QUASI-SIMILARITY OF OPERATORS

BY

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1. Introduction

A natural method for constructing an invariant subspace for an operator on Hilbert space is to find a second, known operator which is similar in some weak sense to the given operator and then to use this second operator and the weak similarity to construct the desired subspace. One such weak similarity is the notion of quasi-similarity introduced by Sz.-Nagy and Foiaş [13].

In what follows, $\mathcal{H}$ will denote a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ will be the algebra of all bounded linear operators on $\mathcal{H}$. (The term operator is meant to imply boundedness and linearity. Unless specified otherwise, all operators are assumed to be acting on the Hilbert space $\mathcal{H}$.) An operator X from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is said to be *quasi-invertible* if X has zero kernel and dense range. Operators S and T acting on $\mathcal{H}$ and $\mathcal{K}$ respectively are *quasi-similar* if there are quasi-invertible operators X from $\mathcal{K}$ to $\mathcal{H}$ and Y from $\mathcal{H}$ to $\mathcal{K}$ which satisfy the equations

$$XT = SX \quad \text{and} \quad TY = YS.$$

It is clear that quasi-similarity is an equivalence relation on the class of all operators. A closed subspace $\mathfrak{M}$ of $\mathcal{H}$ is *invariant* for an operator S in $\mathcal{L}(\mathcal{H})$ if $S(x)$ is in $\mathfrak{M}$ for every x in $\mathfrak{M}$. The subspace $\mathfrak{M}$ is *hyperinvariant* if it is neither the zero subspace nor all of $\mathcal{H}$ and if $\mathfrak{M}$ it is invariant for every operator in $\mathcal{L}(\mathcal{H})$ which commutes with S. Hyperinvariant subspaces have been studied in [3], [4], [8], [13]. If T is an operator which is quasi-similar to an operator with an invariant subspace, then it is not known if T need have an invariant subspace, but the following is proved in [8]:

THEOREM. *If S and T are quasi-similar operators acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and if S has a hyperinvariant subspace, then so does T. If in addition, S is normal, then the lattice of hyperinvariant subspaces for T contains a sublattice which is lattice isomorphic to the lattice of spectral projections for S.*

The purpose of this paper is to discover which properties of operators are preserved by quasi-similarity and which are not. In Section 2 we use the fact that quasi-similar normal operators are unitarily equivalent to show that quasi-similar spectral operators are very closely related. We also investigate

Received July 9, 1970.

[1] This paper consists of part of the author's doctoral dissertation written at the University of Michigan under the direction of Professor C. Pearcy The author is grateful to Professor Pearcy for his many helpful suggestions.

the relationship between our quasi-similarity and the weak similarity notions of Feldzamen [6] and of Tzafriri [14]. Then we show that quasi-similar isometries are unitarily equivalent. The last section contains two examples; one example shows that quasi-similarity preserves neither spectra nor compactness, the second shows that not every quasi-similarity is a direct sum of similarities in sense to be made precise below.

If S is an operator, its spectrum will be denoted by $\sigma(S)$. The symbols $\oplus$ and $\sum \oplus$ will be used to denote orthogonal direct sums of both Hilbert spaces and operators. The closure of some subset U of a topological space will be denoted by U^-.

2. Quasi-similarity of spectral operators

The following theorem is proved by Douglas in [2]. It shows, in particular that quasi-similar normal operators are unitarily equivalent.

THEOREM 2.1. *Let N_1 and N_2 be normal operators acting on the spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and let X be an operator from $\mathcal{H}$ to $\mathcal{K}$ satisfying $XN_1 = N_2X$. If $\mathfrak{M}$ denotes the orthogonal compliment in $\mathcal{H}$ of the kernel of X, and if $\mathfrak{N}$ denotes the closure in $\mathcal{K}$ of the range of X, then $\mathfrak{M}$ and $\mathfrak{N}$ reduce N_1 and N_2 respectively, and $N_1 \mid \mathfrak{M}$ is unitarily equivalent to $N_2 \mid \mathfrak{N}$ via the unitary operator $U \mid \mathfrak{M}$ where $X = UP$ is the polar decomposition of $X(P = (X^*X)^{1/2})$. In particular, if X is quasi-invertible, the N_1 and N_2 are unitarily equivalent.*

Briefly, an operator T on $\mathcal{H}$ is spectral if it has a resolution of the identity much like that of a normal operator. Let E be a σ-homomorphism of the σ-algebra of Borel subsets of the complex plane onto a σ-algebra of uniformly bounded (in norm) idempotents in $\mathcal{L}(\mathcal{H})$ which contains the zero and the identity operators. The map E is a *resolution of the identity for T* if for every Borel set B in the plane, $E(B)T = TE(B)$, and $\sigma(T \mid E(B)(\mathcal{H})) \subseteq B^-$ (the closure of B) where $T \mid E(B)(\mathcal{H})$ denotes the restriction of T to the range of $E(B)$. The operator T is called a *spectral operator* if it has a resolution of the identity.

The following properties of spectral operators are important to our discussion. For a thorough discussion of spectral operators see [5]. First, an operator T is spectral if and only if it can be written in the form $T = N + S$ where N, the scalar part, is similar to a normal operator, S is quasi-nilpotent ($\sigma(S) = \{0\}$), and N commutes with S. This decomposition of spectral operators, called the canonical decomposition, is unique. The invertible operator A for which ANA^{-1} is normal transforms the resolution of the identity E of T onto the spectral measure of ANA^{-1}. The spectrum of T is the spectrum of N, and if R is an operator which commutes with T then for every Borel set B, R commutes with $E(B)$ and hence R commutes with N.

THEOREM 2.2. *Suppose for $i = 1, 2$, $T_i = N_i + S_i$ are spectral operators written in their canonical decomposition. If there is a quasi-invertible operator*

X such that $XT_1 = T_2X$, *then*

(i) $XS_1 = S_2X$; $XN_1 = N_2X$.

(ii) N_1 *is similar to* N_2,

(iii) $\sigma(T_1) = \sigma(T_2)$.

Proof. There are invertible operators A_i such that for $i = 1, 2$, $A_i^{-1}N_i A_i$ is normal. Thus, replacing T_i by $A_i^{-1}T_i A_i$, it suffices to assume that the operators N_i are normal. Consider the following operators acting on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$:

$$Y = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} + \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

Since $XT_1 = T_2 X$, these two operators commute. But T is a spectral operator so Y commutes with the scalar (normal) part of T. It follows that $XN_1 = N_2 X$ and thus $XS_1 = S_2 X$. By Theorem 2.1, N_1 is unitarily equivalent to N_2 and since $\sigma(T_i) = \sigma(N_i)$, $\sigma(T_1) = \sigma(T_2)$.

Part (iii) of Theorem 2.2 is not new; Colojoara and Foias [1] show that quasi-similar decomposable operators have the same spectrum, and the class of decomposable operators properly includes the class of spectral operators.

Two generalizations of similarity have been defined on the class of spectral operators. One of these, introduced by L. Tzafriri [14], has the misfortune of being called quasi-similarity. In order to avoid confusion, we introduce the term weakly similar for Tzafriri's notion. Suppose T_1 and T_2 are spectral operators with resolutions of the identity E_1 and E_2 respectively. We say T_2 is *weakly similar* to T_1 if there is a densely defined closed linear transformation A on $\mathcal{H}$ with densely defined inverse such that

(i) $(AT_1 A^{-1})x = T_2 x$ for every x in the domain of A^{-1} and

(ii) for every Borel set B, there is a constant M_B such that

$$\| (AE_1(B)A^{-1})x \| \leq M_B \| x \|$$

for each x in the domain of A^{-1}.

THEOREM 2.3. *If* T_1 *and* T_2 *are spectral operators with resolutions of the identity* E_1 *and* E_2, *and if* X *is a quasi-invertible operator such that* $XT_1 = T_2 X$, *then* T_2 *is weakly similar to* T_1.

Proof. It suffices to show that for every Borel set B, $XE_1(B)X^{-1}$ is bounded on the domain of X^{-1}, for then the operator X satisfies all the conditions of the definition of weak similarity. For $i = 1, 2$, write T_i as $N_i + S_i$ and we may assume that N_i is normal and that E_i is the spectral measure for N_i. Write X in its polar decomposition $X = UP$ where $P = (X^*X)^{1/2}$ and U is unitary. By Theorem 2.2, $XN_1 = N_2X$ and by Theorem 2.1, U is a unitary equivalence between N_1 and N_2. Consequently, $UE_1(B)U^* = E_2(B)$ for every Borel set B. It also follows by the Putnam-Fuglede Theorem that $N_1 X^* = X^*N_2$

and therefore

$$N_1 X^*X = X^*N_2 X = X^*XN_1$$

or X^*X commutes with N_1. Thus P commutes with N_1 and hence with $E_1(B)$. If x is in the domain of X^{-1}, then

$$(XE_1(B)X^{-1})x = (UPE_1(B)P^{-1}U^*)x = (UE_1(B)U^*)x = E_2(B)x.$$

So, we have shown what we need to show and the proof is complete.

COROLLARY 2.4. *Quasi-similar spectral operators are weakly similar.*

Another generalized similarity notion defined for spectral operators, called semi-similarity, was introduced by A. N. Feldzamen [7]. Suppose T_1 and T_2 are spectral operators which act on the Hilbert space $\mathcal{H}$ and which have resolutions of the identity E_1 and E_2 respectively. Feldzamen calls T_1 and T_2 *semi-similar* if for each $i = 1, 2$, there is a family $\{P^i_\alpha : \alpha \text{ in } A\}$ of pairwise disjoint nonzero idempotents in the weak closure of the range of E_i with $\bigvee_{\alpha\epsilon A} P^i_\alpha = I$ and such that for each α in A, the restriction of T_1 to the range of P^1_α is similar to T_2 restricted to the range of P^2_α. That semi-similarity implies quasi-similarity is a consequence of the following theorem.

THEOREM 2.5. *Suppose that for each α in some index set A, there are Hilbert spaces $\mathcal{H}_\alpha$ and $\mathcal{K}_\alpha$ and operators T_α and S_α in $\mathcal{L}(\mathcal{H}_\alpha)$ and $\mathcal{L}(\mathcal{K}_\alpha)$ respectively which are quasi-similar. Let T be the operator $T = \sum_{\alpha\epsilon A} \oplus T_\alpha$ acting on the Hilbert space which is the direct sum of the spaces $\mathcal{H}_\alpha$, and let $S = \sum_{\alpha\epsilon A} \oplus S_\alpha$ in $\mathcal{L}(\mathcal{K})$ where $\mathcal{K} = \sum_{\alpha\epsilon A} \oplus \mathcal{K}_\alpha$. Then T is quasi-similar to S.*

Proof. Suppose X_α and Y_α are the quasi-invertible operators such that $X_\alpha T_\alpha = S_\alpha X_\alpha$ and $T_\alpha Y_\alpha = Y_\alpha S_\alpha$. If

$$X = \sum_{\alpha\epsilon A} \oplus X_\alpha/\| X_\alpha \| \quad \text{and} \quad Y = \sum_{\alpha\epsilon A} \oplus Y_\alpha/\| Y_\alpha \|,$$

then X and Y are quasi-invertible and satisfy the desired equations.

COROLLARY 2.6. *If T_1 and T_2 are semi-similar spectral operators, they are quasi-similar.*

Proof. The operators T_1 and T_2 are similar to spectral operators whose scalar parts are normal. Thus, since both semi-similarity and quasi-similarity are equivalence relations which generalize similarity [6, Thm. 26], we may assume that the scalar parts of T_1 and T_2 are normal. Let $\{P^i_\alpha : \alpha \epsilon A\}$ be the collections of projections whose existence semi-similarity asserts, and let T^i_α be the restriction of T_i to the range of P^i_α. The idempotents P^i_α are self-adjoint, so their ranges reduce T_i and hence $T_i = \sum_{\alpha\epsilon A} \oplus T^i_\alpha$. Also, T^1_α is similar to T^2_α, so by Theorem 1.7, T_1 is quasi-similar to T_2.

The notion of multiplicity can be generalized from normal operators to spectral operators [6]. For spectral operators with no part of infinite uniform multiplicity, an object called the Weyr characteristic of the operator can be defined. Both Feldzamen and Tzafriri show that this Weyr characteristic is a complete similarity invariant for their special similarity and Tzrafriri's proof is valid for spectral operators on Banach spaces. Thus for spectral operators on Hilbert space with no part of infinite uniform multiplicity, all three generalizations of similarity coincide. It does not seem to be known if this is true in general.

3. Quasi-similarity and isometries

The isometries make up another class of operators in which quasi-similarity reduces to unitary equivalence. Important to our discussion is Wold's characterizations of isometries [13, p. 3]. If $\mathfrak{D}$ is a Hilbert space, then $\mathbf{H}_{\mathfrak{D}}$ denotes the direct sum $\sum_{i \geqq 0} \oplus \mathfrak{D}^i$ where each $\mathfrak{D}^i$ is $\mathfrak{D}$. We identify $\mathfrak{D}$ with the subspace of $\mathbf{H}_{\mathfrak{D}}$ consisting of those vectors f in $\mathbf{H}_{\mathfrak{D}}$ for which $f(i) = 0$ for $i \neq 0$. The forward shift on $\mathbf{H}_{\mathfrak{D}}$ is the operator U defined by

$$(Uf)(0) = 0, \qquad (Uf)(i) = f(i-1) \quad \text{for} \quad i > 0.$$

Wold's theorem says that every isometry is of the form $U \oplus F$ acting on $\mathbf{H}_{\mathfrak{D}} \oplus \mathfrak{F}$ where F is unitary.

THEOREM 3.1. *Quasi-similar isometries are unitarily equivalent.*

Proof. Suppose V_1 and V_2 are quasi-similar isometries and let X and Y be the quasi-invertible operators such that $XV_1 = V_2 X$ and $V_1 Y = YV_2$. Write V_1 and V_2 according to their Wold decompositions: $V_i = U_i \oplus F_i$ acting on $\mathbf{H}_{\mathfrak{D}_i} \oplus \mathfrak{F}_i$. Taking the adjoint of the equation relating X and the V_i, we obtain $V_1^* X^* = X^* V_2^*$, so if f is in $\mathfrak{D}_2$, the kernel V_2^*, then

$$(V_1^* X^*)f = (X^* V_2^*)f = 0.$$

Thus $X^*(\mathfrak{D}_2)$ is in $\mathfrak{D}_1$, the kernel of V_1^*. It follows that the dimension of $\mathfrak{D}_2$ is less than or equal to the dimension of $\mathfrak{D}_1$ and that $X^*(\mathbf{H}_{\mathfrak{D}_2}) \subseteq \mathbf{H}_{\mathfrak{D}_1}$. The adjoint of the last containment is $X(\mathfrak{F}_1) \subseteq \mathfrak{F}_2$. Viewing X as an operator from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ we have $XF_1 = F_2 X$. By Proposition 2.1, $\mathfrak{N} = X(\mathfrak{F}_1)^-$ reduces F_2 and F_1 is unitarily equivalent to the restriction of F_2 to $\mathfrak{N}$.

Using the equation involving Y in the same way the one involving X was used, we get that $\dim \mathfrak{D}_1 \leq \dim \mathfrak{D}_2$ and that F_2 is unitarily equivalent to the restriction of F_1 to some reducing subspace. Since the dimensions of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the same, U_1 is unitarily equivalent to U_2. So it remains to show that F_1 is unitarily equivalent to F_2, but this is accomplished by citing the operator theoretic version of the Schroder-Bernstein theorem [9].

4. Examples

Quasi-similar normal operators, isometries, or even spectral operators are very much alike, but in general quasi-similar operators can be quite different. Sz.-Nagy and Foias show that quasi-similarity need not preserve the spectrum

[13, p. 250]. Unfortunately, their example involves the rather complicated "function caractéristique" and still leaves unanswered the question of whether or not the boundary of the spectra of quasi-similar operators need be the same. In the following example we exhibit quasi-similar operators A and B where $\sigma(A) = \{z : |z| \leq 1\}$ and $\sigma(B) = \{0\}$. It turns out that B is compact while A is not, so we discover that quasi-similarity need not preserve compactness.

Let A_n and B_n be the operators on n-dimensional Hilbert space given by the following $n \times n$ matrices:

$$A_n = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & & & & \\ 0 & 1 & \cdot & & & \\ \cdot & 0 & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \cdot & \\ \cdot & \cdot & & & \cdot & \cdot \\ 0 & \cdot & & & 1 & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1/n & 0 & & & & \\ 0 & 1/n & & & & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ 0 & & & & 1/n & 0 \end{pmatrix}.$$

Then A_n and B_n are similar, in fact A_n is the Jordan canonical form for B_n. If $A = \sum_{n=1}^{\infty} \oplus A_n$ and $B = \sum_{n=1}^{\infty} \oplus B_n$, then by Theorem 2.5, A is quasi-similar to B. Both A and B are weighted shift operators [7, p. 46] and hence their spectra are disks centered at the origin [10, Thm. 5, p. 20]. It remains to compute the spectral radii $r(A)$ and $r(B)$. A simple computation shows that for $k < n$, A_n^k is the $n \times n$ matrix with ones along the k^{th} diagonal below the main diagonal and zeros elsewhere, and $A_n^k = 0$ for $k \geq n$. Thus for $k < n$, $\| A_n^k \| = 1$ and so $\| A^k \| = 1$. It follows that the spectral radius $r(A)$ is

$$r(A) = \lim_k \| A^k \|^{1/k} = 1.$$

$$\| B_n^k \| = \| (A_n/n)^k \| = (1/n)^k \| A_n^k \|.$$

Thus $\| B_n^k \|$ is 0 if $k \geq n$ and is $(1/n)^k$ if $k < n$. Consequently

$$\| B^k \| = \sup_{n>k} (1/n)^k = 1/(k+1)^k$$

and $r(B) = \lim_{k\to\infty} \| B^k \|^{1/k} = 0$. That B is compact while A is not follows from [7, p. 86].

In a sense, this is the best (or worst) possible example of a quasi-similarity.

THEOREM 4.1. *If T and S are quasi-similar operators on $\mathcal{H}$, then $\sigma(T) \cap \sigma(S)$ is non-empty.*

Proof. Suppose X is the quasi-invertible operator such that $TX = XS$. Define an operator $D_{T,S}$ on $\mathcal{L}(\mathcal{H})$ by $D_{T,S}(A) = TA - AS$ for each A in $\mathcal{L}(\mathcal{H})$. By [11], D_{TS} is invertible if $\sigma(T)$ and $\sigma(S)$ are disjoint. But $D_{TS}(X) = 0$, so D_{TS} is not invertible and there must be an α in $\sigma(T) \cap \sigma(S)$.

The quasi-similarity in the above example and in [8] are constructed using

Theorem 2.5 in the case where the summands are not only quasi-similar, but similar. This raises the question of whether or not all quasi-similarities come about in this way. With the following example we show that the answer to this question is no by constructing a non-invertible operator T with no reducing subspaces but which is quasi-similar to a unitary operator. Since T is not invertible, it cannot actually be similar to the unitary operator, and since it has no reducing subspaces, it cannot be a direct sum. Central to this construction is a theorem of Sz.-Nagy and Foiaş, [13, p. 72] which asserts that if T is an operator on $\mathcal{H}$ which is power bounded (i.e., $\| T^n \| \leq M$ for $n = 1, 2, \cdots$) and if for every non-zero f n $\mathcal{H}$, neither $\| T^n f \|$ nor $\| T^{*n} f \|$ converge to zero, then T is quasi-similar to a unitary operator.

If $\mathcal{H}$ is a Hilbert space, let $\mathbf{L}_{\mathcal{H}}$ denote the Hilbert space direct sum $\sum_{i=-\infty}^{\infty} \oplus \mathcal{H}_i$ where each $\mathcal{H}_i$ is $\mathcal{H}$. We shall view the elements of $\mathbf{L}_{\mathcal{H}}$ as $\mathcal{H}$ valued functions defined on the integers and we identify $\mathcal{H}_n$ with the subspace of $\mathbf{L}_{\mathcal{H}}$ consisting of the functions f such that $f(i) = 0$ for $i \neq n$. If A and B are operators on $\mathcal{H}$, define T_{AB} in $\mathcal{L}(\mathbf{L}_{\mathcal{H}})$ by

$$(T_{AB}f)(0) = Af(-1), \qquad (T_{AB}f)(1) = Bf(0),$$
$$(T_{AB}f)(n) = f(n-1) \quad \text{for} \quad n \neq 0, 1.$$

Matricially, T_{AB} is the following two way infinite matrix acting on $\mathbf{L}_{\mathcal{H}}$ in the usual fashion:

$$T_{AB} = \begin{pmatrix} \ddots & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & A & (0) & & \\ & & & & & B & 0 & \\ & & & & & & 1 & 0 \\ & & & & & & & 1 & \ddots \\ & & & & & & & & \ddots \end{pmatrix}.$$

A simple computation shows that for $n \geq 1$,

$$\| T_{AB}^n \| \leq \max \{1, \| A \|, \| B \|, \| BA \|\}$$

and thus T_{AB} is power bounded.

In case A and B are invertible, so is T_{AB} and $T_{AB}^{-1*} = T_{A^{-1*}B^{-1*}}$. Thus

$$\| T_{AB}^{-n} \| = \| (T_{AB}^{-n})^* \| = \| T^n_{A^{-1*}B^{-1*}} \|$$

and T_{AB}^{-1} is power bounded. It follows by a theorem of Sz.-Nagy that T_{AB} is similar to a unitary operator [12].

Now suppose that A and B are only quasi-invertible and let f be any non-zero element of $\mathbf{L}_{\mathcal{H}}$. Then for some integer α, $f(\alpha)$ is non-zero and so for each positive integer n,

$$
\begin{aligned}
(T^n_{AB}f)(\alpha + n) &= f(\alpha) && \text{for } \alpha > 0 \\
&= Af(\alpha) && \text{for } \alpha < 0, \alpha + n = 0 \\
&= BAf(\alpha) && \text{for } \alpha < 0, \alpha + n > 0 \\
&= Bf(\alpha) && \text{for } \alpha = 0 \\
&= f(\alpha) && \text{for } \alpha + n < 0.
\end{aligned}
$$

In any case,

$$
\begin{aligned}
\| T^n_{AB}f \|^2 = \textstyle\sum_{i=-\infty}^{\infty} \| (T^n_{AB}f)(i) \|^2 &\geq \| (T^n_{AB}f)(\alpha + n) \|^2 \\
&\geq \min \{\| f(\alpha) \|^2, \| Af(\alpha) \|^2, \| Bf(\alpha) \|, \| BAf(\alpha) \|\} > 0.
\end{aligned}
$$

Consequently $\| T^n_{AB}f \|$ is bounded away from zero. A similar argument shows that the same is true of $\| T^{*n}_{AB}f \|$ and thus by the theorem of Sz.-Nagy and Foiaş, T_{AB} is quasi-similar to some unitary operator.

Choose A and B to be positive quasi-invertible operators with no common non-trivial reducing subspaces and arrange things so that A is not invertible and both A and B have norm strictly less than one. To be explicit, choose $\mathcal{H}$ to be L^2 of the unit circle with Lebesgue measure, let h be the function in L^∞ defined by $h(e^{it}) = t/4\pi$, $0 \leq t \leq 2\pi$, and let A be the multiplication operator determined by h:

$$Af = hf \quad \text{for} \quad f \text{ in } \mathcal{H}.$$

The reducing subspaces for A are the subspaces of functions which vanish almost everywhere on some specified set. Let $\{e_n\}_{-\infty<n<\infty}$ be the standard orthonormal basis for $\mathcal{H}$, $e_n(z) = z^n$, and let $\{b_n\}_{-\infty<n<\infty}$ be a sequence of distinct positive scalars $0 < b_n \leq 1/2$. Define B by $B(e_n) = b_n e_n$ and extend linearly and continuously to all of $\mathcal{H}$. The reducing subspaces of B are those subspaces of $\mathcal{H}$ spanned by subsets of the e_n. Since A is not invertible, there are vectors f_n in $\mathcal{H}$ such that $\| f_n \| = 1$ and $\| Af_n \|$ converges to zero. If f^1_n are the corresponding vectors in $\mathcal{H}_{-1}$, then $\| Tf^1_n \|$ converges to zero so T_{AB} is not invertible.

Consider the von Neumann algebra $\mathcal{A}$ generated by $T_{AB} = T$. ($\mathcal{A}$ is the smallest weakly closed *-subalgebra of $\mathcal{L}(\mathbf{L}_{\mathcal{H}})$ which contains T.) A simple matricial calculation shows that $(TT^*)^n$ converges in the uniform topology to the projection M onto $\bigvee_{i\neq 0,1} \mathcal{H}_i$. (Here projection means orthogonal projection, and $\bigvee_{i\neq 0,1} \mathcal{H}_i$ is the closed subspace spanned by the $\mathcal{H}_i$, $i \neq 0, 1$. More computations show that $(MT)^n(MT^*)^n$ converges weakly to the projection E onto $\bigvee_{i<0} \mathcal{H}_i$; and $(T^*T)^n(1 - E)$ converges uniformly to F, the projection onto $\bigvee_{i>0} \mathcal{H}_i$.

Let P be any projection in $\mathcal{L}(\mathbf{L}_{\mathcal{H}})$ which commutes with every operator in $\mathcal{A}$.

In particular, P commutes with E, F, and $1 - (E + F)$, so $P = Q \oplus R \oplus S$ where Q, R and S are projections in

$$\mathcal{L}(\textstyle\sum_{i=-\infty}^{-1} \oplus \mathcal{H}_i), \quad \mathcal{L}(\mathcal{H}_0), \quad \text{and} \quad \mathcal{L}(\textstyle\sum_{i=1} \oplus \mathcal{H}_i).$$

But P also commutes with both TT^* and T^*T so R commutes with both BB^* and A^*A. The operators A and B were chosen to be positive operators so R commutes with both A and B. Therefore R is either the zero or the identity operator. If R is the identity, then since P commutes with T, $T^n(\mathcal{H}_0)^- = \mathcal{H}_n$ is in the range of S and consequently S must be the identity operator. Similarly $T^{*n}(\mathcal{H}_0)^- = \mathcal{H}_{-n}$ is in the range of Q, so Q and hence P is the identity operator. If, on the other hand, R is zero, then $1 - P$ is also a projection which commutes with $\mathfrak{A}$, and $1 - P$ is restricted to $\mathcal{H}_0$ will be the identity. Thus $1 - P$ is 1 as above or P is the zero projection.

Any projection which commutes with T will also commute with T^* and hence with $\mathfrak{A}$, so we have shown that T has no non-trivial reducing subspaces. This completes our example.

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