.5 and Theorem 5.1.

Let $\langle u, a\rangle, \langle v, b\rangle \in S$, and suppose that there exists $\langle w, c\rangle \in S$, such that $\langle w, c\rangle\langle u, a\rangle = \langle w, c\rangle\langle v, b\rangle$; i.e., $w\rho_c(u) = w\rho_c(v)$ and $ca = cb$. Since U and T satisfy Sorenson's condition, there is $x \in U$ and $d \in T$, such that

$$\rho_c(u)x = \rho_c(v)x \quad \text{and} \quad ad = bd. \tag{5.5}$$

$\rho_c(u) \sim \rho_c(v)$ and $\bar{\rho}(U) \subset \operatorname{Inj}(U/(R))$ imply $u \sim v$. So there exists $x_1 \in U$ with $ux_1 = vx_1$. Since S is left amenable, $x_1U \cap \rho_{ad}(U) \neq \varnothing$ by Theorem 4.9. Hence we can find $x_2 \in U$ such that $u\rho_{ad}(x_2) = v\rho_{ad}(x_2) = v\rho_{bd}(x_2)$, or

$$u\rho_a(\rho_d(x_2)) = v\rho_b(\rho_d(x_2)). \tag{5.6}$$

It follows from (5.5) and (5.6) that

$$\langle u, a\rangle\langle \rho_d(x_2), d\rangle = \langle v, b\rangle\langle \rho_d(x_2), d\rangle.$$

Thus we have proved that S satisfies Sorenson's condition. But S is left amenable, so we obtain the following result.

LEMMA 5.3. *Let U and T be two semigroups satisfying* SFC, *and let*

$$\rho\colon T \to \operatorname{End}(U)$$

be a homomorphism. If $\bar{\rho}(T) \subset \operatorname{Inj}(U/(R))$ and condition (iii) *of Theorem* 4.9 *holds for ρ, then $S = U \times_\rho T$ satisfies* SFC.

Summing up the above results, we obtain the main theorem of this section.

THEOREM 5.4. *Let U and T be two semigroups satisfying* SFC, *and let*

$$\rho\colon T \to \operatorname{End}(U)$$

be a homomorphism. Let $S = U \times_\rho T$ be the semidirect product. Then the following are equivalent:

(1) *S satisfies* SFC.
(2) *S satisfies* SNFC.
(3) *S satisfies* WNFC.
(4) *S is left amenable and satisfies* WFC.
(5) $\bar{\rho}(T) \subset \operatorname{Inj}(U/(R))$ *and* $\forall u \in U, \forall a \in T$, $u\rho_a(U) \cap \rho_a(U) \neq \varnothing$.

Proof. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from the diagram of implications in §1. Also (4) $\Rightarrow$ (5) is an application of Theorem 4.9 and Theorem 5.1; (5) $\Rightarrow$ (1) is the above lemma. □

If U and T are cancellative, then $\bar{\rho} = \rho$ and $U/(R) = U$, also the left amenability of U and T is equivalent to SFC. By Proposition 4.3, this is a consequence of each of (1), (2), (3) or (4).

COROLLARY 5.5. *Let U and T be two cancellative semigroups, and let*

$$\rho\colon T \to \mathrm{End}(U)$$

be a homomorphism. Let $S = U \times_\rho T$ *be the semidirect product. Then the following are equivalent*:

(1) *S satisfies* SFC.
(2) *S satisfies* SNFC.
(3) *S satisfies* WNFC.
(4) *S is left amenable and satisfies* WFC.
(5) *U and T are left amenable*, $\rho(T) \subset \mathrm{Inj}(U)$, *and* $\forall u \in U$, $\forall a \in T$, $u\rho_a(U) \cap \rho_a(U) \neq \varnothing$.

Problem 5.6. Is there any left amenable semigroup S such that $0 < \varphi(S) < 1$? If not, then all the conditions SFC, SNFC, WNFC and LA + WFC are equivalent. We know that such an example cannot be finite, or abelian, or left cancellative, or a semidirect product of those "better" semigroups. Our Section 3 is in this direction. But we only get a lower bound $1/6$ (Corollary 3.2).

REFERENCES

1. L.N. ARGABRIGHT and C.O. WILDE, *Semigroups satisfying a strong Følner condition*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 587–591.
2. A.H. CLIFFORD and G.B. PRESTON, *The algebraic theory of semigroups I*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R.I., 1961.
3. M.M. DAY, *Amenable semigroups*, Illinois J. Math, vol. 1 (1957), pp. 509–544.
4. ______, *Semigroups and amenability*, *semigroups*, K. Folley, Ed., Academic Press, New York, 1969, pp. 5–53.
5. E. FØLNER, *On groups with full Banach mean values*, Math. Scand., vol. 3 (1955), pp. 243–254.
6. A.H. FREY, JR., *Studies on amenable semigroups*, Thesis, University of Washington, 1960.
7. E.E. GRANIRER, *A theorem on amenable semigroups*, Trans. Amer. Math. Soc., vol. 111 (1964), pp. 367–379.
8. M.M. KLAWE, *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner condition*, Pacific J. Math., vol. 73 (1977), pp. 91–106.
9. I. NAMIOKA, *Følner conditions for amenable semigroups*, Math. Scand., vol. 15 (1964), pp. 18–28.
10. J.P. PIER, *Amenable locally compact groups*, John Wiley and Sons, New York, 1984.

11. W.G. ROSEN, *On invariant means over compact groups*, Proc. Amer. Math. Soc., vol. 7 (1956), pp. 1076–1082.
12. J.R. SORENSON, *Existence of measures that are invariant under a semigroup of transformations*, Thesis, Purdue University, 1966.
13. C.O. WILDE and L.N. ARGABRIGHT, *Invariant means and factor-semigroups*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 226–228.
14. J.C.S. WONG, *On Følner conditions and Følner numbers for semigroups*, to appear.

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