POWER SERIES WHOSE SECTIONS HAVE ZEROS OF LARGE MODULUS. II

BY

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1. Introduction

Given a power series $\sum_{p=0}^{\infty} a_p z^p$, for each positive integer n let $\mathfrak{r}_n$ denote the smallest modulus of a zero of $\sum_{p=0}^{n} a_p z^p$, the n^{th} partial sum. Various growth properties of the sequence $\{\mathfrak{r}_n\}$ were discussed in [2]; since the present paper is an extension of several of these results, some familiarity with [2] is desirable.

If $\sum_{p=0}^{\infty} a_p z^p$ has a zero in the interior of its circle of convergence, then Hurwitz' theorem guarantees that $\{\mathfrak{r}_n\}$ converges to the smallest modulus of such a zero. If we note that $\mathfrak{r}_n \leq | a_0/a_n |^{1/n}$, then Hurwitz' theorem can be used to show that $\mathfrak{r}_n \to \infty$ if and only if $\sum_{p=0}^{\infty} a_p z^p = \exp \{g(z)\}$ for an entire function $g(z)$. There is in this case an interesting connection between the growth of $\{\mathfrak{r}_n\}$ and that of the maximum modulus of $g(z)$. In [2] it was shown that the condition

$$(1.1) \qquad \limsup_{n\to\infty} \frac{\log n}{\mathfrak{r}_n^c} \leq d, \qquad 0 < c < \infty, 0 \leq d < \infty,$$

is satisfied if and only if

$$(1.2) \qquad \sum_{p=0}^{\infty} a_p z^p = \exp \{g(z)\}, \qquad g(z) \textit{ an entire function of growth } (c, d).$$

(The statement that an entire function $g(z)$ is of growth (c, d) means that the order of $g(z)$ does not exceed c, and that the type of $g(z)$ does not exceed d if $g(z)$ is of order c.)

For each $d_1 > d$, (1.1) requires that $\{\mathfrak{r}_n\}$ should grow at least as rapidly as

$$\left[\frac{\log n}{d_1}\right]^{1/c}.$$

We shall investigate the possibility of replacing (1.1) by a weaker condition in which only a certain subsequence of $\{\mathfrak{r}_n\}$ is required to grow this rapidly.

One theorem of this type was obtained in [2]. There it was shown that if $c > 0$ and $\mathfrak{r}_n > n^c$ for infinitely many n, then $\sum_{p=0}^{\infty} a_p z^p = \exp \{P(z)\}$ for some polynomial $P(z)$ of degree $1/c$ or less. *No corresponding result is obtainable if n^c is replaced by a function of slower growth.* Specifically, if $\varphi(n)$ is a positive function such that $\varphi(n) = n^{o(1)}$ as $n \to \infty$, one can construct a power series $\sum_{p=0}^{\infty} a_p z^p$ of arbitrary convergence radius such that $\mathfrak{r}_n > \varphi(n)$ for infinitely many n. Such a construction is carried out in §3.

Results similar to (1.2) are obtainable if it is assumed that the values of n

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for which $\mathfrak{r}_n$ is large are not too sparsely distributed. In this direction we prove two theorems.

THEOREM 1. *Suppose that* $\sum_{p=0}^{\infty} a_p z^p$ *is a power series,* $c > 0$ *and* $d > 0$. *If the inequality*

$$(1.3) \qquad \mathfrak{r}_n > \left[\frac{\log n}{d}\right]^{1/c}$$

holds for a sequence of indices $n = n_1 < n_2 < n_3 < \cdots$ *which satisfies*

$$(1.4) \qquad \lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1,$$

then $\sum_{p=0}^{\infty} a_p z^p = \exp\{g(z)\}$ *for an entire function* $g(z)$ *of growth* (c, d).

The previously mentioned equivalence of (1.1) and (1.2) implies that the conclusion of Theorem 1 cannot be strengthened.

THEOREM 2. *Suppose that* $\sum_{p=0}^{\infty} a_p z^p$ *is a power series and* $c > 0$. *If the series*

$$(1.5) \qquad \sum_{n=1}^{\infty} \frac{1}{n\mathfrak{r}_n^c}$$

converges, then $\sum_{p=0}^{\infty} a_p z^p = \exp\{g(z)\}$ *for an entire function* $g(z)$ *of growth* $(c, 0)$. *Furthermore, the infimum of numbers* c *for which* (1.5) *converges is equal to the order of* $g(z)$.

2. Proof of Theorems 1 and 2

For a given power series $\sum_{p=0}^{\infty} a_p z^p$ with $a_0 = 1$, it is not hard to show that there is exactly one power series $g(z) = \sum_{k=1}^{\infty} b_k z^k$ which satisfies the formal power series identity

$$(2.1) \qquad \exp\{g(z)\} = \sum_{p=0}^{\infty} [g(z)]^p/p! = \sum_{p=0}^{\infty} a_p z^p.$$

We shall suppose from now on that $a_0 = 1$ and that the series $g(z) = \sum_{k=1}^{\infty} b_k z^k$ satisfies (2.1). The sequence $\{b_k\}$ thus obtained allows us to state concisely an upper bound for $\mathfrak{r}_n$ which was obtained in [2]:

LEMMA. *If* $0 < k \leq n$ *and* $b_k \neq 0$, *then*

$$(2.2) \qquad \mathfrak{r}_n \leq \left\{\frac{n}{k \mid b_k \mid}\right\}^{1/k}.$$

This is the only information about $\mathfrak{r}_n$ which will be required for the proofs of Theorems 1 and 2.

Proof of Theorem 1. Rewriting (2.2), we have

$$(2.3) \qquad k \mid b_k \mid^{c/k} \leq k\mathfrak{r}_n^{-c}(n/k)^{c/k},$$

which is valid for all $k \leq n$. Using the sequence $\{n_p\}$, we choose $q = q(k)$ so that

$$\log n_q \leq k/c < \log n_{q+1},$$

and let $n = n(k) = n_q$.

For this choice of n we have $k < n$ for large k, and, from (1.4),

$$\log n \sim k/c, \qquad k \to \infty. \tag{2.4}$$

Making use of (1.3) and (2.3), we have

$$k \mid b_k \mid^{c/k} \leq \frac{kd}{\log n}\left(\frac{n}{k}\right)^{c/k}.$$

A short computation using (2.4) shows that

$$\lim_{k \to \infty} \frac{k}{\log n}\left(\frac{n}{k}\right)^{c/k} = ce,$$

so that

$$\limsup_{k \to \infty} k \mid b_k \mid^{c/k} \leq cde.$$

Therefore $g(z)$ is an entire function of growth (c, d) [1, p. 11].

Proof of Theorem 2. Suppose $A > 0$. The series

$$\sum_{n=1}^{\infty} \frac{1}{n\mathfrak{r}_n^c} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

are convergent and divergent respectively; consequently the inequality

$$\mathfrak{r}_n^c \geq A \log n \tag{2.5}$$

is satisfied for infinitely many n. Let

$$n = n_1 < n_2 < n_3 < \cdots$$

denote the values of n for which (2.5) is satisfied. We shall prove that this sequence satisfies (1.4). If $n_p + 1 < n_{p+1}$, let $j = n_p + 1$ and $i = n_{p+1} - 1$. Then

$$\sum_{n=j}^{i} \frac{A}{n\mathfrak{r}_n^c} > \sum_{n=j}^{i} \frac{1}{n \log n}, \tag{2.6}$$

since (2.5) is false for $j \leq n \leq i$. Comparison of the right hand side of (2.6) with the integral

$$\int_{n_p}^{n_{p+1}} \frac{1}{x \log x}\, dx$$

shows that

$$\sum_{n=j}^{i} \frac{1}{n \log n} = \log\left[\frac{\log n_{p+1}}{\log n_p}\right] + o(1), \qquad p \to \infty.$$

The Cauchy convergence criterion implies that the left hand side of (2.6) tends to zero as $p \to \infty$. Therefore we have

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1.$$

Comparing (2.5) with (1.3), we see from Theorem 1 that

$$\sum_{p=0}^{\infty} a_p z^p = \exp\{g(z)\}$$

for an entire function $g(z)$ of growth $(c, 1/A)$. Since A is arbitrary, $g(z)$ is of growth $(c, 0)$.

Let ρ denote the order of $g(z)$. To justify the last sentence of Theorem 2 it is necessary to show that (1.5) *does* converge if $c > \rho$. If we choose a so that $\rho < a < c$ and note that $g(z)$ is of growth $(a, \frac{1}{2})$, then from (1.1) we have

$$\frac{1}{n\mathfrak{r}_n^c} < \frac{1}{n(\log n)^{c/a}}$$

for all sufficiently large n. Therefore (1.5) converges.

3. An example

Let $\varphi(n)$ be a positive function such that $\varphi(n) \to \infty$ and $\varphi(n) = n^{o(1)}$ as $n \to \infty$. Given $0 \le R \le \infty$, we shall construct a power series $\sum_{p=0}^{\infty} a_p z^p$ with convergence radius R which has the property that $\mathfrak{r}_n > \varphi(n)$ for infinitely many n.

Let $\{b_k\}$ be a sequence of complex numbers distinct from zero which satisfies

$$\lim_{k\to\infty} | b_k |^{-1/k} = R.$$

We shall select a subsequence $\{b_{k_p}\}$ in such a way that the power series

$$\sum_{p=0}^{\infty} a_p z^p = \exp\{\textstyle\sum_{p=1}^{\infty} b_{k_p} z^{k_p}\}$$

has the desired properties. Let $k_1 = 1$; suppose now that $k_1, k_2, \cdots, k_q$ have been chosen. We note that if $n < k_{q+1}$, then $\sum_{p=0}^{n} a_p z^p$ is also the n^{th} partial sum of the power series for

$$\exp\{\textstyle\sum_{p=1}^{q} b_{k_p} z^{k_p}\}.$$

We can therefore apply the lower bound for $\mathfrak{r}_n$ which was obtained in [2, Theorem 4.1]. From this we have

$$\mathfrak{r}_n > A_{k_q} n^{1/k_p} \quad \textit{if} \quad n < k_{q+1},$$

where A_{k_q} is a positive number which depends only on $b_{k_1}, b_{k_2}, \cdots, b_{k_q}$. The hypothesis on $\varphi(n)$ guarantees that

$$A_{k_q} n^{1/k_q} > \varphi(n) \tag{3.1}$$

for all n sufficiently large. Let n_q be a value of n which satisfies (3.1) and is not less than k_q. Let $k_{q+1} = n_q + 1$. This determines the sequence $\{k_p\}$, and we have

$$\mathfrak{r}_n > \varphi(n) \quad \textit{for} \quad n + 1 = k_2, k_3, k_4, \cdots .$$

Let R_1 denote the convergence radius of $\sum_{p=0}^{\infty} a_p z^p$. It remains to show that $R_1 = R$. Clearly $R_1 \geq R$. Suppose $R_1 > 0$; since $\limsup \mathfrak{r}_n = \infty$, it follows from Hurwitz' theorem that $\sum_{p=0}^{\infty} a_p z^p$ has no zero in the disc $|z| < R_1$. Therefore

$$\sum_{p=1}^{\infty} k_p b_{k_p} z^{k_p - 1}, \tag{3.2}$$

the logarithmic derivative of $\sum_{p=0}^{\infty} a_p z^p$, converges for all $|z| < R_1$. The radius of convergence of (3.2) is equal to R, so that $R \geq R_1$. Therefore $R_1 = R$.

References

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2. J. D. Buckholtz, *Power series whose sections have zeros of large modulus*, Trans. Amer. Math. Soc., to appear.

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