SOME ABSTRACT PROPERTIES OF THE SET OF INVARIANT SETS OF A FLOW[1]

BY

C. C. CONLEY

The results contained here belong somewhere near the beginning of the abstract study of flows on compact metric spaces. The theorems are general, quite easy to prove, and help make up a setting for deeper results.

Specifically, given some connected compact metric space X we assign to each flow on X the collection of closed invariant sets of the flow. This collection is viewed as a subspace of the space of closed subsets of X with the Hausdorff metric. The larger space shares with X the properties of connectedness and compactness, and the collection of invariant sets is likewise compact. However, it needn't be connected. The lack of connectivity corresponds to the existence of "isolated" invariant sets and so inspires the definition of a second function which assigns to each flow the (closure of) the collection of isolated invariant sets.

Two of the three theorems proved concern these set-valued functions; they are shown to be continuous "almost everywhere" with respect to various topologies on the various sub-sets of the sets of flows (Theorems 2.3 and 4.2).

The third theorem (3.5) relates the notions of isolated and quasi-isolated invariant sets of a flow to connectivity properties of the sapce of closed invariant sets. Isolated invariant sets were defined by T. Ura in [1] and have been discussed more recently in [2], [3] and [4]. In the latter references and in [5] some "local" perturbation results for isolated invariant sets were described.[2] The emphasis in these theorems is on algebraic (cohomological) properties of a single isolated invariant set which are preserved under perturbation. Thus they complement the more "global" and topological results listed here.

In Section 1, results concerning the space of closed subsets of a compact metric space are listed; included is the theorem about continuity points of semi-continuous functions. These known results are included separately for ready reference as well as to set the tone.

1. The space of closed subsets of X

Throughout the paper X denotes a connected compact metric space with metric ρ. For $S \subset X$, $N_\varepsilon(S)$ is the open set of points within ε of S. Most of

Received June 25, 1970.

[1] This work was partially supported by a Sloan Foundation grant to the Courant Institute of Mathematical Sciences, New York University, and partially supported by a National Science Foundation grant to the University of Wisconsin.

[2] These results were proved for flows on smooth manifolds; however they have been generalized to fit the present setting in as yet unpublished work of R. Churchill.

the statements below can be found in references [6]. The proofs in the present setting are all elementary.

1.1 DEFINITION. $C(X)$ is the collection of non-empty closed subsets of X. If A and B are in $C(X)$, then

$$\rho^*(A, B) \equiv \inf\{\varepsilon \mid N_\varepsilon(A) \supset B\}, \qquad \rho(A, B) \equiv \max[\rho^*(A, B), \rho^*(B, A)].$$

1.2 LEMMA. *The function ρ is a metric on $C(X)$ and the resulting topological space is compact and connected (since X is).*

1.3 DEFINITION. For $S \subset C(X)$ let $\pi(S) \subset X$ denote the union of elements of S.

1.4 LEMMA. *If S is open in $C(X)$ then $\pi(S)$ is open in X; if S is closed in $C(X)$, then $\pi(S)$ is closed in X.*

Also if U is open (closed) in X then the collection of closed subsets of X contained in U is open (closed) in $C(X)$.

1.5 DEFINITION. With E a topological space and Y a compact metric space, a function $\mathcal{A} : E \to C(Y)$ is called upper semi-continuous (u.s.c.) at $e \in E$ if given $\varepsilon > 0$ there is an open set W about e such that $e' \varepsilon W$ implies $\rho^*(\mathcal{A}(e), \mathcal{A}(e')) < \varepsilon$.

Similarly, $\mathcal{A}$ is lower semi-continuous (l.s.c.) at e if given $\varepsilon > 0$ there is an open set W about e such that $e' \varepsilon W$ implies $\rho^*(\mathcal{A}(e'), \mathcal{A}(e)) < \varepsilon$. $\mathcal{A}$ is u.s.c. (l.s.c.) if it is at every point of E.

1.6 DEFINITION. E is called a Baire space if the intersection of a countable number of dense open sets in E is dense. Such an intersection is called a dense G_δ of E.

1.7 LEMMA. *If E is a Baire space and $E' \subset E$ is a dense G_δ of E then E' is a Baire Space (in the relative topology) and any dense G_δ of E' is a dense G_δ of E.*

1.8 LEMMA. *If $\mathcal{A} : E \to C(Y)$ is u.s.c. (l.s.c.) and E is a Baire space, then $\mathcal{A}$ is continuous on a dense G_δ of E.*

2. The set of invariant sets of a flow

The collection of flows on X will be denoted F; it is a subset of the set of continuous functions from $X \times R \to X$ where R denotes the real numbers. With respect to the compact-open (c-o) topology on the latter space, F is a closed subspace and a Baire-space (in fact a complete metric space).

Since we are concerned with different topologies on varying subsets of F, we use E for a subset and E_τ for the space consisting of E with topology τ; if $E' \subset E$ then E'_τ is the subspace of E_τ corresponding to E'. The set F with the relativized c-o topology is F_{co}.

With $f \varepsilon F$ understood, $f(x, t)$ is denoted $x \cdot t$; for $A \subset X$ and $J \subset R$, $A \cdot J$ is the set of points $x \cdot t$ where $x \varepsilon A$ and $t \varepsilon J$.

2.1 Definition. Given $f \varepsilon F$, an invariant set, I, of f is a closed subset of X (i.e. $\varepsilon C(X)$) such that $I \cdot R = I$. The collection of invariant sets of f is $\mathscr{I}(f)$ and is a subset of $C(X)$.

2.2 Lemma. *$\mathscr{I}(f)$ is closed in $C(X)$, thus $\mathscr{I}$ is a function from F to $C(C(X))$.*

Proof. Suppose $\lim_{n\to\infty} I_n = I$ where $I_n \varepsilon \mathscr{I}(f)$ for all n. Then $\lim_{n\to\infty} I_n \cdot t = I \cdot t$ for all $t \varepsilon R$ and so $I \cdot R = I$ and $I \in \mathscr{I}(f)$.

2.3 Theorem. *$\mathscr{I} : F_{co} \to C(C(X))$ is upper semi-continuous.*

Proof. Suppose $H \varepsilon C(X)$ is not an invariant set of $f \varepsilon F$; thus there is a point $x \varepsilon H$ and a real number t such that $x \cdot t \notin H$. Consequently, there is a compact neighborhood, C, of x and disjoint open sets $U \supset H$ and U′ containing $x \cdot t$ such that $C \cdot t \subset U'$. The compact set $C \times \{t\}$ and open set U' determine a neighborhood V of f in F. Also, the open set U and neighborhood C determine a neighborhood W of $H \in C(X)$, namely $I' \in W$ if $I' \cap C \neq \emptyset$ and $I' \subset U$. It follows that $f' \in V$, then $W \cap \mathscr{I}(f') = \emptyset$ since any invariant set of f' which meets C cannot be contained in U.

Now suppose $\varepsilon > 0$ is given. The complement, K, of $N_\varepsilon(\mathscr{I}(f))$ in $C(X)$ is compact and for $H \in K$ there are open sets V_H and W_H such that $f' \in V_H$ implies $W_H \cap \mathscr{I}(f') = \emptyset$. A finite collection of the W_H cover K and the intersection of the corresponding V_H is a neighborhood, V, of f such that $f' \in V$ implies $\mathscr{I}(f') \cap K = \emptyset$. In other words $N_\epsilon(\mathscr{I}(f)) \supset \mathscr{I}(f')$ or $\rho^*(\mathscr{I}(f), \mathscr{I}(f')) < \varepsilon$. This proves the theorem.

2.4 Corollary. *Suppose E is a subset of F and τ is a topology on E which is stronger than the relativized* c-o *topology and with which E_τ is a Baire space. Then the set E' of points of continuity of $\mathscr{I}$ is a dense G_δ in E_τ.*

Note that with different topologies τ, E' may vary—for example, if τ is the discrete topology, $E = E'$.

3. Isolated and quasi-isolated invariant sets

3.1 Definition. *A set $I \in \mathscr{I}(f)$ is isolated if there is a neighborhood U of I (in X) such that if $I' \subset U$ then $I' \subset I$.*

If an invariant set I is the intersection of isolated invariant sets it is called quasi-isolated.

It is clear that the existence of isolated invariant sets is related to the connectivity of $\mathscr{I}(f)$; the precise result is given after the following lemmas.

3.2 Lemma. *Let $K \subset \mathscr{I}(f)$ be compact and connected and suppose $I' \in \mathscr{I}(f)$. Let $K \cup I' \equiv \{I \cup I' \mid I \in K\}$. Then $K \cup I'$ is contained in $\mathscr{I}(f)$ and is connected.*

Since $\mathscr{I}(f)$ is closed under finite unions, $K \cup I' \subset \mathscr{I}(f)$. Also, for any

$A, B, C \in C(X)$, $\rho(A \cup C, B \cup C) \leq \rho(A, B)$; thus $K \cup I'$ is the continuous image of K under the map sending $I \to I \cup I'$.

3.3 LEMMA. *If K is a component of $\mathcal{I}(f)$ then $\pi(K)$ (see 1.3) is the unique maximal element of K.*

Proof. Since K is compact, it admits some maximal element H. Namely since K is partially ordered under the inclusion relation, there is a maximal well-ordered chain $\{I_\alpha\}$ in K. If $\varepsilon(\alpha, \beta)$ is the distance from I_α to I_β, then by compactness, given $\varepsilon > 0$, at most countably many of the numbers $\varepsilon(\alpha, \beta) \geq \varepsilon$. It follows that only countably many of the numbers $\varepsilon(\alpha, \beta)$ are positive and in particular that $\mathrm{I} \equiv \mathrm{Cl}\,(\bigcup_\alpha I_\alpha)$ is in K. Since $\{I_\alpha\}$ was a maximal chain, I is itself a maximal element of K. Suppose $I' \in K$; then $K \cup I'$ is connected and meets K in I'. Thus $\mathrm{K} \cup I' \subset K$ and so $H \cup I' \in K$. But then $H \cup I' \supset H$ implies $H \cup I' = H$. Since I' was arbitrary, $H = \pi(K)$.

A similar property of $\mathcal{I}(f)$ is the following.

3.4 LEMMA. *Suppose K is a component of $\mathcal{I}(f)$ and that I and I'' are in K. If $I' \in \mathcal{I}(f)$ and $I \subset I' \subset I''$ then $I' \in K$.*

Proof. $K \cup I'$ is again connected and meets K in $I' \cup I'' = I''$. Thus $K \cup I' \subset K$. Also $I' = I \cup I' \in K \cup I'$; thus $I' \in K$.

3.5 THEOREM. *A set $I \in \mathcal{I}(f)$ is isolated if and only if I is the maximal element in some closed and open subset of $\mathcal{I}(f)$.*

A set $I \in \mathcal{I}(f)$ is quasi-isolated if and only if I is the maximal element of some component $K \subset \mathcal{I}(f)$.

Proof. Suppose $I \in \mathcal{I}(f)$ is isolated and let U be a neighborhood of I in X such that $I' \subset U$ and $I' \in \mathcal{I}(f)$ implies $I' \subset I$. Let U^* be the subset of $\mathrm{C}(X)$ consisting of closed sets contained in U, and let H^* be that subset of $C(X)$ consisting of closed sets contained in I. Then U^* is open and H^* is closed and so $U^* \cap \mathcal{I}(f) = H^* \cap \mathcal{I}(f) \equiv H$ is open and closed in $\mathcal{I}(f)$. Also $I = \pi(H)$.

Now suppose $\mathrm{H} \subset \mathcal{I}(f)$ is both open and closed and let $\varepsilon > 0$ be such that, in $C(X)$, $N_\varepsilon(H) \cap \mathcal{I}(f) = \mathrm{H}$. Let I be maximal in H and let $U = N_\varepsilon(I)$. Then $I' \subset U$ implies $I \cup I' \in N_\varepsilon(H)$; thus $I \cup I' \in H$, and by maximality of I, $\mathrm{I} \cup \mathrm{I}' = I$. This proves I is isolated.

Now if K is a component of $\mathcal{I}(f)$ then K is the intersection of closed and open sets H_α. But then $\pi(K)$ is the intersection of the isolated invariant sets corresponding to the maximal elements of H_α containing $\pi(K)$ (cf. 3.3).

On the other hand suppose I is quasi-isolated and let K be the component of $\mathcal{I}(f)$ containing I. If $\pi(K) \neq I$, let U be an open set about I which does not contain $\pi(K)$. Since I is quasi-isolated, there is an isolated invariant set I' in U. Now $C(I') \cap \mathcal{I}(f)$ is a closed and open subset of $\mathcal{I}(f)$ which contains I and so the component K of $\mathcal{I}(f)$ which contains I. On the other hand,

$C(I')$ does not contain $\pi(K) \in K$. This contradiction finishes the proof of 3.5.

4. The closure of the set of isolated invariant sets

The last theorem concerns the object of the following definition.

4.1 Definition. For each $f \in F$, let $\mathfrak{I}_S(f)$ be the closure in $C(X)$ of the set of quasi-isolated invariant sets.

Also $\mathfrak{I}_S(f)$ is characterized as the closure of the isolated invariant sets since they are dense in the quasi-isolated ones.

It is easy to see by example that $\mathfrak{I}_S \mid F_{co}$ is neither upper nor lower semi-continuous; however, it is lower semi-continuous at points where $\mathfrak{I}$ is continuous and so we have

4.2 Theorem. *Let $E \subset F$ be a Baire-Space with respect to a topology τ which is at least as strong as the c-o topology (relativized). Let E'_τ be the set of points of continuity of $\mathfrak{I} \mid E_\tau$.*

Then the set E''_τ of points of continuity of both $\mathfrak{I} \mid E_\tau$ and $\mathfrak{I}_S \mid E'_\tau$ is a dense G_δ in E_τ.

Proof. By 2.3, 1.8 and 1.7 it suffices to show that $\mathfrak{I}_S \mid E'_\tau$ is l.s.c. Thus, let $f \in E'_\tau$. Let $\varepsilon > 0$ be given and choose isolated invariant sets $I_1, \cdots, I_n$ such that $I \in \mathfrak{I}_S(f)$ implies $\min_{i=1,\cdots,n} \{\rho(I, I_i)\} < \varepsilon/2$. Now for each i, let H_i be a closed and open subset of $\mathfrak{I}(f)$ such that $\pi(H_i) = I_i$. By continuity of $\mathfrak{I}$ at f, choose a neighborhood $U \subset E'_\tau$ of f such that if $f' \in U$ then for each $i = 1, \cdots, n$, $\mathfrak{I}(f')$ admits a closed and open subset H_i^* within $\varepsilon/2$ of H_i. Then H_i^* contains a maximal element, hence an isolated invariant set, within $\varepsilon/2$ of I_i. It follows that $\rho^*(\mathfrak{I}_S(f'), \mathfrak{I}_S(f)) < \varepsilon$ and, since ε and f were arbitrary, that $\mathfrak{I}_S$ is l.s.c. In view of the initial remarks, Theorem 3.4 is proved.

Conclusion

The statement that $\mathfrak{I}$ and $\mathfrak{I}_S$ are continuous almost everywhere is not very strong; for example, any closed subset of $C(X)$ can be approximated by one consisting of a finite set of points. The only "algebraic" consequence of the theorems proved here is that usually the number of components of an isolated invariant set doesn't decrease. Somewhat stronger algebraic results can be proven [2]–[5] but in the c-o topology the results are necessarily weak. With more restricted classes of flows and stronger topologies one might hope to considerably strengthen such theorems as are proved here and in the references cited above.

Bibliography

1. T. Ura, *On the flow outside a closed invariant set; stability, relative stability, and saddle sets*, Contributions to Differential Equations, vol. 3 (1964), pp. 249–294.

2. C. Conley and R. Easton, "On isolated invariant sets and isolating submanifolds," *Advances in Differential and Integral Equations*, edited by John Nohel, W. A. Benjamin, New York, pp. 97–104.
3. ———, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc.,
4. C. Conley, *Isolated invariant sets which carry a one-form*, J. Differential Equations,
5. R. Easton, *On the existence of invariant sets inside a submanifold convex to the flow*, J. Differential Equations, vol. 7 (1970) pp. 54–68.
6. E. A. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., vol. 71 (1951), pp. 152–182.

University of Wisconsin
Madison, Wisconsin