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AFFINE SURFACES FIBERED BY AFFINE LINES OVER THE PROJECTIVE LINE

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0. Pinchuk's example and Peretz' follow-up

The classical Jacobian conjecture asserts that if k is a field of characteristic zero and $\varphi\colon \mathbb{A}_k^2 \to \mathbb{A}_k^2$ is a polynomial map whose Jacobian determinant is a non-zero constant, then φ has a polynomial inverse. A related conjecture, the "real Jacobian conjecture", asserted that if $k = \mathbb{R}$ and the Jacobian determinant of φ is non-vanishing, then φ is a global homeomorphism on $\mathbb{R}^2$. This latter statement was shown by S. Pinchuk to be false by virtue of the following counter-example:

Pinchuk's example. Let X and Y be variables, and let

$$\begin{aligned} t &= XY - 1 \\ h &= t(Xt+1) \\ f &= (Xt+1)^2\left(\frac{h+1}{X}\right). \end{aligned}$$

Furthermore, let $p, q \in \mathbb{R}[X, Y]$ be defined by

$$\begin{aligned} p &= f + h \\ q &= -t^2 - 6th(h+1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75h^3 f - \frac{75}{4}h^4. \end{aligned}$$

Then

$$\frac{\partial(p,q)}{\partial(X,Y)} = t^2 + [t + (13+15h)f]^2 + f^2. \tag{1}$$

(This equation can be verified by a symbolic algebra computer program.) One quickly sees that $Xf \equiv 1 \pmod{t}$, hence $\partial(p,q)/\partial(X,Y)$ has no real zeros; i.e., the map $\varphi\colon \mathbb{A}_{\mathbb{R}}^2 \to \mathbb{A}_{\mathbb{R}}^2$ defined by (p,q) is unramified at all real points. The locus $p = 0$ contains the component $Xt + 1 = 0$, which can be written as $Y = (X-1)/X^2$, which is disconnected. It follows that $p = 0$ is not both smooth and connected, hence φ is not a diffeomorphism on $\mathbb{R}^2$. Thus this polynomial map is a counter-example to the "real Jacobian conjecture." The reader is referred to [11] for details.

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Follow-up by Peretz. In [10], Ronen Peretz observed that the polynomials p and q in Pinchuk's example lie in the subring $\mathbb{R}[t, h, f] \subset \mathbb{R}[Y, XY, X^2Y - X]$. He recognized the latter ring with $\mathbb{R}$ replaced by $\mathbb{C}$ as "merely a special case of the type of rings that arise in the theory of assymptotics of polynomials" [10, §2]. Peretz showed there does not exist a pair of polynomials $p, q \in \mathbb{C}[Y, XY, X^2Y - X]$ with $\partial(p, q)/\partial(X, Y)$ non-vanishing (i.e., constant) on $\mathbb{A}^2_{\mathbb{C}}$. This fact is essentially the special case $m = 2$ of the following more general theorem, which appears as Theorem 4 in [10]:

THEOREM 0.1 (PERETZ). *There does not exist a pair of polynomials*

$$p, q \in \mathbb{C}[Y, XY, X^2Y + \alpha X, X^3Y + \alpha X^2, \ldots, X^mY + \alpha X^{m-1}],$$

where $\alpha \in \mathbb{C}^*$,*with* $\frac{\partial(p,q)}{\partial(X,Y)}$ *non-vanishing (i.e., constant) on* $\mathbb{A}^2_{\mathbb{C}}$.

In §3 of this paper we will generalize Peretz' theorem by giving a larger class of subrings of $\mathbb{C}[X, Y]$ which could not contain such p and q (Theorem 3.3). We will furthermore show that the rings in this larger class are precisely the affine coordinate rings of affine surfaces which are $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^1_{\mathbb{C}}$, which are studied in §2. In §4 we provide some evidence that such objects are significant in the study of the Jacobian conjecture.

1. Geometric interpretation of the case $m = 2$

Let k be a field of characteristic zero. We first consider the ring $k[Y, XY, X^2Y - X]$, which, for $k = \mathbb{R}$, contains the polynomials p and q of Pinchuk's example. For $k = \mathbb{C}$ this is the ring that appears in the above theorem of Peretz, for $m = 2$. We will give geometric reasons why no polynomials p, q from this ring could have constant nonzero jacobian determinant.

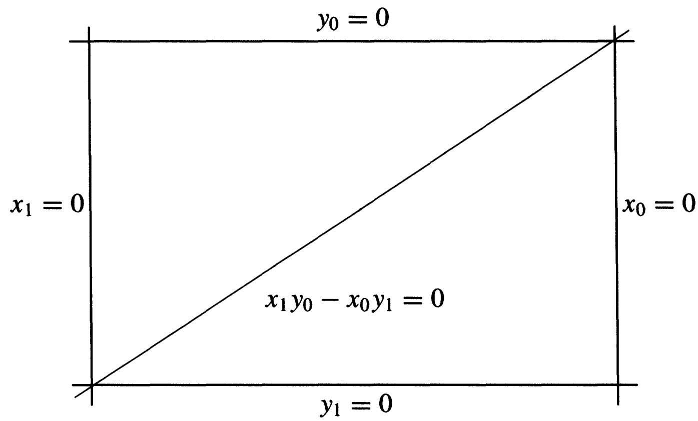
PROPOSITION 1.1. *Let k be a field, and let $V = \mathbb{P}^1_k \times \mathbb{P}^1_k - \Delta$, where Δ is the diagonal. V is an affine variety, and the ring $k[Y, XY, X^2Y - X]$ can be realized as its coordinate ring in such a way that the containment $k[X, Y] \supset k[Y, XY, X^2Y - X]$ corresponds to the open embedding of $\mathbb{A}^2_k$ in V which identifies $\mathbb{A}^2_k$ with the complement of a fiber of one of the standard projections $V \to \mathbb{P}^1$.*

Proof. We will appeal to two facts which will be proved later in this paper. That V is affine follows from Theorem 2.3.[1] Realizing $\mathbb{P}^1_k \times \mathbb{P}^1_k$ as $\{(x_0 : x_1), (y_0 : y_1)\}$,

[1]In this case, V is embedded in $\mathbb{P}^1_k \times \mathbb{P}^1_k$,which is the Nagata-Hirzebruch surface $\mathcal{F}_0$. In the notation of Theorem 2,3, we have $T = \Delta \sim D_0 + F$. This tells us $n = 0$ and $k = 1$, so the affineness of V follows from (3) $\Longrightarrow$ (1).

the diagonal Δ is defined by $x_1y_0 - x_0y_1 = 0$. Let U_0 be the complement in $V = \mathbb{P}^1_k \times \mathbb{P}^1_k - \Delta$ of $x_0 = 0$, and let U_1 be the complement in V of $x_1 = 0$. Then $V = U_0 \cup U_1$. This is all depicted in the following diagram.

$$\mathbb{P}^1_k \times \mathbb{P}^1_k = \{(x_0 : x_1), (y_0 : y_1)\}$$



Let $X = \frac{x_1}{x_0}$, and let $A_0 = k[X]$. The complement of $x_0 = 0$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is $\operatorname{Proj} A_0[y_0, y_1]$, and Δ is defined here by the equation $y_0X - y_1 = 0$ (homogeneous in y_0, y_1). Note that U_0 is the complement of Δ in $\operatorname{Proj} A_0[y_0, y_1]$, and since $A_0[y_0, y_1] = A_0[y_0, y_0X - y_1]$ we have

$$\begin{aligned} U_0 &= \operatorname{Spec} A_0\left[\frac{y_0}{y_0X - y_1}\right] \\ &= \operatorname{Spec} k[X, Y], \quad \text{where} \quad Y = \frac{y_0}{y_0X - y_1}. \end{aligned}$$

Setting $X' = \frac{x_0}{x_1} = X^{-1}$ and $A_1 = k[X']$, we similarly have

$$\begin{aligned} U_1 &= \operatorname{Spec} A_1\left[\frac{y_1}{y_0 - y_1X'}\right] \\ &= \operatorname{Spec} k[X', Y'], \quad \text{where} \quad Y' = \frac{y_1}{y_0 - y_1X'}. \end{aligned}$$

An easy computation shows $Y' = X^2Y - X$. Since $V = U_0 \cup U_1$, we have

$$\begin{aligned} \Gamma(V) &= \Gamma(U_0) \cap \Gamma(U_1) = k[X, Y] \cap k[X', Y'] \\ &= k[X, Y] \cap k[X^{-1}, X^2Y - X]. \end{aligned}$$

From Theorem 3.1, with $m = 2$, we obtain $\Gamma(V) = k[Y, XY, X^2Y - X]$, and $V = \operatorname{Spec} k[Y, XY, X^2Y - X]$. The containment $k[X, Y] \supset k[Y, XY, X^2Y - X]$ obviously corresponds to the embedding $U_0\,(\cong \mathbb{A}^2_k) \subset V$, so the proposition is proved. $\square$

Remark. For the case $k = \mathbb{R}$, we have

$$V = \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} - \Delta = \operatorname{Spec} \mathbb{R}[Y, XY, X^2Y - X]$$

Identifying $\mathbb{A}^2_{\mathbb{R}}$ with U_0 as above, we see that Pinchuk's map $\varphi = (p, q)$ extends to a map $\widetilde{\varphi}$: $V \to \mathbb{A}^2_{\mathbb{R}}$. The following proposition shows that the extended map $\widetilde{\varphi}$ "folds" (i.e., has vanishing jacobian determinant) along the complement $V - U_0$.

PROPOSITION 1.2. *The map $\widetilde{\varphi}$: $V \to \mathbb{A}^2_{\mathbb{R}}$ defined by Pinchuk's polynomials (p, q) has jacobian determinant zero at all points (real or complex) of $V - U_0$.*

Proof. In the notation of the previous proof, we have $V - U_0 \subset U_1 = \operatorname{Spec} \mathbb{R}[X', Y']$, where we calculate the jacobian determinant with respect to the variables X' and Y'. Since $X = X'^{-1}$ and $Y = X'^2Y' + X'$, we have

$$\begin{aligned}
\text{(2)} \qquad \frac{\partial(p,q)}{\partial(X',Y')} &= \frac{\partial(p,q)}{\partial(X,Y)} \cdot \frac{\partial(X,Y)}{\partial(X',Y')} \quad \text{(by the chain rule)} \\
&= \frac{\partial(p,q)}{\partial(X,Y)} \cdot \frac{\partial(X'^{-1}, X'^2Y' + X')}{\partial(X',Y')} \\
&= \frac{\partial(p,q)}{\partial(X,Y)} \cdot \begin{vmatrix} -\frac{1}{X'^2} & 0 \\ 2X'Y' + 1 & X'^2 \end{vmatrix} \\
&= \frac{\partial(p,q)}{\partial(X,Y)} \cdot (-1) \\
&= -(t^2 + [t + (13 + 15h)f]^2 + f^2) \quad \text{(by (1))}.
\end{aligned}$$

Writing t and f in terms of X' and Y', we get

$$t = X'Y', \quad f = (Y' + 1)^2[X'Y'(Y' + 1) + 1]X',$$

which shows that X' divides t and f in $k[X', Y']$. Therefore X'^2 divides $\partial(p, q)/\partial(X', Y')$. Since $X' = 0$ defines the complementary fiber in U_1, we see that $\partial(p, q)/\partial(X', Y')$ vanishes along it. □

In light of Proposition 1.1, Peretz' Theorem (0.1) is equivalent to the following unpublished theorem:

THEOREM 1.3 (KUMAR-MURTHY-NORI). *There does not exist $\widetilde{\varphi}$: $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} - \Delta \to \mathbb{A}^2_{\mathbb{C}}$ such that $\widetilde{\varphi}|_{U_0}$ is étale.*

Sketch of proof. The statement $\widetilde{\varphi}|_{U_0}$ is étale is equivalent to the assertion that $\partial(p, q)/\partial(X, Y)$ is a non-zero constant, i.e., $\widetilde{\varphi}|_{U_0}$ is unramified; flatness is automatic under this hypothesis [2, Ch. V, Prop. 3.5]. Therefore, by Proposition 1.1, this theorem is the $m = 2$ case of Theorem 3.3, so we only sketch the proof as conceived by Kumar, Murthy, and Nori. Let $V = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} - \Delta$, $A = \mathbb{C}[Y, XY, X^2Y - X] = \Gamma(V)$, and let X', Y', U_0, and U_1 be as in the proof of Proposition 1.1. Such a $\widetilde{\varphi}$ is given by $p, q \in A$ with $\partial(p,q)/\partial(X,Y) \in \mathbb{C}^*$. Kumar-Murthy-Nori observed that $\widetilde{\varphi}$ must in fact be étale on all of V. This results from the fact that $\partial(p,q)/\partial(X',Y') = -\partial(p,q)/\partial(X,Y)$ (as in (2) of the proof of Proposition 1.2). This also shows that $dp \wedge dq = dX \wedge dY = -dX' \wedge dY'$, and this 2-form is a generator for $\Omega^2_{A/\mathbb{C}}$, since it generates on both of the open sets U_0 and U_1. Hence $\Omega^2_{A/\mathbb{C}}$ is free. The containment $\mathbb{C}[p,q] \subset A$ induces from the De Rham sequences of $\mathbb{C}[p,q]$ and A the commutative diagram

$$\begin{array}{ccc} \Omega^1_{\mathbb{C}[p,q]/\mathbb{C}} & \xrightarrow[adb \mapsto da \wedge db]{d} & \Omega^2_{\mathbb{C}[p,q]/\mathbb{C}} \\ \downarrow & & \downarrow \\ \Omega^1_{A/\mathbb{C}} & \xrightarrow{d} & \Omega^2_{A/\mathbb{C}} \end{array}$$

in which we have

$$\begin{array}{ccc} p\,dq & \longmapsto & dp \wedge dq \\ \downarrow & & \downarrow \\ \omega & \longmapsto & dX \wedge dY \end{array}$$

for some $\omega \in \Omega^1_{A/\mathbb{C}}$. It is then shown that the equation $d\omega = dX \wedge dY$ is impossible because $dX \wedge dY$ is not integrable. This uses the graded structure on $A = \mathbb{C}[Y, XY, X^2Y - X]$ and $\Omega_{A/\mathbb{C}}$ determined by setting $\deg X = -1$, $\deg Y = 1$. Since $dX \wedge dY$ is homogeneous of degree 0, if it is integrable it should lift to a homogeneous 1-form of degree zero, which can be shown by a direct argument not to be the case. □

We conclude this section by again pointing out that $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} - \Delta$ is an affine variety, by Theorem 2.3, and observing that it is an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$ (via either of its two canonical projections onto $\mathbb{P}^1_{\mathbb{C}}$). In the next section we will describe the coordinate rings of all affine $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^1_{\mathbb{C}}$, and see that these include rings of the type which appear in Theorem 0.1, i.e. those of the form $\mathbb{C}[Y, XY, X^2Y + \alpha X, X^3Y + \alpha X^2, \ldots, X^mY + \alpha X^{m-1}]$, $m \geq 2$. We will then prove Theorem 3.3, which includes Theorem 0.1 and generalizes Theorem 1.3, replacing $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} - \Delta$ by a larger class of $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^1_{\mathbb{C}}$.

2. $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^1_{\mathbb{C}}$

We begin with some preliminaries. Let V be a variety over $\mathbb{C}$ (which in this discussion includes being reduced, irreducible, and separated). Given another variety X and a morphism $\pi\colon V \to X$, we say that V is an $\mathbb{A}^1_{\mathbb{C}}$-*bundle* over X (via π) if X

has a cover $\{X_i\}$ such that $\pi^{-1}(X_i)$ is compatibly isomorphic to $X_i \times \mathbb{A}^1_{\mathbb{C}}$ for all i. An obvious weaker condition is that π is a flat morphism and for each point $p \in X$, the scheme-theoretic fiber $\pi^{-1}(p)$ is isomorphic to $\mathbb{A}^1_{k(p)}$, $k(p)$ being the residue field at p, in which case we say V is an $\mathbb{A}^1$*-fibration* over X. In turn, a stronger condition is that V is a rank one vector bundle, or *line bundle*, over X. The main result of [7] asserts that if V is an $\mathbb{A}^1$-fibration over X, then it is an $\mathbb{A}^1_{\mathbb{C}}$-bundle.[2] Let us also note that if X is 1-dimensional, as in the case $X = \mathbb{P}^1_{\mathbb{C}}$, flatness is automatic, so that V is an $\mathbb{A}^1_{\mathbb{C}}$-bundle if and only if each fiber of π is an $\mathbb{A}^1$. The main result of [3] says that n-space bundles are vector bundles in the case where X is affine. The following easy theorem futher clarifies the relationship between $\mathbb{A}^1_{\mathbb{C}}$-bundles and line bundles:

LEMMA 2.1. *Let V be a variety with a map $\pi\colon V \to X$ making V an $\mathbb{A}^1_{\mathbb{C}}$-bundle over X. Then V is a line bundle if and only if π admits a section.*

Proof. A line bundle has a zero section (and possibly other global sections), so one implication is trivial. Conversely, assume V is an $\mathbb{A}^1_{\mathbb{C}}$-bundle and let $\{X_i\}$ be a cover of X such that $\pi^{-1}(X_i) \cong X_i \times \mathbb{A}^1_{\mathbb{C}}$. Then $\pi^{-1}(X_i)$ is the trivial line bundle over X_i, since its coordinate ring is a polynomial ring in one variable over $\Gamma(X_i)$, and any section of $\pi|_{X_i}$ gives rise to a choice of variable, unique up to multiplication by a unit in $\Gamma(X_i)$. We can view the $\mathbb{A}^1_{\mathbb{C}}$-bundle V as being constructed from gluing data over intersections $X_i \cap X_j$. The existence of a section provides a compatible choice of variable (i.e., a canonical "origin"), giving rise to sheaf of rank one projective $\mathcal{O}_X$-modules making V is a line bundle over X. □

The following is a well-known fact about on ruled surfaces.

LEMMA 2.2. *Let S be a non-singular projective surface, B a non-singular curve. Let $\tilde{\pi}\colon S \to B$ be a morphism making S a birationally ruled surface, i.e., S is birationally equivalent to $B \times \mathbb{P}^1_{\mathbb{C}}$ with $\tilde{\pi}$ being compatible with the projection $B \times \mathbb{P}^1_{\mathbb{C}} \to B$. Then the general fiber of $\tilde{\pi}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Every fiber not isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ is singular. Every singular fiber is a connected union of curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. In each singular fiber there exists a component having self-intersection -1. If E is a component having multiplicity one in a singular fiber, then there exists a component $E' \neq E$ of the same fiber with $(E'^2) = -1$.*

Proof. Compatibility of $\tilde{\pi}$ with the projection onto B implies that each exceptional curve for the birational map from S to $B \times \mathbb{P}^1_{\mathbb{C}}$ must be contained in some fiber of $\tilde{\pi}$. If we take remove from B the finite set of points whose fibers contain exceptional curves and/or fundamental points, we get an open set $B_0 \subset B$ such that

[2]This result is not known to be true for n-space fibrations for $n \geq 2$, except when $n = 2$ and π is an affine morphism [12].

$\tilde{\pi}^{-1}(B_0) \cong B_0 \times \mathbb{P}^1_{\mathbb{C}}$; hence the general fiber of $\tilde{\pi}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Thus $\tilde{\pi}$ satisfies the hypothesis of [9, Lemma 2.2, p. 115], which tells us all the facts asserted above (and more) regarding singular fibers. Finally, a non-singular fiber must have arithmetic genus zero, since arithmetic genus is constant amongst fibers of a flat morphism, hence it is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. □

THEOREM 2.3. *Let V be a variety with a map $\pi\colon V \to \mathbb{P}^1_{\mathbb{C}}$ making V an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$. Then V can be embedded as an open subvariety in a Nagata-Hirzebruch surface $\mathcal{F}_n$ in such a way that $V = \mathcal{F}_n - T$ where T is a section, and the canonical projection $\mathcal{F}_n \to \mathbb{P}^1_{\mathbb{C}}$ extends π. In this situation, $T \sim D_n + kF$ (D_n being the special section in $\mathcal{F}_n$, F a fiber). Moreover, the following conditions are equivalent*:

(1) *V is affine.*
(2) *V is not a line bundle (i.e., by virtue of Lemma 2.1, π does not admit a section).*
(3) $k \geq n + 1$.

The integers n and k are uniquely determined by V and π.

Proof. We assume the reader is familiar with the Nagata-Hirzebruch surfaces and their properties, as well as basic surface theory. We may embed V as an open subvartiety of a projective surface S, and by blowing up some points not in V we may assume π extends to a map $\tilde{\pi}\colon S \to \mathbb{P}^1_{\mathbb{C}}$, putting us in the situation of Lemma 2.2.

We claim that each reducible fiber of $\tilde{\pi}$ has a component not intersecting V with self-intersection -1. Let F be a reducible fiber, and let E be the (unique) component intersecting V. Since V is an $\mathbb{A}^1_{\mathbb{C}}$-bundle, E has multiplicity one in F, and Lemma 2.2 asserts the existence of another component with self-intersection -1, proving the claim.

We can contract the component whose existence is established above, and continue until this fiber, and every fiber, is irreducible, thus isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, by Lemma 2.2. The resulting surface S is then a $\mathbb{P}^1_{\mathbb{C}}$-bundle,[3] hence is isomorphic to one of the Nagata surfaces $\mathcal{F}_n$. Recall that the Picard group of $\mathcal{F}_n$ is freely generated by the classes of D_n (the special section) and F (a fiber), and that $(D_n^2) = -n$, $(F \cdot D_n) = 1$, and of course $(F^2) = 0$. This determines the intersection theory in $\mathcal{F}_n$. One sees that the complement $T = \mathcal{F}_n - V$ has one point in each fiber of $\tilde{\pi}$, hence it maps isomorphically to $\mathbb{P}^1_{\mathbb{C}}$. Clearly $(T, F) = 1$, and from this and the above information, one deduces that $T \sim D_n + kF$ for some integer k. Note, then, that $(T \cdot D_n) = ((D_n + kF) \cdot D_n) = (D_n^2) + k(F \cdot D_n) = -n + k$.

If V is affine it cannot contain a subvariety isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, hence π does not admit a section. Hence (1) $\Longrightarrow$ (2).

[3]This is because S is geometrically ruled; see [8, Ch V, §2].

Assume $k \leq n$. Then $(T \cdot D_n) \leq 0$. If $(T \cdot D_n) < 0$, then $T = D_n$ (since both are prime divisors). Therefore $V = \mathcal{F}_n - D_n$, which is known to be a line bundle. If $(T \cdot D_n) = 0$, then $D_n \subset V$, which shows that π admits a section. This establishes (2) $\Longrightarrow$ (3).

Assume $k \geq n + 1$. We know that V is affine if its complement T is the support of an ample divisor,[4] which, since T is irreducible, means T itself is ample. By the Nakai-Moishezon Criterion [8, Ch. V, Thm. 1.10, p. 365], we must show $(T \cdot C) > 0$ for all irreducible curves C. Since $T \sim D_n + kF$, we have $(T^2) = -n + 2k > 0$, so the condition holds for $C = T$. Also, $(T \cdot D_n) = -n + k > 0$ and $(T \cdot F) = 1$, verifying the condition when C is D_n or any fiber. Any other C must have positive intersection with a fiber and non-negative intersection with D_n. From this it follows that $(T \cdot C) > 0$. Therefore V is affine. This shows (3) $\Longrightarrow$ (1), completing the circle.

Lastly we establish the essential uniqueness the embedding $V \hookrightarrow \mathcal{F}_n$ extending π, from which will follow the uniqueness of n and k. Suppose V is also embedded in $\mathcal{F}_m$, as in the theorem, with $V = \mathcal{F}_m - T'$, T' being a section, with $T' \sim D_m + \ell F'$ (D_m the special section in $\mathcal{F}_m$, F' a fiber). This determines a π-compatible birational map ϕ: $\mathcal{F}_n \dashrightarrow \mathcal{F}_m$. The fact that ϕ is π-compatible implies that T (being a section for π) is not an exceptional curve for ϕ; i.e., T does not collapse. Hence T maps to T', and π is an isomorphism at all but finitely many points of T. These points are precisely the fundamental points of ϕ. We show no such fundamental points exist. Assuming x were such a point, we proceed to minimally resolve ϕ at x by blowing up x and its infinitely near exceptional points; the birational map that ϕ induces on this surface will again be called ϕ. Let E denote the union of rational curves obtained in the process and note that ϕ must collapse E to a single point x' on T'. (This follows from the π-compatibility and the fact that all other points in the fiber of x' lie in V, as embedded in $\mathcal{F}_m$.) This shows that, in fact, the last blow-up was redundant, contradicting the minimality of the resolution. Hence ϕ is an isomorphism (so $m = n$), and $\phi(T) = T'$ (so $\ell = k$), concluding the proof. □

3. Coordinate rings of affine $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^1_{\mathbb{C}}$

The connection between Theorem 0.1 and affine varieties which are $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{P}^1_{\mathbb{C}}$ is illuminated when we examine the "gluing data" which patches together two copies of $\mathbb{A}^2_{\mathbb{C}}$ to construct such a bundle V as their union. This leads to an explicit description of $\Gamma(V)$ as a subring of $\mathbb{C}[X, Y]$, corresponding to the containment of one of the $\mathbb{A}^2_{\mathbb{C}}$ s in V.

Let V be an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$ with structure map π: $V \to \mathbb{P}^1_{\mathbb{C}}$. Choose X such that the function field of $\mathbb{P}^1_{\mathbb{C}}$ is $\mathbb{C}(X)$, and let $U_0 = \pi^{-1}(\operatorname{Spec} \mathbb{C}[X])$, $U_1 =$

[4] According to Goodman's criterion for surfaces [5, Thm. 2], this condition is also necessary for V to be affine.

$\pi^{-1}(\operatorname{Spec}\mathbb{C}[X^{-1}])$. Then $V = U_0 \cup U_1$. Both U_0 and U_1 are $\mathbb{A}^1_{\mathbb{C}}$-bundles over $\mathbb{A}^1_{\mathbb{C}}$, and since these are known to be trivial, we have $U_0 \cong \mathbb{A}^2_{\mathbb{C}}$ and $U_1 \cong \mathbb{A}^2_{\mathbb{C}}$.

THEOREM 3.1. *Let $V = U_0 \cup U_1$ and X be as above and assume further that V is affine. There exists $Y \in \Gamma(V)$ such that*

$$U_0 = \operatorname{Spec}\mathbb{C}[X, Y] \qquad U_1 = \operatorname{Spec}\mathbb{C}[X', Y'] \tag{3}$$

where

$$X' = X^{-1}, \qquad Y' = X^m Y + \alpha_1 X^{m-1} + \alpha_2 X^{m-2} + \cdots + \alpha_{m-1} X \tag{4}$$

where $m \geq 2$ and $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, not all zero. Moreover, letting $A = \Gamma(V)$, we have

$$A = \mathbb{C}[t_0, t_1, \ldots, t_m]$$

where

$$\begin{aligned} t_0 &= Y \\ t_1 &= XY \\ t_2 &= X^2Y + \alpha_1 X \\ t_3 &= X^3Y + \alpha_1 X^2 + \alpha_2 X \\ &\;\vdots \\ t_m &= X^mY + \alpha_1 X^{m-1} + \alpha_2 X^{m-2} + \cdots + \alpha_{m-1} X \ (= Y'). \end{aligned} \tag{5}$$

In fact, A is a free module over $\mathbb{C}[t_0, t_m]$ with basis $\{1, t_1, \ldots, t_{m-1}\}$.

Conversely, given $t_0, \ldots, t_m$ as in (5) *with $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$, not all zero, then letting $A = \mathbb{C}[t_0, \ldots, t_m]$, and letting X' and Y' be defined by* (4), *we have* $\operatorname{Spec} A = U_0 \cup U_1$, *where U_0 and U_1 are as in* (3), *and* $\operatorname{Spec} A$ *is an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}} =$* $\operatorname{Spec}\mathbb{C}[X] \cup \operatorname{Spec}\mathbb{C}[X']$ *by virtue of the containments $\mathbb{C}[X] \subset \mathbb{C}[X, Y]$, $\mathbb{C}[X'] \subset \mathbb{C}[X', Y']$.*

Proof. Certainly we can choose $Y \in \Gamma(U_0)$, $Y' \in \Gamma(U_1)$ such that $U_0 = \operatorname{Spec}\mathbb{C}[X, Y]$, $U_1 = \operatorname{Spec}\mathbb{C}[X', Y']$, where $X' = X^{-1}$. These preliminary choices, however, will need to be modified. Note that $U_0 \cap U_1 = \operatorname{Spec}\mathbb{C}[X, X^{-1}, Y] = \operatorname{Spec}\mathbb{C}[X, X^{-1}, Y']$, whence $\mathbb{C}[X, X^{-1}, Y] = \mathbb{C}[X, X^{-1}, Y']$ (both viewed as subrings of the function field $\mathbb{C}(V)$). From this it follows that $Y' = \beta X^m Y + f(X, X^{-1})$, where $\beta \in \mathbb{C}^*$, $m \in \mathbb{Z}$, and $f(X, X^{-1}) = \sum \nu_i X^i$ is a Laurant polynomial in X with coefficients in $\mathbb{C}$. If $f(X, X^{-1}) = 0$, the retractions $\mathbb{C}[X, Y] \to \mathbb{C}[X]$, $\mathbb{C}[X', Y'] \to \mathbb{C}[X']$ sending Y and Y', respectively, to 0 are compatible and determine a section for the structure map π. Since V is affine, this violates Theorem 2.3's condition (2) for affineness; hence $f(X, X^{-1}) \neq 0$. Replacing Y by βY, we may assume $\beta = 1$. Now replace Y by $Y + \sum_{i\geq 0} \nu_{i+m} X^i$ (a legitimate replacement for Y in $\mathbb{C}[X, Y]$) to effect $\nu_i = 0$ for $i \geq m$. In similar fashion, after replacing Y' by $Y' - \sum_{i\leq 0} \nu_i X^i = Y' - \sum_{i\leq 0} \nu_i X'^{-i}$ we have $\nu_i = 0$ for $i \leq 0$. Note that if $m \leq 1$ all coefficients ν_i are zero, i.e., $f(X, X^{-1}) = 0$, which is impossible, as shown above. Hence $m \geq 2$ and, letting $\alpha_i = \nu_{m-i}$, we have arranged (4).

Since $V = U_0 \cup U_1$, we have $A = \Gamma(U_0) \cap \Gamma(U_1) = \mathbb{C}[X, Y] \cap \mathbb{C}[X', Y']$. Clearly the elements $t_0, \ldots, t_m$ as defined in (5) lie in $\mathbb{C}[X, Y]$. The equations $t_m = Y', t_{i-1} = X't_i - \alpha_i$, $i = 1, \ldots, m$, and $t_0 = X't_1$ show that $t_0, \ldots, t_m \in \mathbb{C}[X', Y']$ as well. So letting $R = \mathbb{C}[t_0, \ldots, t_m]$ we have $R \subseteq A$ and thus the following series of ring containments:

$$\mathbb{C}[t_0, t_m] \subseteq R \subseteq A \subseteq \mathbb{C}[X, Y]. \tag{6}$$

We claim that R is a free C-module with basis $\{1, t_1, \ldots, t_{m-1}\}$. Toward proving this, we first calculate the rank of R as a $\mathbb{C}[t_0, t_m]$-module by adjoining $1/t_0$ to all the rings in (6). Since $X = t_1/t_0$, $R[t_0^{-1}]$ contains X, Y, and Y^{-1} and we have

$$\mathbb{C}[t_0, t_0^{-1}, t_m] \subseteq R[t_0^{-1}] = A[t_0^{-1}] = \mathbb{C}[X, Y, Y^{-1}]. \tag{7}$$

Note that $\mathbb{C}[t_0, t_0^{-1}, t_m] = \mathbb{C}[Y, Y^{-1}, t_m]$ and that t_m has degree m as a polynomial in X over the ring $\mathbb{C}[Y, Y^{-1}]$, the leading coefficient being Y, a unit. It follows that the rank of $\mathbb{C}[X, Y, Y^{-1}]$ over $\mathbb{C}[t_0, t_0^{-1}, t_m]$, and hence the rank of R over $\mathbb{C}[t_0, t_m]$, is m. To prove the claim it suffices to show that$\{1, t_1, \ldots, t_{m-1}\}$ generate R as a $\mathbb{C}[t_0, t_m]$-module. Since R is generated as a $\mathbb{C}[t_0, t_m]$-module by monomials in $\{1, t_1, \ldots, t_{m-1}\}$, it suffices to show that for $i, j \in \{1, \ldots, m-1\}$, $t_it_j = \sum_\ell h_\ell t_\ell$ with $h_\ell \in \mathbb{C}[t_0, t_m]$. This, in turn will follow if we can show $t_it_j = t_{i-1}t_{j+1} + \sum_\ell h_\ell t_\ell$ with $h_\ell \in \mathbb{C}[t_0, t_m]$. Note from (5) that $t_i = X(t_{i-1} + \alpha_{i-1})$ (setting $\alpha_0 = 0$) and $t_{j+1} = X(t_j + \alpha_j)$, whence $t_i(t_j + \alpha_j) = t_{j+1}(t_{i-1} + \alpha_{i-1})$. This can be written as $t_it_j = t_{i-1}t_{j+1} - \alpha_j t_i + \alpha_{i-1} t_j$, accomplishing the goal and proving the claim.

It remains to show $R = A$. By the first equality in (7), it suffices to show that if $f \in A$ and $t_0 f \in R$, then $f \in R$. Given such an f, then using the basis $\{1, t_1, \ldots, t_{m-1}\}$, we write $t_0 f = b_0 + \sum_{i=1}^{m-1} b_i t_i$, where $b_0, b_1, \ldots, b_{m-1} \in \mathbb{C}[t_0, t_m]$. For $i = 0, \ldots, m-1$, write $b_i = c_i + t_0 d_i$ with $c_i \in \mathbb{C}[t_m]$, $d_i \in \mathbb{C}[t_0, t_m]$, and set

$$f_1 = f - d_0 - \sum_{i=1}^{m-1} d_i t_i. \tag{8}$$

Then

$$\begin{aligned} t_0 f_1 &= t_0 f - t_0 d_0 - \sum_{i=1}^{m-1} d_i t_0 t_i \\ &= b_0 + \sum_{i=1}^{m-1} b_i t_i - t_0 d_0 - \sum_{i=1}^{m-1} t_0 d_i t_i \end{aligned}$$

$$= (b_0 - t_0 d_0) + \sum_{i=1}^{m-1} (b_i - t_0 d_i) t_i$$
$$= c_0 + \sum_{i=1}^{m-1} c_i t_i.$$

We restate the resulting equation:

$$t_0 f_1 = c_0 + \sum_{i=1}^{m-1} c_i t_i. \tag{9}$$

Now we observe that $f_1 \in A \subset \mathbb{C}[X', Y']$, and we view equation (9) as a polynomial equation in the indeterminants X' and Y'. From (5) we have $t_m = Y'$, $t_0 = X'^m Y' - \alpha_1 X' - \alpha_2 X'^2 - \cdots - \alpha_{m-1} X'^{m-1}$, and $t_i = X'^{m-i} Y' - \alpha_i - \alpha_{i+1} X' - \cdots - \alpha_{m-1} X'^{m-1-i}$ for $i = 1, \ldots, m-1$. One sees that t_0 has degree m as a polynomial in X', and, for $i = 1, \ldots, m-1$, t_i has degree $m - i$. Since $c_1, \ldots, c_{m-1} \in \mathbb{C}[t_m] = \mathbb{C}[Y']$, the left side of equation (7) has degree $\geq m$ while the right side of the equation has degree $\leq m-1$. It follows that $f_1 = 0$, i.e., $f = d_0 + \sum_{i=1}^{m-1} d_i t_i$ (see (8)); hence $f \in R$ as desired.

We now prove the converse. Denoting by $(U_0)_X$ the principal open set in U_0 defined by the function X, we have $(U_0)_X = \operatorname{Spec} \mathbb{C}[X, X^{-1}, Y] = (U_1)_{X'}$. Hence a prevariety $V = U_0 \cup U_1$ can be glued together. The containments $\mathbb{C}[X] \subset \mathbb{C}[X, Y]$ and $\mathbb{C}[X'] \subset \mathbb{C}[X', Y']$ define a map $\pi\colon V \to \mathbb{P}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[X] \cup \operatorname{Spec} \mathbb{C}[X']$ making V an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$. This morphism shows that V is separated (i.e., a variety) since it separates points in $U_0 - U_1$ from points in $U_1 - U_0$. We claim that π does not admit a section. Such a section would give compatible ring retractions $\phi_0\colon \mathbb{C}[X, Y] \to \mathbb{C}[X]$, $\phi_1\colon \mathbb{C}[X', Y'] \to \mathbb{C}[X']$. This is impossible, for if $\phi_0(Y) = h(X)$, then we would have $\phi_1(Y') = \phi_1(X^m Y + \alpha_1 X^{m-1} + \alpha_2 X^{m-2} + \cdots + \alpha_{m-1} X) = X^m h(X) + \alpha_1 X^{m-1} + \alpha_2 X^{m-2} + \cdots + \alpha_{m-1} X$, which cannot lie within $\mathbb{C}[X']$. The claim is proved, and Theorem 2.3 tells us that V is affine. Just as before, we can show that $\Gamma(U_0) \cap \Gamma(U_1) = A = \mathbb{C}[t_0, \ldots, t_m]$. Therefore $V = \operatorname{Spec} A$. $\square$

Relationship between Theorem 2.3 *and Theorem* 3.1. It is natural to ask: For V an affine $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$, what is the relationship between the data of Theorem 3.1 (the integer m and the polynomial $\alpha_1 X^{m-1} + \alpha_2 X^{m-2} + \cdots + \alpha_{m-1} X$) and that of Theorem 2.3 (the integers n and k). The author has established that $m = n + 2d$, where $d = k - n$ (which is necessarily ≥ 1 by (1) $\Longrightarrow$ (3) of Theorem 2.3). The proof is a calculation and will not be given here. However,the author has not found a good way to recover n and k from m and the polynomial $\alpha_1 X^{m-1} + \alpha_2 X^{m-2} + \cdots + \alpha_{m-1} X$.

Peretz' Theorem (Thm. 0.1) is related to the following conjecture:

CONJECTURE 3.2 (GEOMETRIC FORMULATION). *Let V be an affine variety which is an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$, $U = V - F$, where F is a fiber in V. There does* **not** *exist $f\colon V \to \mathbb{A}^2_{\mathbb{C}}$ such that $f|_U$ is étale.*

Equivalently, by virtue of Theorem 3.1:

CONJECTURE 3.2 (ALGEBRAIC FORMULATION). *There does not exist a pair of polynomials*

$$p, q \in \mathbb{C}[t_0, t_1, \ldots, t_m] \subset \mathbb{C}[X, Y],$$

where $t_0, t_1, \ldots, t_m$ *are as in* (5) *of Theorem* 3.1 ($\alpha_1, \ldots \alpha_{m-1} \in \mathbb{C}$, *not all zero*), *with* $\frac{\partial(p,q)}{\partial(X,Y)}$ *non-vanishing (i.e. constant) on* $\mathbb{A}^2_{\mathbb{C}}$.

The following theorem proves a special case of the above conjecture.

THEOREM 3.3. *Conjecture* 3.2 *holds in the case where the coefficient* α_1 *is nonzero.*

Remark. This statement may seem a bit peculiar, but it includes Peretz' result (Theorem 0.1), which is precisely the case $\alpha_1 \neq 0, \alpha_2 = \cdots = \alpha_{m-1} = 0$.

Proof. Letting $A = \mathbb{C}[t_0, t_1, \ldots, t_m]$, we are in the situation of Theorem 3.1, and we will freely refer to its various notations. Suppose there exists $p, q \in A$ with $\frac{\partial(p,q)}{\partial(X,Y)} \in \mathbb{C} - \{0\}$. We can easily arrange that $\frac{\partial(p,q)}{\partial(X,Y)} = 1$, so that in $\Omega^2_{\mathbb{C}[X,Y]/\mathbb{C}}$ we have $dp \wedge dq = dX \wedge dY$. In the diagram below, the rows are from the De Rham sequence for A and $\mathbb{C}[X, Y]$, respectively. These sequences are exact by [6, Thm. 1][5] (We will only need the exactness of the second row.) The fact that $A \hookrightarrow \mathbb{C}[X, Y]$ induces an open embedding of affine varieties insures that the vertical maps are injective (hence they are denoted as containments).

$$\begin{array}{ccccccc}
A & \xrightarrow{d} & \Omega^1_{A/\mathbb{C}} & \xrightarrow{d} & \Omega^2_{A/\mathbb{C}} & \ni & dp \wedge dq \\
\cap & & \cap & & \cap & & \| \\
\mathbb{C}[X,Y] & \xrightarrow{d} & \Omega^1_{\mathbb{C}[X,Y]/\mathbb{C}} & \xrightarrow{d} & \Omega^2_{\mathbb{C}[X,Y]/\mathbb{C}} & \ni & dX \wedge dY \\
 & & & & & & \\
h & \mapsto & X\,dY - p\,dq & \mapsto & 0. & &
\end{array}$$

Since $d(XdY - p\,dq) = dX \wedge dY - dp \wedge dq = 0$, there exists (by exactness) $h \in \mathbb{C}[X, Y]$ with $dh = XdY - p\,dq$. Along the fiber $F = V - U_0$, $p\,dq$ is clearly holomorphic since p and q lie in A. However, we claim that $X\,dY$ has a

[5]The theorem of Grothendieck quoted asserts that the cohomology in the middle positions calculate the complex cohomology $H^1(W, \mathbb{C})$, for $W = \operatorname{Spec} A$ and $W = \mathbb{A}^2_{\mathbb{C}}$ respectively, so the exactness follows from the simple-connectivity of W in each case. Simple-connectivity (in fact, contractibility) of $\mathbb{A}^2_{\mathbb{C}}$ is well known. Although, as we point out above, exactness of the top row is not needed here, we note that simple-connectivity of the bundle $V = \operatorname{Spec} A$ follows from the surjectivity of the map of fundamental groups $\pi_1(\mathbb{A}^2_{\mathbb{C}}) \to \pi_1(V)$ arising from the open embedding $\mathbb{A}^2_{\mathbb{C}} \subset V$; this is surjective because complement of $\mathbb{A}^2_{\mathbb{C}}$ in V has real codimension two.

pole of order 1 along F. This derives from the fact that F is defined by $X' = 0$ on $U_1 = \operatorname{Spec} \mathbb{C}[X', Y']$, where X' and Y' are as in (4). One easily sees from (4) that $X = X'^{-1}$ and $Y = X'^m Y' - \alpha_1 X' - \alpha_2 X'^2 - \cdots - \alpha_{m-1} X'^{m-1}$, so that

$$\begin{aligned} X\,dY &= \frac{1}{X'} d\left(X'^m Y' - \alpha_1 X' - \alpha_2 X'^2 - \cdots - \alpha_{m-1} X'^{m-1}\right) \\ &= \frac{1}{X'}\big[(mX'^{m-1}Y' - \alpha_1 - 2\alpha_2 X' - \cdots - (m-1)\alpha_{m-1}X'^{m-2})dX' \\ &\qquad - X'^m\, dY'\big] \\ &= \left(mX'^{m-2}Y' - \alpha_1 X'^{-1} - 2\alpha_2 - \cdots - (m-1)\alpha_{m-1}X'^{m-3}\right) dX' \\ &\quad - X'^{m-1}\, dY'. \end{aligned}$$

The presence of term $-\alpha_1 X'^{-1}$ in the last expression together with the fact that $\alpha_1 \neq 0$ shows that $X\,dY$ has a pole of order 1 along F, establishing the claim. It follows that $X\,dY - p\,dq = dh$ also has a pole of order 1 along F. Thus h must have a pole along F as well. Considering h as an element of $\mathbb{C}[X', X'^{-1}, Y']$, this says $h \notin \mathbb{C}[X', Y']$, i.e., as a Laurant polynomial in X', h has negative order. But then $\frac{\partial h}{\partial X'}$ has order ≤ -2, and since

$$dh = \frac{\partial h}{\partial X'} dX' + \frac{\partial h}{\partial Y'} dY',$$

we see that dh must have a pole of order ≥ 2 along F, contradicting our previous conclusion that the order of this pole is 1. □

Remark. Theorem 3.3 answers Conjecture 3.2 affirmatively in the case $m = 2$ (since we must have $\alpha_1 \neq 0$ in this case), so the simplest unresolved case is when $m = 3$, $\alpha_1 = 0$. Here we can easily arrange that $\alpha_2 = 1$ (replace Y by $\alpha_2 Y$), leading us to consider:

SIMPLEST UNRESOLVED CASE OF CONJECTURE 3.2. *There does not exist a counterexample(p, q) to the Jacobian conjecture with $p, q \in \mathbb{C}[Y, XY, X^2Y, X^3Y + X]$.*

Note that, setting $\deg X = -1$ and $\deg Y = 2$, the ring $\mathbb{C}[Y, XY, X^2Y, X^3Y + X]$ is a graded ring, giving an action of the algebraic group $\mathbb{G}_a$ on $V = \operatorname{Spec} \mathbb{C}[Y, XY, X^2Y, X^3Y + X]$. This structure may be useful in solving this special case.

4. Connection to the Jacobian conjecture

The two-dimensional Jacobian conjecture, which asserts that an étale map $f = (p, q)\colon \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ is an isomorphism, remains unproved, even (to the author's knowledge) in the case where the integral closure of $\mathbb{C}[p, q]$ in $\mathbb{C}[X, Y]$ is smooth. We will refer to this latter condition as the case of "smooth integral closure".

We begin by establishing a criterion for affineness which will be needed in the proof of Theorem 4.3.

PROPOSITION 4.1. *Let $W = \operatorname{Spec} A$ be an affine scheme, with A a normal Noetherian domain. Let Z be an irreducible subvariety of codimension one in W which is locally defined by one equation, set-theoretically. Then $W - Z$ is affine.*

Proof. Set $V = W - Z$. Let $\mathfrak{a}$ be the radical ideal in A defining Z, and let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the height one primes of A containing $\mathfrak{a}$; these correspond to the irreducible components of Z. Let $B = \Gamma(V, \mathcal{O}_W)$. Normality implies that

$$B = \bigcap_{\substack{\operatorname{ht}\mathfrak{q}=1 \\ \mathfrak{q}\neq \mathfrak{q}_1,\dots,\mathfrak{q}_r}} A_{\mathfrak{q}}.$$

We claim that $\mathfrak{a}B = B$. If not, choose a prime ideal $\mathfrak{P}$ in B containing $\mathfrak{a}B$, and let $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{p} \supseteq \mathfrak{a}$. We have a local containment $A_{\mathfrak{p}} \subset B_{\mathfrak{P}}$. Our assumption about Z says that there exists $f \in A_{\mathfrak{p}}$ such that $\sqrt{fA_{\mathfrak{p}}} = \mathfrak{a}A_{\mathfrak{p}}$. This says f has zeros only along the components Z in $\operatorname{Spec} A_{\mathfrak{p}}$, i.e, those divisors of $\operatorname{Spec} A_{\mathfrak{p}}$ corresponding to the height one primes $\mathfrak{q}_i A_{\mathfrak{p}}$ (for those $\mathfrak{q}_i$ contained in $\mathfrak{p}$). Noting that all height one localizations of $B_{\mathfrak{P}}$ are height one localizations of B and of $A_{\mathfrak{p}}$, we see that f has no zeros in the divisors of $\operatorname{Spec} B_{\mathfrak{P}}$, hence $1/f \in B_{\mathfrak{P}}$. But this is impossible since $f \in \mathfrak{a}A_{\mathfrak{p}} \subset \mathfrak{P}B_{\mathfrak{P}}$, establishing the claim.

Choose generators $f_1, \dots, f_t$ for $\mathfrak{a}$. The principal open sets W_{f_j} cover $V = W - Z$ and $V_{f_j} = W_{f_j}$, so we have $V = V_{f_1} \cup \cdots \cup V_{f_t}$. It follows from [8, Ex. 2.28, p. 81] that V is affine. □

COROLLARY 4.2. *Let W be an irreducible normal affine surface over $\mathbb{C}$ which contains $\mathbb{A}^2_{\mathbb{C}}$ as an open subvariety. Let Z be a subvariety of pure codimension one in W. Then $W - Z$ is affine.*

Proof. By Proposition 4.1, we need only to show that all curves on W are locally defined by one equation, set-theoretically. We only need to check this property at the singular points of W, which are discrete. Let p be a singular point. According to [9, Thm. 6.6 (1)], p is a rational singularity, which implies that the divisor class group of the local ring $\mathcal{O}_{p,W}$ is a torsion group [4, Thms. 1.4 and 1.5]. Hence $\mathcal{O}_{p,W}$ has the property that all height one primes are the radicals of principal ideals, which is the needed result. □

Note. The assumption "W contains $\mathbb{A}^2_{\mathbb{C}}$ as an open subvariety" can be replaced by the assumption "W contains a cylinderlike open subvariety", since this is precisely what is needed to evoke [9, Thm. 6.6 (1)].

The following theorem shows that a counter-example to the Jacobian conjecture would lead to a situation resembling the one whose non-existence is asserted by Conjecture 3.2 (geometric formulation).

THEOREM 4.3. *If the Jacobian conjecture is false, there exists a normal affine variety V containing $U = \mathbb{A}^2_{\mathbb{C}}$ as an open subvariety having the following properties*: (1)$F = V - U$ *is a rational curve whose normalization is* $\mathbb{A}^1_{\mathbb{C}}$ *and each singular point of F has a one-point desingularization*; (2) *there is a map* $\pi\colon V \to \mathbb{P}^1_{\mathbb{C}}$ *such that F is the set-theoretic fiber of a point* $z \in \mathbb{P}^1_{\mathbb{C}}$, *and the restriction map* $\pi|_U\colon U \to \mathbb{P}^1_{\mathbb{C}} - \{z\} = \mathbb{A}^1_{\mathbb{C}}$ *is the projection onto a coordinate line*; *and* (3) *there is a map* $f\colon V \to \mathbb{A}^2_{\mathbb{C}}$ *such that* $f|_U$ *is étale*; . *If the Jacobian conjecture is false in the case of "smooth integral closure", V can be chosen to be smooth and* $F \cong \mathbb{A}^1_{\mathbb{C}}$.

Proof. Let $f = (p, q)\colon U \to U'$, where $U = U' = \mathbb{A}^2_{\mathbb{C}}$, be an étale morphism, and let $\tilde f\colon S \to \mathbb{P}^2_{\mathbb{C}}$ be a minimal resolution of the birational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ determined by f. The minimality of the resolution assures that the only possible exceptional curve for $\tilde f$ having self-intersection -1 is the proper transform $\tilde L$ of the line at infinity L in $\mathbb{P}^2_{\mathbb{C}}$. One easily verifies that $S - U$ is a simply connected union of smooth rational curves, having normal crossings, and containing $\tilde L$. Moreover, $\tilde L$ must map into the complement of U'.

Let $W = \tilde f^{-1}(U')$. Note that W contains U as an open subvariety (because the resolution of $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ does not blow up any points of U), and that $\tilde f$ restricts to a proper morphism $W \to U'$. The situation is depicted in the diagram below:

$$\begin{tikzcd}
\tilde f^{-1}(U') \arrow[r, phantom, "="] & W \arrow[r, phantom, "\subset"] & S \arrow[d] \arrow[dr, "\tilde f"] & & \\
 & \cup & \mathbb{P}^2_{\mathbb{C}} \arrow[r, dashed] & \mathbb{P}^2_{\mathbb{C}} & \\
 & & \cup & \cup & \\
 & U \arrow[r, phantom, "="] & \mathbb{A}^2_{\mathbb{C}} \arrow[r, "f"] & \mathbb{A}^2_{\mathbb{C}} \arrow[r, phantom, "="] & U' \\
 & & \mathbb{C}[X, Y] & \mathbb{C}[p, q]. \arrow[l] &
\end{tikzcd}$$

Let $\overline{U'}$ be the normalization of U' in W. Then $\overline{U'} = \operatorname{Spec} B$, where B is the integral closure of $\mathbb{C}[p, q]$ in $\mathbb{C}[X, Y]$. We know that U is an open subvariety of $\overline{U'}$ [12, Prop. 3.1]. We have maps $\bar f\colon \overline{U'} \to U'$ extending f and $g\colon W \to \overline{U'}$ such that $\tilde f|_W = \bar f \circ g$. The map g is birational. Any curve collapsed by g must have the property that its closure in S lies entirely within W, since $W = \tilde f^{-1}(U')$, and outside of U. Also, any such curve must map via $\tilde f$ to a point in U', by the commutativity $\tilde f|_W = \bar f \circ g$. Therefore, by the remarks above, $\tilde L$ is not among these curves. All such curves are exceptional curves for $\tilde f$ as well, hence have self-intersection ≤ -2. It follows that the image of the exceptional locus of g is the singular locus of $\overline{U'}$. In particular, the integral closure $\overline{U'}$ is smooth if and only if g is an isomorphism (i.e., $\overline{U'} = W$), and this holds precisely when W is affine, as affineness precludes the existence of any exceptional curves for g, since these are complete curves contained in W.

These considerations insure that the contractions which map W to $\overline{U'}$ also map S to a complete surface $\overline{S}$ containing $\overline{U'}$, with $S - W$ mapping isomorphically to $\overline{S} - \overline{U'}$. Since $\overline{U'}$ is affine, $\overline{S} - \overline{U'}$ is connected [5, Corollary to Thm. 1], hence so is $S - W$.

Let $D_1, \ldots, D_r$ be the connected components (note: *not* the irreducible components) of $W - U$. The removal of $W - U$ from $S - U$ leaves $S - W$, which is connected. From the simple-connectivity of $S - U$ we conclude that each D_i has precisely one point in its closure which is not in D_i, and that point lies on $S - W$. Therefore D_i contains precisely one non-complete component F_i, this component's closure containing the missing point. We must have $F_i \cong \mathbb{A}^1_{\mathbb{C}}$ and all other components of D_i isomorphic to $\mathbb{P}^2_{\mathbb{C}}$. It follows from the discussion above that F_i maps birationally and injectively to an affine curve $\overline{F_i}$ (possibly singular) which is closed in $\overline{U'}$, and that all other components of D_i contract to points of $\overline{F_i}$ which are singular points of $\overline{U'}$. These points are the only possible singularities of $\overline{F_i}$. All singularities of $\overline{F_i}$ have one-point desingularizations, and $\overline{F_i}$ has one point at infinity. We have $\overline{U'} - U = \cup \overline{F_i}$. Observe that in the case of "smooth integral closure" ($\overline{U'} = W$), we have $D_i = F_i$, so that $\overline{U'} - U$ is the disjoint union of the curves F_i, which are isomorphic to $\mathbb{A}^1_{\mathbb{C}}$.

If the two-dimensional Jacobian conjecture is false there exists $f = (p, q)$ as above which is not an isomorphism. It is well-known (see [13, Thm. 3.3], for example) that this is equivalent to the condition $\mathbb{C}[X, Y]$ is not integral over $\mathbb{C}[p, q]$, i.e., the union $\overline{U'} - U = \cup \overline{F_i}$ is non-empty. According to a theorem of Abhyankar [1, Cor. 18.15], the polynomials p and q can be chosen so that the curves $p = 0$ and $q = 0$ each have two points at infinity in $\mathbb{P}^2_{\mathbb{C}}$. These two points, call them x and y, must lie on both curves. Let us note that these two points are precisely the points of indeterminacy for the birational map $f\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$, hence the resolution of f blows up only these two points and "infinitely near" points above them. We conclude that each component D_i of $W - U$ maps entirely to one of these two points on $\mathbb{P}^2_{\mathbb{C}}$, since D_i does not contain $\widetilde{L}$. Assume that D_1 maps to y.

Let V be the surface $\overline{U'} - (\overline{F_2} \cup \cdots \cup \overline{F_r})$. By Corollary 4.2, V is affine, and $V = U \cup \overline{F_1}$. Without loss of generality we may assume that x and y are the points at infinity on the lines $X = 0$ and $Y = 0$, respectively, and that the component D_1 of $W - U$ contracts to the point y. We may also assume that the first blow-up in the resolution is centered at x. This blow-up resolves the "projection from x", giving a morphism to $\mathbb{P}^1_{\mathbb{C}}$ extending the map $U \to \mathbb{A}^1_{\mathbb{C}}$ corresponding to the containment $\mathbb{C}[X] \to \mathbb{C}[X, Y]$. This morphism sends the proper transform of the line at infinity L on $\mathbb{P}^2_{\mathbb{C}}$ to the point at infinity in $\mathbb{P}^1_{\mathbb{C}}$ and induces morphisms from all subsequent surfaces obtained in the resolution process to $\mathbb{P}^1_{\mathbb{C}}$. In particular, we get a morphism $\widetilde{\pi}\colon S \to \mathbb{P}^1_{\mathbb{C}}$. Since the component D_1 of $W - U$ contracts to x on $\mathbb{P}^2_{\mathbb{C}}$, it maps to the point at infinity on $\mathbb{P}^1_{\mathbb{C}}$. It follows that $\widetilde{\pi}$ factors through the contractions which collapse D_1 to $\overline{F_1}$, giving a morphism $\pi\colon V \to \mathbb{P}^1_{\mathbb{C}}$ with $\pi^{-1}(\text{point at } \infty) = \overline{F_1}$, set-theoretically. (The fiber may be reduced.) In the case of "smooth integral closure", $\overline{F_1} = F_1$, and this curve is non-singular.

Setting $F = \overline{F_1}$, we have:

$$\begin{array}{ccccc} & & F & \to & \text{pt at } \infty \\ & & & & \\ & & V & \xrightarrow{\pi} & \mathbb{P}^1_{\mathbb{C}} \\ & & \cup & & \cup \\ \mathbb{A}^2_{\mathbb{C}} & = & U & \xrightarrow{\pi} & \mathbb{A}^1_{\mathbb{C}}. \end{array}$$

These observations conclude the proof of Theorem 4.3. □

Remark. We do not know that F has multiplicity one in the fiber, even in the case of "smooth integral closure". If, however, F is smooth and $\pi^{-1}(\text{point at } \infty) = F$ *scheme*-theoretically, then V is an $\mathbb{A}^1_{\mathbb{C}}$-bundle over $\mathbb{P}^1_{\mathbb{C}}$ via the map $V \xrightarrow{\pi} \mathbb{P}^1_{\mathbb{C}}$. Hence we are in the situation of Conjecture 2.4, which would rule out this possibility.

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REFERENCES

[1] S. Abhyankar, *Expansion techniques in algebraic geometry*, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
[2] A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Math., 146, Springer-Verlag, New York, 1970.
[3] H. Bass, E. H. Connell, and D. Wright, *Locally polynomial algebras are symmetric algebras*, Inventiones Math. **38** (1977), 279–299.
[4] E. Brieskorn, *Rationale Singularitäten komplexer Flächen*, Invent. Math. **4** (1968), 336–358.
[5] J. E. Goodman, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. **89** (1969), 160–183.
[6] A. Grothendieck, *On the De Rham cohomology of algebraic varieties*, Publ. Math. I. H. E. S. no. 31, 1966.
[7] T. Kambayashi and D. Wright, *Flat families of affine lines are affine line bundles*, Illinois J. Math. **29** (1985), 672–681.
[8] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
[9] M. Miyanishi, *Lectures on curves on rational and unirational surfaces*, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, New York, 1978.
[10] R. Peretz, *On counterexamples to Keller's problem*, Illinois J. of Math. **40** (1996), 293–303.
[11] S. Pinchuk, *A counterexample to the strong real Jacobian conjecture*, Math. Z. **217** (1994), 1–4.
[12] A. Sathaye, *Polynomial ring in two variables over a D. V. R.: a criterion*, Inventiones Math. **74** (1983), 159–168.
[13] D. Wright, *On the Jacobian conjecture*, Illinois J. Math. **25** (1981), 423–440.

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