INJECTIVES AND HOMOTOPY[1]

BY

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Any functor P on a category $\mathcal{A}$ determines an equivalence relation on the morphisms of $\mathcal{A}$ which is compatible with composition in $\mathcal{A}$. We call any relation determined by a functor a *homotopy* and develop the ideas of cylinder and cone functors in this general context. An appropriate generalization of the homotopy extension axiom implies that a cone functor is essentially a functorial injective for the category. This structure occurs in the cases of the usual homotopy on CW-complexes and the Eckmann-Hilton injective homotopy of modules which we present in a generalized categorical context.

For any category $\mathcal{A}$ together with a class of its morphisms M, the projection functor $P : \mathcal{A} \to \mathcal{A}/M$ yields a homotopy, where $\mathcal{A}/M$ is the Gabriel-Zisman category of fractions of $\mathcal{A}$ by M. Indeed for the "right" choice of M, P yields the well-known homotopies above: to be precise, take M to be the class of coretracts $i : X \to Y$ for all X, Y such that $Y/i(X)$ is injective. This enables us to determine the "right" homotopy from the knowedge of the contractible objects (injectives) alone.

1. Categorical preliminaries

If $\mathcal{A}$ is a category, a *homotopy* (or congruence) on $\mathcal{A}$ is an equivalence relation $\sim$ on each of the sets $\mathcal{A}(X, Y)$ of morphisms between objects of $\mathcal{A}$ which is compatible with composition; that is, $f \sim g$ implies $fh \sim gh$ and $kf \sim kg$ for all h, k for which the compositions are defined. If $\mathcal{A}$ has a homotopy $\sim$ the *homotopy category* of $\mathcal{A}$ with respect to $\sim$ is the category $\mathcal{A}/\sim$ whose objects are those of $\mathcal{A}$ and whose morphisms are the equivalence classes under $\sim$ of $\mathcal{A}(X, Y)$ together with the projection functor $\rho : \mathcal{A} \to \mathcal{A}/\sim$ which is the identity on objects and maps each morphism f to its equivalence class $[f]$ under $\sim$. The functor ρ determined by the homotopy is clearly universal with respect to all functors $F : \mathcal{A} \to \mathcal{B}$ such that $f \sim g$ implies $F(f) = F(g)$. Conversely any functor $F : \mathcal{A} \to \mathcal{B}$ defines a homotopy by $f \sim g$ iff $F(f) = F(g)$ and f, g have the same domain and codomain, though $\mathcal{B}$ need not then be $\mathcal{A}/\sim$, e.g. when F is not one-to-one on objects.

Three examples indicate the applicability and generality of these techniques. These are the usual homotopy of continuous functions on CW-complexes, the fibre homotopy in the category of functions to a fixed base space, and the Eckmann-Hilton injective homotopy of modules of which a development is given in Section 4. For more details see Hilton [4].

Received April 10, 1970.

[1] This research was supported in part by grants from the Research Corporation and the ETH.

Let $\mathfrak{A}$ be any category and M any class of its morphisms. We denote the *category of fractions of* $\mathfrak{A}$ *by* M [1, 3] by $\mathfrak{A}/M$ and denote the canonical projection by $P : \mathfrak{A} \to \mathfrak{A}/M$.

For any class M, the homotopy determined by the functor $P : \mathfrak{A} \to \mathfrak{A}/M$ is called *M-homotopy*. If two different classes of morphisms M and N yield the same homotopy we write $[M] = [N]$. Theorem 1.1 (Theorem 2.3 of [1]) is useful in determining when $[M] = [N]$.

THEOREM 1.1 (Bauer and Dugundji). *Let* $\mathfrak{A}$ *be a category,* M *a class of morphisms of* $\mathfrak{A}$ *with projection* $P : \mathfrak{A} \to \mathfrak{A}/M$, *and* $\sim$ *a homotopy on* $\mathfrak{A}$ *with* $Q : \mathfrak{A} \to \mathfrak{A}/\sim$ *the projection to the homotopy category. Suppose that if* $f_0 \sim f_1 : X \to Y$ *then there exists an object* Z_X *of* $\mathfrak{A}$ *with morphisms*

$$i_0, i_1 : X \to Z_X, \quad r : Z_X \to X \quad \text{and} \quad F : Z_X \to Y$$

such that $r \circ i_j = 1$ *and* $f_j = F \circ i_j$, $j = 0, 1$. *Then*:

(a) *If* $P(r)$ *is invertible in* $\mathfrak{A}/M$,

$$Q(f_0) = Q(f_1) \Rightarrow P(f_0) = P(f_1).$$

(b) *If* $Q(f)$ *is invertible for every* $f \in M$, *then*

$$P(f_0) = P(f_1) \Rightarrow Q(f_0) = Q(f_1).$$

We shall restrict our discussion to categories having conull objects. For CW-complexes this is the one point space, for modules the zero module, and for morphisms to a fixed base space the conull object is the identity morphism of the base space. Using this conull object we may define what we mean by a quotient. If $f : X \to Y$ is any morphism of the category $\mathfrak{A}$, we shall denote by $Y/f(X)$ the pushout (if it exists) of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

If for all morphisms f of $\mathfrak{A}$ the pushout above exists we shall say the category $\mathfrak{A}$ has quotients. If f is a monomorphism we shall call $Y/f(X)$ the quotient of Y by the subobject f, and where the context makes clear which monomorphism is meant we shall merely write Y/X and refer to the quotient of Y by (the subobject) X.

Finally we shall want the following definition [9].

DEFINITION 1.2. Let $f_0, f_1 : X \to Y$ be morphisms of $\mathfrak{A}$. We call $\gamma : Y \to Z$ a *weak equalizer* of f_0 and f_1 if

(a) $\gamma \circ f_0 = \gamma \circ f_1$ and

(b) if $\gamma' : Y \to Z'$ is any morphism such that $\gamma' \circ f_0 = \gamma' \circ f_1$ then there is a morphism (not necessarily unique) $g : Z \to Z'$ such that $\gamma' = g \circ \gamma$.

2. Cylinders and cones

In the two topological examples the following definition is illustrated by the cylinder over X, that is, $X \times I$ for a CW-complex X and $f : X \times I \to B$ where $f(x, t) = p(x)$ for a morphism $p : X \to B$. For an abelian category with an injective cogenerator the "standard" cylinder over X is the sum of X with $C(X)$ where C is a functorial injective (cf. Section 4).

DEFINITION 2.1. A *cylinder functor* for a category $\mathfrak{A}$ is a quadruple (Z, i_0, i_1, r) where $Z : \mathfrak{A} \to \mathfrak{A}$ is a functor and

$$i_0, i_1 : \mathrm{Id} \to Z \quad \text{and} \quad r : Z \to \mathrm{Id}\ \mathfrak{A}$$

are natural transformations such that $r \circ i_j = \mathrm{id}, j = 0, 1$ [6].

Any cylinder functor (Z, i_0, i_1, r) gives rise to an obvious relation on morphisms, written $f \sim_Z g$, which extends to an equivalence relation which is a homotopy, §1. The relation given directly by the cylinder may fail to be an equivalence relation, and moreover not every homotopy arises from a cylinder [8]. For a homotopy which does come from a cylinder to be given directly by that cylinder we mean that $f, g : X \to Y$ are homotopic iff there exists $H : Z(X) \to Y$ such that $H \circ i_0 = f_0, H \circ i_1 = f_1$. Henceforth we will assume that every homotopy is given directly by a cylinder functor and leave the extensions to the reader. It should be noted that Theorem 1.1 enables one to tell when the homotopy category resulting from a category with cylinder is a category of fractions.

When a given homotopy comes from a cylinder, we have the following:

PROPOSITION 2.2. *Let $\mathfrak{A}$ be a category with direct limits and a homotopy $\sim$ given directly by a cylinder functor (Z, i_0, i_1, r). Then the homotopy category $\mathfrak{A}/\sim$ has weak equalizers in the sense of Definition* 1.2.

Proof. A categorical version of the proof in [9, page 406].

It is easy to construct cases where the weak equalizer is not unique but this result is sufficient for its application in representability of cohomology theories.

Now if $\mathfrak{A}$ has a homotopy given directly by a cylinder functor (Z, i_0, i_1, r) then a morphism $f : X \to Y$ is homotopic to a null morphism (one factoring through 0) iff there is a morphism

$$F : Z(X)/i_1(X) \to Y$$

such that $F \circ p \circ i_0(X) = f$ where $p : Z(X) \to Z(X)/i_1(X)$ is the morphism from $Z(X)$ to $Z(X)/i_1(X)$ in the definition of $Z(X)/i_1(X)$ as a pushout. Now $Z(X)/i_1(X)$ defines a functor $C : \mathfrak{A} \to \mathfrak{A}$ and $p \circ i_0(X)$ defines a natural transformation $i : \mathrm{Id} \to C$. We call any pair (C, i) where $C : \mathfrak{A} \to \mathfrak{A}$ is a functor and $i : \mathrm{Id} \to C$ is a natural transformation a *pre-cone*. If for some object X, id_X is homotopic to a null morphism we say X is *contractible*. If the homotopy is given by a cylinder functor, X is contractible iff id_X may be factored as

$$X \xrightarrow{i(X)} C(X) \xrightarrow{F} X$$

for some X where $C(X) = Z(X)/i_1(X)$.

In any category with a pre-cone (C, i) we say an object X is *contractible* if id_X may be factored through $i(X)$. In the usual cases CX is contractible for every X, and indeed the existence of a "natural" contraction is equivalent to the existence of a natural transformation $p : C^2 \to C$ such that $p \circ i(C) = \mathrm{id}_C$. We call such a triple (C, i, p), where $C : \mathfrak{A} \to \mathfrak{A}$ is an endofunctor on the category $\mathfrak{A}$ and $i : \mathrm{id}_{\mathfrak{A}} \to C$ and $p : C^2 \to C$ are natural transformations such that $p \circ i(C) = \mathrm{id}_C$, a *cone-functor.*

3. Homotopy extension and injectives

Let $\mathfrak{A}$ be any category having a homotopy $\sim$ with projection $\rho : \mathfrak{A} \to \mathfrak{A}/\sim$ and a cone functor (C, i, p).

DEFINITION. We say a monomorphism $i : A \to X$ in $\mathfrak{A}$ has the *homotopy extension property* (HEP) if for each pair of morphisms $f : X \to Y$ and $g' : A \to Y$ such that $f' = f \circ i$ is homotopic to g' there exists $g : X \to Y$ such that $g' = g \circ i$ and g and f are homotopic. If every monomorphism satisfies HEP we say the homotopy satisfies HEP.

THEOREM 3.1. *If for every contractible object K of $\mathfrak{A}$, $\rho(\mathfrak{A}(X, K))$ is a one element set and if $\sim$ satisfies HEP then the contractible objects of $\mathfrak{A}$ are injective objects. Furthermore, if I is injective and $i(I) : I \to C(I)$ is a monomorphism then I is contractible.*

Proof. If $i(I)$ is a monomorphism and I is injective then $i(I)$ is a coretract so I is contractible. On the other hand let $j : A \to X$ be any monomorphism and $f' : A \to K$ any morphism. Since $\rho(\mathfrak{A}(X, K))$ is a one element set there exists $g : X \to K$ for which $g \circ j \sim f'$. By HEP there exists $f : X \to K$ such that $f \circ j = f'$ so K is injective. QED

Examples. With the usual homotopy and cone in the category of CW-complexes we see that the contractible objects are precisely the injective objects. We shall see in the next section that the same statement applies in the category of modules over a ring, using the notion of injective homotopy developed by Eckmann and Hilton [4]. Thus these categories have enough injectives and the contractible objects are precisely the absolute retracts in the categories.

4. Injective homotopy in an abelian category

In this section we shall discuss the structure of the homotopies on an abelian category which are given by a cylinder functor and which satisfy HEP. We shall see that such a homotopy is essentially unique and thus is the injective homotopy presented by Hilton [4].

For any abelian category $\mathfrak{A}$ with enough injectives the Eckmann-Hilton injective homotopy is defined as follows: $f, g \in \mathfrak{A}(X, Y)$ are homotopic iff f-g factors through some injective, or equivalently iff f-g factors through the inclusion of X into some injective.

We shall adopt the following notation. The morphism from $X \to A \times B$, the direct product of A and B, defined by $f : X \to A$ and $g : X \to B$ will be denoted by $(f, g)^{\mathrm{T}}$ (where T denotes transpose) and likewise we shall use (f, g) for the morphism from $A \oplus B \to X$ defined by $f : A \to X$, $g : B \to X$. In this notation matrix multiplication then yields the composition of the morphisms involved.

Let (Z, i_0, i_1, r) be a cylinder functor for $\mathfrak{A}$. Then since $p \circ i_0 = \mathrm{id}$ we may write $Z = I \oplus K$ where I is the identity functor and i_0, i_1, r become $(\mathrm{id}, 0)^{\mathrm{T}} : I \to I \oplus K$, $(\mathrm{id}, t)^{\mathrm{T}} : I \to I \oplus K$ and $(\mathrm{id}, 0) : I \oplus K \to I$ respectively for some natural transformation $t : I \to K$. We then see immediately that (Z, i_0, i_1, r) yields a homotopy defined by $f \sim g : X \to Y$ iff f-g factors through $t(X) : X \to K(X)$ and that this is precisely the relation $\sim_Z$ given by (Z, i_0, i_1, r). Thus, in this case, $\sim_Z$ is always an equivalence relation without extension. It should also be noted that (K, t) is a pre-cone and that the above shows that f-g factors through K iff f-g is null homotopic. Further if (C, i) is any pre-cone then

$$(I \oplus C, (\mathrm{id}, 0)^{\mathrm{T}}, (\mathrm{id}, i)^{\mathrm{T}}, (\mathrm{id}, 0))$$

is a cylinder functor and $\sim_{I \oplus C}$ is precisely the relation given by $f \sim_{I \oplus C} g$ iff f-g is nullhomotopic with respect to (C, i). Thus in abelian categories the study of pre-cone functors is equivalent to the study of cylinder functors.

Before proceeding to a general consideration of homotopies in an abelian category we shall note that in certain cases cylinder functors do exist which give the injective homotopy for an abelian category. We consider the dual of the construction given in [5]. Let $G(X) = \mathfrak{A}(X, U)$ where U is an injective cogenerator for $\mathfrak{A}$, be the usual contravariant hom functor. Then G has a right adjoint P which is a contravariant (!) product functor. For details see [8]. We may conclude (by duality) from Huber [6] that P and G are adjoint and that therefore $PG = C$, which is a covariant functor (!), has the structure of a triple (C, i, p). Also we note that C is an injective monofunctor iff U is an injective object in $\mathfrak{A}$ and $i(X)$ is a monomorphism for all X iff U is a cogenerator for $\mathfrak{A}$.

Definition. If for a cylinder functor (Z, i_0, i_1, r), $r(X) : ZX \to X$ is a homotopy equivalence with $i_0(X)$ as homotopy inverse we call (Z, i_0, i_1, r) a *natural cylinder functor.*

Proposition. $(I \oplus K, (\mathrm{id}, 0)^{\mathrm{T}}, (\mathrm{id}, t)^{\mathrm{T}}, (\mathrm{id}, 0))$ *is a natural cylinder functor iff* (K, t) *is a cone functor.*

Proof. A direct mechanical computation. For details see [8].

Now applying the above results together with the results of Section 3 we see that in an abelian category any homotopy given by a natural cylinder functor, or equivalently by a cone functor, which satisfies HEP is precisely the Eckmann-Hilton injective homotopy.

5. The injective homotopy in a category

We present here a construction of a homotopy for any category with quotients which depends only on the injectives of the category. Since this homotopy is the usual homotopy in the three examples we have been considering, we conclude that this is the "right" way to reclaim a homotopy from the knowledge of the contractible objects (injectives).

DEFINITION. Let M be the set of morphisms i of $\mathcal{A}$ such that $i : A \to B$ is a coretract and $B/i(A)$ is injective, construct the category $\mathcal{A}/M$ together with the projection functor $P : \mathcal{A} \to \mathcal{A}/M$ and define $f, g : X \to Y$ to be homotopic (written $f \sim g$) if $P(f) = P(g)$. We call $\sim$ the injective homotopy for $\mathcal{A}$.

THEOREM 5.1. (a) *If $\mathcal{A}$ is an abelian category with an injective cogenerator, $\sim$ is the usual Hilton-Eckmann injective homotopy.*

(b) *If $\mathcal{A}$ = CW-complexes with continuous functions, $\sim$ is the usual homotopy of continuous functions.*

Proof. (a) Let (Z, i_0, i_1, r) be the cylinder functor for $\mathcal{A}$ discussed in Section 4. By Theorem 1.1 it is sufficient to prove that $P(r) \in \mathcal{A}/M$ is invertible and that each $i \in M$ is invertible in $\mathcal{A}_H$, the homotopy category of $\mathcal{A}$ with respect to Hilton-Eckmann homotopy. If $i \in M$, $i : A \to B$, let $\rho : B \to A$ be such that $\rho \circ i = \mathrm{id}_A$. Then

$$A \xrightarrow{i} B \xrightarrow{\rho} A$$

may be factored as

$$A \xrightarrow{(\mathrm{id}, i')^T} A \oplus I \xrightarrow{(\mathrm{id}, 0)} A$$

for some $i' : A \to I$ where I is an injective. Then clearly i is invertible in $\mathcal{A}_H$. Now in $\mathcal{A}/M$ i_0 has an inverse j hence, since $i_0 \circ r \circ i_0 = i_0$, we have $i_0 \circ r = \mathrm{id}$ in $\mathcal{A}/M$. So $P(r)$ is indeed invertible in $\mathcal{A}/M$ with i_0 (or i_1) as inverse.

(b) Let (Z, i_0, i_1, r) be the usual cylinder functor for topological spaces. That $P(r)$ is invertible follows exactly as above. Let $i \in M$ where

$$A \xrightarrow{i} B \xrightarrow{\rho} A$$

with $\rho \circ i = \mathrm{id}_A$. It remains to show that $i \circ \rho \sim \mathrm{id}_B$. Since this is a homotopy problem we may assume that i is a cellular map. Since $B/i(A)$ is injective it is contractible and

$$\tilde{H}^n(B/i(A); G) \cong \tilde{H}^n(B, A; G) = 0 \quad \text{for all } n, G$$

(including local coefficients G). Then by obstruction theory the partial homotopy which is given by $r : A \times I \to A \subset B$ on $A \times I$, the identity on $B \times \{0\}$, and $i \circ \rho$ on $B \times \{1\}$ may be extended to $B \times I$. So $i \circ \rho \sim \mathrm{id}_B$. QED

6. Summary

The previous section has shown that in two classical cases it is indeed possible to reconstruct the homotopy from the knowledge of the contractible objects (injectives) alone. This appears to answer a conjecture of R. L. Knighten [7].

For an abelian category a category of fractions may be described equivalently as a quotient category in which identification occurs only within morphism classes; instead of specifying the class of morphisms to be inverted by an additive functor, one specifies the corresponding class of morphisms to be identified with (appropriate) null morphisms. Thus for an abelian category Freyd [2] obtains (dually) via quotient categories various homotopy categories. Freyd's central construction enables one to reconstruct an abelian category from any "ample class", that is a full subcategory of a resolving set of relative projectives. The analogous problem of reconstructing a topological category from the full subcategory of contractible objects runs into the usual problem of choosing the "right" category of topological spaces: for example, the category of CW-complexes is not closed under "kernels" of morphisms between injectives.

Although computational difficulties in $\mathcal{A}/M$, which allows no calculus of fractions since homotopy does not preserve limits, make it difficult to prove that injective homotopy has in general such properties as HEP, we hope eventually to investigate the above and more of the general properties of the injective homotopy. However, the following remarks indicate that some at least of the expected results do hold in our general context.

It can be shown that if $\alpha : X \to Y$ has a left inverse ρ, and $\alpha \in M$ then $P(\alpha)$ has $P(\rho)$ as its inverse. Hence if α is a coretract in M with $\rho \circ \alpha = 1_X$ we may add ρ to M and $[M \cup \{\rho\}] = [M]$. We should also note that if $\alpha : 0 \to I$ is a morphism where 0 is conull and I is injective, then $\alpha \in M$. Thus for injective homotopy every injective is contractible.

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