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COMODULE AND COPRODUCT STRUCTURES FOR H_*MU

BY

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1. Introduction

We observed in [4] that when H_*X is torsion free, there is a natural coaction

$$\psi: H_*X \to H_*H \otimes H_*X$$

where H is the integral Eilenberg-MacLane spectrum. In this paper we study ψ in the case $X = MU$. We begin in §2 by deriving the basic properties of this coaction in terms of the canonical polynomial generators of H_*MU. In §3 we define a coproduct on H_*MU which is a natural one for algebraic reasons. In addition we observe in §4 that this coproduct makes H^*MU isomorphic to the Landweber-Novikov algebra. In §3, use the conjugation of H_*MU to derive the coaction and coproduct on the polynomial generators

$$m_n = \frac{1}{n+1}[CP^n]$$

of H_*MU, and then in §4 we compute the Hopf algebras H^*MU and $H^*(MU;\, Q)$. In §5 we give explicit formulas for three sequences of algebraically independent elements of PH_*MU, the H_*H primitives of H_*MU. The methods are analogous to those applied to $H_*(MO{:}Z_2)$ in [3]. In §6, we compute PH_*MU in terms of the elements of §5. We compare PH_*MU with the image of the Hurewicz homomorphism h in §7. We find that Image $h \subset_{\neq} PH_*MU$; i.e., the algebraic structures of H_*MU contain less information than is required to understand the ring $\pi_*MU = \Omega_*^U$ of geometrical origin. We show that none of the sequences of §5 are in the image of the Hurewicz homomorphism, and we compare one of them with the Hazewinkel generators.

All the results of this paper except §7 have analogues for H_*MSp. There is also an analogous theory for $H_*(MO;\, Z_2)$. In this case the analogous coaction is the A_*-coaction ψ and the analogous coproduct is given by

$$H_*(MO; Z_2) \cong \mathsf{A}_* \otimes Z_2[V_n \mid n \neq 2^t - 1]$$

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with all the V_n primitive. Thus in this case Image $h = P_\psi H_*(MO; Z_2)$.

Throughout this paper ψ, Δ will always denote a coaction, coproduct respectively, and PH_*MU will always denote the primitive elements under the H_*H-comodule structure on H_*MU.

2. The Coaction on H_*MU

Recall that $H_*MU = Z[b_1, \ldots, b_n, \ldots]$ where

$$H_*MU(1) = H_*CP^\infty = Z\{1, b_0, \ldots, b_n, \ldots\}$$

and $b_n \in H_{2n+2}MU(1)$ determines an element of $H_{2n}MU$. We will use the following three nontrivial properties of ψ: $H_*MU \to H_*H \otimes H_*MU$ from [4]. First, H_*H is a "Hopf algebra" and ψ is coassociative. (The coproduct Δ of H_*H is defined on a subalgebra of H_*H such that $(\Delta \otimes 1) \circ \psi$ is defined.) Second, H_*H has no p^2 torsion for any prime p. Third, ψ is an algebra homomorphism because MU is a ring spectrum. Thus the coaction on H_*MU is determined by $\psi(b_n)$, $n \geqslant 1$. We begin by determining the coaction on the b_n.

LEMMA 2.1. (a) $\bigoplus_{n=0}^\infty Zb_n$ *is a subcomodule of* H_*MU. *Thus write*

$$\psi(b_n) = \sum_{k=0}^{n} \theta_{n,k} \otimes b_k \quad \textit{with } \theta_{n,k} \in H_{2n-2k}H.$$

(b) $\theta_{n,k} = (1 + \theta_{1,0} + \theta_{2,0} + \cdots + \theta_{t,0} + \cdots)^{k+1}_{2n-2k}$ *where* X^k_h *means the component of the nonhomogeneous element* X^k *in degree* h.

(c) $\Delta(\theta_{n,k}) = \sum_{i=k}^{n} \theta_{n,i} \otimes \theta_{i,k}$ *in* H_*H.

Proof. (a) This fact follows from the naturality of ψ applied to the canonical map $SCP^\infty \to MU$.

(b) Let p be a prime. Let γ: $H \to HZ_p$ be the canonical map to the mod p Eilenberg-MacLane spectrum. Then the following diagram commutes:

$$(*) \qquad \begin{array}{ccc} H_*CP^\infty & \xrightarrow{\psi} & H_*H \otimes H_*CP^\infty \\ \downarrow \gamma_* & & \downarrow \gamma_* \otimes \gamma_* \\ H_*(CP^\infty; Z_p) & \xrightarrow{\psi'} & A_* \otimes H_*(CP^\infty; Z_p) \end{array}$$

Note that we have identified $H_*(HZ_p; Z_p)$ with A_*, the dual of the mod p Steenrod algebra. Write

$$H_*(CP^\infty; Z_p) = Z_p\{1, b'_0, \ldots, b'_n, \ldots\}$$

It is well known that

$$\psi'(b'_n) = \sum_{k=0}^{n} \theta'_{n,k} \otimes b'_k$$

where

$$\theta'_{n,k} = (1 + \theta'_{1,0} + \theta'_{2,0} + \cdots + \theta'_{t,0} + \cdots)^{k+1}_{2n-2k}$$

Thus $\theta_{n,k}$ and $(1 + \theta_{1,0} + \cdots + \theta_{t,0} + \cdots)^{k+1}_{2n-2k}$ have the same mod p reductions for all primes p. Since $H_{2n-2k}H$ is a finite group with no p^2-torsion for any prime p, it follows that $\theta_{n,k}$ and $(1 + \theta_{1,0} + \cdots + \theta_{t,0} + \cdots)^{k+1}_{2n-2k}$ are equal.

(c) This formula follows from the coassociativity formula

$$(\Delta \otimes 1) \circ \psi(b_n) = (1 \otimes \psi) \circ \psi(b_n)$$

because

$$(\Delta \otimes 1) \circ \psi(b_n) = \sum_{k=0}^{n} \Delta(\theta_{n,k}) \otimes b_k$$

while

$$(1 \otimes \psi) \circ \psi(b_n) = \sum_{i=0}^{n} \sum_{k=0}^{i} \theta_{n,i} \otimes \theta_{i,k} \otimes b_k.$$

The argument used in [3] to construct primitive elements requires analogues $\phi_{n,k}$ in H_*MU of the $\theta_{n,k}$ in H_*H.

LEMMA 2.2. *Define $\phi_{n,k} \in H_{2n-2k}MU$ by $\phi_{n,k} = (1 + b_1 + \cdots + b_t + \cdots)^{k+1}_{2n-2k}$. These elements have the following properties:*

(a) $$\psi(\phi_{n,k}) = \sum_{i=k}^{n} \theta_{n,i} \otimes \phi_{i,k},$$

(b) $$\phi_{n,p^k-1} = p\phi'_{n,p^k-1} \quad \text{when } n \not\equiv -1 \bmod p^k,$$

(c) $$\phi_{n,p^k-1} = p\phi'_{n,p^k-1} + b^{p^k}_{t-1} \quad \text{when } n = tp^k - 1.$$

Proof. (a)

$$\begin{aligned}
\psi(\phi_{n,k}) &= [1 + \psi(b_1) + \cdots + \psi(b_t) + \cdots]^{k+1}_{2n-2k} \\
&= \left(\sum_{t=0}^{\infty} \sum_{j=0}^{t} \theta_{t,j} \otimes b_j\right)^{k+1}_{2n-2k} \\
&= \left(\sum_{t=0}^{\infty} \sum_{j=0}^{t} A^{j+1}_{t-j} \otimes b_j\right)^{k+1}_{2n-2k} \qquad \text{where } A = 1 + \theta_{1,0} + \cdots + \theta_{t,0} + \cdots \\
&= \left(\sum_{j=0}^{\infty} A^{j+1} \otimes b_j\right)^{k+1}_{2n-2k} \\
&= \left(\sum_{j_1,\ldots,j_{k+1} \geq 0} A^{j_1+\cdots+j_{k+1}+k+1} \otimes b_{j_1} \cdots b_{j_{k+1}}\right)_{2n-2k} \\
&= \sum_{s=0}^{n-k} A^{s+k+1}_{2n-2k-2s} \otimes B^{k+1}_{2s} \qquad \text{where } B = 1 + b_1 + \cdots + b_t + \cdots \\
&= \sum_{s=0}^{n-k} \theta_{n,k+s} \otimes \phi_{k+s,k}.
\end{aligned}$$

(b) and (c) These formulas follow from applying the multinomial expansion to the definition of $\phi_{n,k}$. Observe that the ϕ'_{n,p^k-1} are uniquely determined by (b) and (c) because H_*MU is a free abelian group.

Before deriving the analogues of Lemma 2.2 (b), (c) for the θ_{n,p^k-1} we investigate the $\theta_{n,0}$.

LEMMA 2.3. (a) *If $n + 1$ is not a power of a prime then $\theta_{n,0} = 0$.*
(b) *If p is prime then $\theta_{p^n-1,0} \neq 0$ and $p\theta_{p^n-1,0} = 0$.*

Proof. Fix a prime p and use the notation of the proof of Lemma 2.1 (b). It follows from [6] that $\theta'_{n,0}$ is zero for $n \neq p^t - 1$. Thus we see from the diagram (*) in Lemma 2.1 that $\gamma_*(\theta_{n,0}) = 0$ if and only if $n \neq p^t - 1$. Therefore p does not divide the order of $\theta_{n,0}$ or p divides $\theta_{n,0}$ when $n \neq p^t - 1$. Since $H_{2n}H$ is a finite abelian group with no q^2 torsion it follows that $\theta_{n,0} = 0$ if $n \neq q^s - 1$ for all primes q and positive integers s. In addition for q prime, $\theta_{q^s-1,0}$ must be nonzero and must have order q.

We can expand the expression for $\theta_{n,k}$ in Lemma 2.1 (b) by the multinomial expansion where we remove the terms which are zero by Lemma 2.3. We thus obtain an analogue of Lemma 2.2 (b), (c).

LEMMA 2.4. (a) *$\theta_{n,p^k-1} = p\theta'_{n,p^k-1}$ when n is not of the form $p^t - 1$ with p prime and $t \geqslant k$.*
(b) *$\theta_{p^t-1,p^k-1} = p\theta'_{p^t-1,p^k-1} + \theta^{p^k}_{p^{t-k}-1,0}$ where p is prime and $t \geqslant k$.*

Observe that since H_*H has torsion, the θ'_{n,p^k-1} are not uniquely determined by the formulas of Lemma 2.4.

3. A Hopf Algebra Structure on H_*MU

In Section 2 we defined analogues $\phi_{n,k}$ in H_*MU of the $\theta_{n,k}$ in H_*H. We proved that

$$\psi(\phi_{n,k}) = \sum_{i=k}^{n} \theta_{n,i} \otimes \phi_{i,k}.$$

To imitate the methods of [3] for constructing elements of PH_*MU we require a coproduct Δ on H_*MU which is an analogue of ψ in the sense that $\Delta(\phi_{n,k})$ is obtained from the formula for $\psi(\phi_{n,k})$ above by replacing each $\theta_{n,i}$ by $\phi_{n,i}$. Clearly there is at most one such coproduct Δ, and the following theorem shows that such a Δ exists. We then study the conjugation on H_*MU in Theorem 3.2 and determine $\psi(m_n)$, $\Delta(m_n)$ in Theorem 3.4. In Theorem 4.1 we will show that H_*MU with the coproduct Δ is isomorphic as a Hopf algebra to the dual S_* of the Landweber-Novikov algebra S. Thus some of the results of this section such as Theorems 3.2(a) and 3.4(a) are what one expects from the known Hopf algebra structure of S_* [1, Theorem 11.3].

For $K = (k_1, \ldots, k_s)$ define

$$b_K = b_1^{k_1} \cdots b_s^{k_s} \quad \text{and} \quad \theta_K = \theta_{1,0}^{k_1} \cdots \theta_{s,0}^{k_s}.$$

Let $\theta_{i,0}^0 = 1$.

THEOREM 3.1. *Let H_*MU have the Hopf algebra structure induced by defining*

$$\Delta(b_n) = \sum_{k=0}^{n} \phi_{n,k} \otimes b_k.$$

Then Δ has the following properties:

(a) $\Delta(\phi_{n,k}) = \sum_{i=k}^{n} \phi_{n,i} \otimes \phi_{i,k}$;

(b) *If $X \in H_*MU$ and $\Delta(X) = \sum_{I,J} \alpha_{I,J}\, b_I \otimes b_J$ for integers $\alpha_{I,J}$ then*

$$\psi(X) = \sum_{I,J} \alpha_{I,J}\, \theta_I \otimes b_J.$$

Proof. (a) The proof of this fact is analogous to the proof of Lemma 2.2 (a).

(b) If $\Delta(X_i) = \sum_{I,J} \alpha_{I,J}^{(i)}\, b_I \otimes b_J$ for $i = 1, 2$ then

$$\Delta(X_1X_2) = \sum_{I_1,I_2,J_1,J_2} \alpha_{I_1,J_1}^{(1)} \alpha_{I_2,J_2}^{(2)}\, b_{I_1+I_2} \otimes b_{J_1+J_2}.$$

If (b) is true for X_1 and X_2 then

$$\begin{aligned}\psi(X_1X_2) &= \psi(X_1)\psi(X_2)\\ &= \left(\sum_{I_1,J_1} \alpha_{I_1,J_1}^{(1)}\, \theta_{I_1} \otimes b_{J_1}\right)\left(\sum_{I_2,J_2} \alpha_{I_2,J_2}^{(2)}\, \theta_{I_2} \otimes b_{J_2}\right)\\ &= \sum_{I_1,I_2,J_1,J_2} \alpha_{I_1,J_1}^{(1)} \alpha_{I_2,J_2}^{(2)}\, \theta_{I_1+I_2} \otimes b_{J_1+J_2}.\end{aligned}$$

Thus (b) is true for X_1X_2. Therefore it suffices to prove that (b) is true for $X = b_n$. This follows from the definition of Δ and Lemma 2.1.

Since H_*MU is now a Hopf algebra we study its conjugation χ. Recall from [1, p. 64] that $H_*MU = Z[m_1, \ldots, m_n, \ldots]$ where $f(t) = t + m_1t^2 + \cdots + m_nt^{n+1} + \cdots$ is the inverse power series of $g(t) = t + b_1t^2 + \cdots + b_nt^{n+1} + \cdots$.

THEOREM 3.2. *The conjugation χ of H_*MU has the following properties:*

(a) $\chi(b_n) = m_n$;

(b) $\chi(\phi_{n,k}) = \mu_{n,k}$ *where* $\mu_{n,k} = (1 + m_1 + \cdots + m_s + \cdots)_{2n-2k}^{k+1}$.

Proof. Note that $b_n = \phi_{n,0}$ and $m_n = \mu_{n,0}$. We prove that $\chi(\phi_{n,k}) = \mu_{n,k}$ by induction on $\deg \phi_{n,k} = 2n - 2k$: Now $\chi(b_1) = -b_1 = m_1$. Assume that the theorem is true in degrees less than $2s$. If $n - k = s$, and for fixed $2n - 2k$ we use induction on k, then

$$\begin{aligned}\chi(\phi_{s,0}) &= \chi(b_s)\\ &= -b_s - \sum_{i=1}^{s-1} \chi(\phi_{s,i})b_i \quad \text{from } \Delta(b_s)\\ &= -b_s - \sum_{i=1}^{s-1} \mu_{s,i}b_i \quad \text{by induction}\\ &= m_s.\end{aligned}$$

The last step follows from the observation that the coefficient of t^{s+1} in $g(f(t)) = t$ is

$$\sum_{j=0}^{s} b_j(1 + m_1 + \cdots + m_r + \cdots)^{j+1}_{2s-2j} = m_s + \sum_{j=1}^{s-1} \mu_{s,j}b_j + b_s$$

which must be zero.

If $k > 0$ then

$$\begin{aligned}\chi(\phi_{n,k}) &= \chi[(1 + b_1 + \cdots + b_t + \cdots)^{k+1}_{2n-2k}]\\ &= (1 + \chi(b_1) + \cdots + \chi(b_t) + \cdots)^{k+1}_{2n-2k}\\ &= (1 + m_1 + \cdots + m_s)^{k+1}_{2n-2k} \quad \text{by induction}\\ &= \mu_{n,k}.\end{aligned}$$

COROLLARY 3.3.

(a) $$m_s = -b_s - \sum_{j=1}^{s-1} \mu_{s,j}b_j;$$

(b) $$b_s = -m_s - \sum_{j=1}^{s} \phi_{s,j}m_j.$$

Proof. The formula in (a) was derived in the proof of Theorem 3.2. Now (b) follows from (a) by Cramer's rule as in [3, Lemma 2.2].

THEOREM 3.4 (a) $\Delta(m_n) = \sum_{k=0}^{n} m_k \otimes \mu_{n,k}$.
(b) *There are nonzero elements $\alpha_{p,t} \in H_{2(p^t-1)}H$ for p prime, $t > 0$, such that*

$$p\alpha_{p,t} = 0 \quad \textit{and} \quad \psi(m_n) = 1 \otimes m_n + \sum_{n=p^t(s+1)-1} \alpha_{p,t} \otimes m_s^{p^t}.$$

Proof. (a) By Theorem 3.2 and [7; Prop. 8.6],

$$\begin{aligned}\Delta(m_n) &= \Delta\chi(b_n)\\ &= (\chi \otimes \chi) \circ T \circ \Delta(b_n)\\ &= (\chi \otimes \chi)\left(\sum_{k=0}^{n} b_k \otimes \phi_{n,k}\right)\\ &= \sum_{k=0}^{n} m_k \otimes \mu_{n,k}.\end{aligned}$$

(b) By Corollary 3.3 (a),

$$\Delta(m_n) = \sum_{k=0}^{n}\left(-b_k - \sum_{j=1}^{k-1} \mu_{k,j}b_j\right) \otimes \mu_{n,k}.$$

Let

$$\nu_{k,j} = (1 + \nu^{1,0} + \cdots + \nu_{t,0} + \cdots)^{j+1}_{2k-2j}$$
$$\text{where } \nu_{t,0} = -\theta_{t,0} - \sum_{i=1}^{t-1} \nu_{t,i}\theta_{i,0}.$$

By Lemma 2.3, $\nu_{t,0}$ is zero unless $t = p^r - 1$ for some prime p and $p\nu_{p^r-1,0} = 0$. By Theorem 3.1 (b),

$$\psi(m_n) = \sum_{k=0}^{n}\left(-\theta_{k,0} - \sum_{j=1}^{k-1} \nu_{k,j}\theta_{j,0}\right) \otimes \mu_{n,k}.$$

If $s \neq p^t - 1$ for some prime p then $\theta_{s,0} = 0$ by Lemma 2.3 (a). In addition

$$\nu_{k,p^t-1}\theta_{p^t-1,0} = (1 + \nu^{p^t}_{1,0} + \cdots + \nu^{p^t}_{r,0} + \cdots)_{2k-2p^t+2}\theta_{p^t-1,0}$$

which is zero unless $k = p^u - 1$ with $u \geqslant t$. Thus, in the above formula for $\psi(m_n)$ the summands with $k \neq p^t - 1$ for some prime p are zero. Hence

$$\begin{aligned}\psi(m_n) &= \sum_{p^t-1\leqslant n}\left(-\theta_{p^t-1,0} - \sum_{j=1}^{t-1} \nu_{p^t-1,p^j-1}\theta_{p^j-1,0}\right) \otimes \mu_{n,p^t-1} \\ &= \sum_{p^t-1\leqslant n}\left(-\theta_{p^t-1,0} - \sum_{j=1}^{t-1} \nu^{p^j}_{p^{t-j}-1,0}\theta_{p^j-1,0}\right) \otimes m^{p^t}_{(n-p^t+1)/p^t}\end{aligned}$$

where m_k is zero when k is not an integer. Thus define

$$\alpha_{p,t} = -\theta_{p^t-1,0} - \sum_{j=1}^{t-1} \nu^{p^j}_{p^{t-j}-1,0}\theta_{p^j-1,0}.$$

Observe that $\alpha_{p,t}$ is nonzero because $\alpha_{p,t}$ reduces modulo p and decomposables to $-\xi_t$ when p is odd and to ξ_t^2 when $p = 2$.

4. The Hopf Algebras H^*MU and $H^*(MU;Q)$

We begin in Theorem 4.1 by showing that H^*MU is isomorphic as a Hopf algebra to the Landweber-Novikov algebra. We then give a novel explicit computation of the Landweber-Novikov algebra H^*MU in Theorem 4.2. The usual description of H^*MU in terms of Landweber-Novikov operations is analogous to describing the Steenrod algebra in terms of the Milnor basis. (See [1; Part I, §6].) The description of H^*MU in Theorem 4.2 is analogous to describing the Steenrod algebra in terms of admissable monomials and Adem relations. As a corollary of our computation, we determine $H^*(MU;Q)$ in Corollary 4.3.

Recall from [1] that $MU_*MU = MU_*[B_1, \ldots, B_n, \ldots]$ with coproduct induced by

$$\Delta(B_n) = \sum_{k=0}^{n} (1 + B_1 + \cdots + B_t + \cdots)_{2n-2k}^{k+1} \otimes B_k.$$

$B_n \in MU_{2n}MU$ is determined by $B_n = (w^{n+1})^* \in MU^{2n+2}CP^\infty$ where $MU^*CP^\infty = MU^*[[w]]$. The Landweber-Novikov algebra S is the Hopf algebra which is generated as an abelian group by all dual basis elements of monomials in the B_n. The canonical map

$$f\colon MU \to H$$

induces

$$f_*\colon MU_*MU \to H_*MU$$

with $f_* \mid MU_*$ the augmentation and $f_*(B_n) = b_n$. The map $f_*\colon MU^*MU \to H^*MU$ restricts to a coalgebra isomorphism on S.

THEOREM 4.1. (a) *$f_*\colon MU_*MU \to H_*MU$ and $f_*\colon MU^*MU \to H^*MU$ are maps of Hopf algebras.*

(b) *$f_* \mid S\colon S \to H^*MU$ is an isomorphism of Hopf algebras.*

Proof. Observe that $(f_* \otimes f_*) \circ \Delta(B_n)$

$$\begin{aligned}
&= (f_* \otimes f_*)\left[\sum_{k=0}^{n} (1 + B_1 + \cdots + B_t + \cdots)_{2n-2k}^{k+1} \otimes B_k\right] \\
&= \sum_{k=0}^{n} (1 + b_1 + \cdots + b_t + \cdots)_{2n-2k}^{k+1} \otimes b_k \\
&= \sum_{k=0}^{n} \phi_{n,k} \otimes b_k \\
&= \Delta(b_n) \\
&= \Delta \circ f_*(B_n).
\end{aligned}$$

Since Δ and f_* are algebra homomorphisms it follows that $(f_* \otimes f_*) \circ \Delta = \Delta \circ f_*$ which proves (a). Now (b) follows from the remarks preceding the theorem.

Warning. Do not be misled by the following commutative diagram:

$$\begin{array}{ccccc}
MU\wedge MU & \xrightarrow{\simeq} & MU\wedge S\wedge MU & \xrightarrow{1\wedge\eta\wedge 1} & MU\wedge MU\wedge MU \\
\downarrow f\wedge 1 & & \downarrow f\wedge 1\wedge 1 & & \downarrow f\wedge 1\wedge 1 \\
H\wedge MU & \xrightarrow{\simeq} & H\wedge S\wedge MU & \xrightarrow{1\wedge\eta\wedge 1} & H\wedge MU\wedge MU.
\end{array}$$

The top row induces the coproduct on MU_*MU while the bottom row induces a coproduct Δ' on H_*MU. However, $f_*\colon MU_*MU \to [H_*MU, \Delta']$

is not a map of Hopf algebras because the following diagram does not commute:

$$\begin{array}{ccc} MU_*(MU\wedge MU) & \xleftarrow{\cong} & MU_*MU \otimes_{MU_*} MU_*MU \\ \downarrow f_* & & f_* \otimes f_* \downarrow \\ H_*(MU\wedge MU) & \xleftarrow{\cong} & H_*MU \otimes_Z H_*MU. \end{array}$$

In fact Δ' is the trivial coproduct $\Delta'(Y) = 1 \otimes Y$ for all $Y \in H_*MU$.

Since H_*MU is commutative and highly noncocommutative it follows that H^*MU is cocommutative and highly noncommutative. We take advantage of the noncommutativity in the following description of H^*MU. We use the notation

$$\mathrm{ad}(x)(y) = [x, y] = xy - (-1)^{\deg x \deg y} yx$$

and

$$\mathrm{ad}^n(x)(y) = [x, \mathrm{ad}^{n-1}(x)(y)] \quad \text{for } n \geq 2.$$

THEOREM 4.2. *Let $\alpha = b_1^* \in H^2MU$ and let $\beta = b_2^* \in H^4MU$. Define $\mathscr{P}_n \in H^{2n}MU$ by $\mathscr{P}_1 = \alpha$, $\mathscr{P}_2 = \beta$ and*

$$\mathscr{P}_n = \frac{1}{(n-2)!} \mathrm{ad}^{n-2}(\alpha)(\beta) \quad \textit{for } n \geq 3.$$

Then the Hopf algebra structure of H^*MU is determined by the following results.

(a) $$\mathscr{P}_n = \sum_{k=0}^{n-2} (-1)^{k+n} \frac{1}{k!(n-k-2)!} \alpha^k \beta \alpha^{n-k-2} \quad \textit{for } n \geq 2.$$

(b) $$PH^*MU = \bigoplus_{n=1}^{\infty} Z\mathscr{P}_n.$$

(c) $\mathscr{P}_m\mathscr{P}_n - \mathscr{P}_n\mathscr{P}_m = (n - m)\mathscr{P}_{m+n}$ *for $m, n > 0$.*

(d) *$\tilde{H}^*MU$ is a free abelian group with basis*

$$\left\{ \frac{1}{e_1! \cdots e_t!} \mathscr{P}_{n_1}^{e_1} \cdots \mathscr{P}_{n_t}^{e_t} \;\middle|\; 0 < n_1 < \cdots < n_t \textit{ and } 0 < e_i \textit{ for all } i \right\}.$$

Proof. Observe that we can use induction on n to prove that α^n is divisible by $n!$. We have

$$\langle \alpha^n, b_n \rangle = \langle \alpha \otimes \alpha^{n-1}, \Delta(b_n) \rangle = n\langle \alpha, b_1 \rangle \langle \alpha^{n-1}, b_{n-1} \rangle$$

which by the induction hypothesis is divisible by $n \cdot (n-1)! = n!$. If

$b_I = b_{I'}b_{I''}$ with $\deg b_{I'} = a > 0$ and $\deg b_{I''} = n - a > 0$ then

$$\begin{aligned}\langle \alpha^n, b_I\rangle &= \langle \Delta(\alpha^n), b_{I'} \otimes b_{I''}\rangle \\ &= \sum_{k=0}^{n} \tbinom{n}{k}\langle \alpha^k, b_{I'}\rangle\langle \alpha^{n-k}, b_{I''}\rangle \\ &= \tbinom{n}{a}\langle \alpha^a, b_{I'}\rangle\langle \alpha^{n-a}, b_{I''}\rangle\end{aligned}$$

which by the induction hypothesis is divisible by

$$\tbinom{n}{a}a!(n-a)! = n!.$$

Thus $\mathscr{P}_n$ is defined in $H^{2n}MU$ by the formula in (a).

We prove that all the $\mathscr{P}_n$ are primitive. Let $n \geqslant 3$. Then

$$\begin{aligned}\Delta(\mathscr{P}_n) &= \sum_{k=0}^{n-2}(-1)^{k+n}\frac{1}{k!(n-k-2)!}\Delta(\alpha^k\beta\alpha^{n-k-2}) \\ &= \sum_{k=0}^{n-2}\sum_{s=0}^{k}\sum_{t=0}^{n-k-2}(-1)^{k+n}\frac{1}{k!(n-k-2)!}\binom{k}{s}\binom{n-k-2}{t} \\ &\qquad [\alpha^s\beta\alpha^t \otimes \alpha^{n-s-t-2} + \alpha^{n-s-t-2} \otimes \alpha^s\beta\alpha^t] \\ &= \sum_{s=0}^{n-2}\sum_{t=0}^{n-s-2}(-1)^{s+n}\frac{1}{s!t!(n-s-t-2)!} \\ &\qquad \left[\sum_{k=s}^{n-t-2}(-1)^{k-s}\binom{n-s-t-2}{k-s}\right] \\ &\qquad [\alpha^s\beta\alpha^t \otimes \alpha^{n-s-t-2} + \alpha^{n-s-t-2} \otimes \alpha^s\beta\alpha^t].\end{aligned}$$

If $s + t < n - 2$ then

$$\begin{aligned}\sum_{k=s}^{n-t-2}(-1)^{k-s}\binom{n-s-t-2}{k-s} &= \sum_{h=0}^{n-s-t-2}(-1)^h\binom{n-s-t-2}{h} \\ &= (1-1)^{n-s-t-2} = 0.\end{aligned}$$

Thus the nonzero terms in $\Delta(\mathscr{P}_n)$ have $t = n - s - 2$ and $\Delta(\mathscr{P}_n) = \mathscr{P}_n \otimes 1 + 1 \otimes \mathscr{P}_n$.

Next we use induction on n to prove that $\langle \mathscr{P}_n, b_n\rangle = 1$. Observe that Pascal's formula implies that

$$(n-2)!\mathscr{P}_n = -(n-3)!\mathscr{P}_{n-1}\alpha + (n-3)!\alpha\mathscr{P}_{n-1}.$$

Thus,

$$\begin{aligned}\langle \mathscr{P}_n, b_n\rangle &= \frac{1}{n-2}\langle -\mathscr{P}_{n-1} \otimes \alpha + \alpha \otimes \mathscr{P}_{n-1}, \Delta(b_n)\rangle \\ &= \frac{1}{n-2}(-2+n) = 1.\end{aligned}$$

This proves (b) because $QH_*MU = \bigoplus_{n=1}^{\infty} Zb_n$. Observe that $\mathscr{P}_n = b_n^*$.

To prove (c) observe that $\mathcal{P}_m\mathcal{P}_n - \mathcal{P}_n\mathcal{P}_m$ is primitive and hence must be $\varepsilon_{m,n}\mathcal{P}_{m+n}$ for some $\varepsilon_{m,n} \in Z$. Moreover,

$$\begin{aligned}\varepsilon_{m,n} &= \langle \mathcal{P}_m\mathcal{P}_n - \mathcal{P}_n\mathcal{P}_m, b_{m+n}\rangle = \langle \mathcal{P}_m \otimes \mathcal{P}_n - \mathcal{P}_n \otimes \mathcal{P}_m, \Delta(b_{m+n})\rangle \\ &= (n+1) - (m+1) = n - m.\end{aligned}$$

To prove (d) we show by induction on degree that

$$\mathcal{P}_{n_1}^{e_1} \cdots \mathcal{P}_{n_t}^{e_t} = e_1! \cdots e_t!\left[\left(b_{n_1}^{e_1} \cdots b_{n_t}^{e_t}\right)^* + \sum_I \lambda_I b_I^*\right] \quad \text{where}$$

$$\lambda_I \in Z,\ b_I^* = (b_{m_1}^{f_1} \cdots b_{m_s}^{f_s})^*$$

and the sum is taken over all b_I^* with $f_1 + \cdots + f_s < e_1 + \cdots + e_t$. Let

$$N = e_1n_1 + \cdots + e_tn_t.$$

We have

$$\begin{aligned}\langle \mathcal{P}_{n_1}^{e_1} \cdots \mathcal{P}_{n_t}^{e_t}, b_N\rangle &= e_t\langle \mathcal{P}_{n_1}^{e_1} \cdots \mathcal{P}_{n_t}^{e_t-1} \otimes \mathcal{P}_{n_t}, \Delta(b_N)\rangle \\ &= e_t\langle \mathcal{P}_{n_1}^{e_1} \cdots \mathcal{P}_{n_t}^{e_t-1}, (1 + b_1 + \cdots + b_s + \cdots)_{N-n_t}^{n_t+1}\rangle \\ &= e_t \cdot e_1! \cdots e_{t-1}!(e_t - 1)!\lambda \\ &= e_1! \cdots e_t!\lambda.\end{aligned}$$

Let $f_1 + \cdots + f_s \geqslant 2$ and let $0 < m_1 < \cdots < m_s$. Then

$$\begin{aligned}\langle \mathcal{P}_{n_1}^{e_1} \cdots \mathcal{P}_{n_t}^{e_t}, b_{m_1}^{f_1} \cdots b_{m_s}^{f_s}\rangle &= \langle \Delta(\mathcal{P}_{n_1}^{e_1} \cdots \mathcal{P}_{n_t}^{e_t}), b_{m_1}^{f_1} \cdots b_{m_s}^{f_s-1} \otimes b_{m_s}\rangle \\ &= \sum_{i=1}^{t}\sum_{\varepsilon_i=0}^{e_i}\binom{e_1}{\varepsilon_1}\cdots\binom{e_t}{\varepsilon_t}\langle \mathcal{P}_{n_1}^{e_1-\varepsilon_1} \cdots \mathcal{P}_{n_t}^{e_t-\varepsilon_t}, b_{m_1}^{f_1} \cdots b_{m_s}^{f_s-1}\rangle\langle \mathcal{P}_{n_1}^{\varepsilon_1} \cdots \mathcal{P}_{n_t}^{\varepsilon_t}, b_{m_s}\rangle \\ &= \sum_{i=1}^{t}\sum_{\varepsilon_i=0}^{e_i}\binom{e_1}{\varepsilon_1}\cdots\binom{e_t}{\varepsilon_t}[(e_1-\varepsilon_1)! \cdots (e_t-\varepsilon_t)!\lambda_{\varepsilon_1,\ldots,\varepsilon_t}][\varepsilon_1! \cdots \varepsilon_t!\mu_{\varepsilon_1,\ldots,\varepsilon_t}] \\ &\qquad\qquad \text{where } \lambda_{\varepsilon_1,\ldots,\varepsilon_t}, \mu_{\varepsilon_1,\ldots,\varepsilon_t} \in Z \\ &= e_1! \cdots e_t!\left[\sum_{i=1}^{t}\sum_{\varepsilon_i=0}^{e_i}\lambda_{\varepsilon_1,\ldots,\varepsilon_t}\mu_{\varepsilon_1,\ldots,\varepsilon_t}\right].\end{aligned}$$

If $\mu_{\varepsilon_1,\ldots,\varepsilon_t} \neq 0$ then $\varepsilon_1 + \cdots + \varepsilon_t \geqslant 1$. Hence by the induction hypothesis if

$$b_{m_1}^{f_1} \cdots b_{m_s}^{f_s} \neq b_{n_1}^{e_1} \cdots b_{n_t}^{e_t} \quad \text{and} \quad \lambda_{\varepsilon_1,\ldots,\varepsilon_t}\mu_{\varepsilon_1,\ldots,\varepsilon_t} \neq 0$$

then

$$e_1 + \cdots + e_t > e_1 - \varepsilon_1 + \cdots + e_t - \varepsilon_t > f_1 + \cdots + f_s - 1.$$

Thus

$$e_1 + \cdots + e_t > f_1 + \cdots + f_s.$$

If

$$b_{m_1}^{f_1} \cdots b_{m_s}^{f_s} = b_{n_1}^{e_1} \cdots b_{n_t}^{e_t}$$

then

$$\mu_{\varepsilon_1,\dots,\varepsilon_t} = 0 \quad \text{for } (\varepsilon_1, \dots, \varepsilon_t) \neq (0, \dots, 1) \text{ and } \mu_{0,\dots,0,1} = \lambda_{0,\dots,0,1} = 1.$$

Since we defined the $\mathcal{P}_n$ by the formula in (a), we must prove that

$$\mathcal{P}_n = \frac{1}{(n-2)!} \operatorname{ad}^{n-2}(\alpha)(\beta) \quad \text{for } n \geqslant 3.$$

We use induction on n. We have $\mathcal{P}_3 = -\beta\alpha + \alpha\beta = \operatorname{ad}(\alpha)(\beta)$. Inductively,

$$\mathcal{P}_n = \frac{1}{n-2}[\mathcal{P}_1, \mathcal{P}_{n-1}] = \frac{1}{n-2}\left[\alpha, \frac{1}{(n-3)!} \operatorname{ad}^{n-3}(\alpha)(\beta)\right]$$
$$= \frac{1}{(n-2)!} \operatorname{ad}^{n-2}(\alpha)(\beta).$$

COROLLARY 4.3. *Let* $\gamma_1 = \alpha$, $\gamma_2 = \beta$ *and* $\gamma_n = \operatorname{ad}^{n-2}(\alpha)(\beta) = (n-2)!\mathcal{P}_n$ *for* $n \geqslant 3$. *Then* $H^*(MU;Q)$ *has the following Hopf algebra structure.*

(a) $H^*(MU;Q)$ *is a primitively generated Hopf algebra which is generated as an algebra by* α *and* β.

(b) $PH^*(MU;Q)$ *has* $\{\gamma_n \mid n \geqslant 1\}$ *as a* Q*-basis.*

(c) *For* $m, n \geqslant 1$,

$$\gamma_m\gamma_n - \gamma_n\gamma_m = \frac{n-m}{(m-1)(n-1)(m-1, n-1)} \gamma_{m+n}.$$

(d) $\tilde{H}^*(MU;Q)$ *has a* Q*-basis* $\{\gamma_{n_1} \cdots \gamma_{n_s} \mid 0 < n_1 \leqslant \cdots \leqslant n_s\}$.

5. Polynomial Subalgebras of PH_*MU

We can not apply [3, Theorem 2.1] to H_*MU using the b_n or the m_n because all the $\theta_{p^t-1,0}$ and $\alpha_{p,t}$ are nonzero, However $p\theta_{p^t-1,0} = p\alpha_{p,t} = 0$. We will therefore modify the argument of [3, Theorem 2.1] as follows. Instead of converting

$$\{b_n \mid n \geqslant 1\} \quad \text{and} \quad \{m_n \mid n \geqslant 1\}$$

into primitive elements we will convert

$$\{b_n \mid n \neq p^t - 1\} \cup \{pb_{p^t-1}\} \quad \text{and} \quad \{m_n \mid n \neq p^t - 1\} \cup \{pm_{p^t-1}\}$$

into primitive elements. We will thus obtain two polynomial subalgebras of PH_*MU.

THEOREM 5.1. *Choose integers* $\lambda(e, p)$ *for all positive integers* e *and primes* p *with* $p \leqslant e$ *such that:*

(i) *If q is prime, $q \neq p$ and $q \leq e$ then q divides $\lambda(e, p)$.*
(ii) $\lambda(e, p) \equiv 1 \bmod p$.

The following recursive formula defines $V_n[e] \in H_{2n}MU$ for $e \geq n + 1$:

$$V_n[e] = b_n - \sum_{k=1}^{n-1} \phi_{n,k} V_k[e] + \sum_{n=p^k(s+1)-1} \lambda(e, p) b_s^{p^k} V_{p^k-1}[e].$$

Define

$$V_n = \begin{cases} pV_n[n+1] & \text{if } n = p^t - 1, p \text{ prime} \\ V_n[n+1] & \text{otherwise.} \end{cases}$$

Then $Z[V_1, \ldots, V_n, \ldots] \subset PH_*MU$.

Proof. We can choose the $\lambda(e, p)$ as follows. Let m be the product of all the primes q with $q \leq e$ and $q \neq p$. Then $(p, m) = 1$ so we can find integers s, t with $sm + tp = 1$. Choose $\lambda(e, p)$ to be sm.

Since ψ is an algebra homomorphism, PH_*MU is a subalgebra of H_*MU. Thus it suffices to show that the V_n are primitive. By Lemma 2.3, it suffices to show that

$$\psi(V_n[e]) = \theta_{n,0} \otimes 1 + 1 \otimes V_n[e]$$

by induction on n. If $n = p^k(s + 1) - 1$ then

$$\begin{aligned} &\psi(\lambda(e, p) b_s^{p^k} V_{p^k-1}[e]) \\ &\quad = \lambda(e, p)\left(\sum_{i=0}^{s} \theta_{s,i} \otimes b_i\right)^{p^k} (1 \otimes V_{p^k-1}[e] + \theta_{p^k-1,0} \otimes 1) \\ &\quad = \lambda(e, p)\left(\sum_{i=0}^{s} \theta_{s,i}^{p^k} \otimes b_i^{p^k}\right)(1 \otimes V_{p^k-1}[e] + \theta_{p^k-1,0} \otimes 1) \\ &\qquad\qquad (\text{because } p\lambda(e, p)\theta_{\beta,\alpha} = 0 \text{ for } 0 \leq \alpha < \beta \leq s) \\ &\quad = \lambda(e, p)\left(\sum_{i=0}^{s} \theta_{n,p^k(i+1)-1} \otimes b_i^{p^k}\right)(1 \otimes V_{p^k-1}[e] + \theta_{p^k-1,0} \otimes 1) \end{aligned}$$

because

$$\begin{aligned} \theta_{s,i}^{p^k} &= [(1 + \theta_{1,0} + \ldots)_{2s-2i}^{i+1}]^{p^k} = (1 + \theta_{1,0} + \ldots)_{p^k(2s-2i)}^{p^k(i+1)} \\ &= \theta_{p^k(s+1)-1,p^k(i+1)-1} \qquad\qquad = \theta_{n,p^k(i+1)-1} \end{aligned}$$

Thus,

$$\begin{aligned} &\psi(\lambda(e, p) b_s^{p^k} V_{p^k-1}[e]) \\ &= \sum_{i=0}^{s} \lambda(e, p)\theta_{n,p^k(i+1)} \otimes b_i^{p^k} V_{p^k-1}[e] + \sum_{i=0}^{s} \theta_{n,p^k(i+1)}\theta_{p^k-1,0} \otimes \phi_{(i+1)p^k-1,p^k-1} \end{aligned}$$

Now,

$$\begin{aligned}\psi(V_n[e]) &= \sum_{k=0}^{n} \theta_{n,k} \otimes b_k - \sum_{k=1}^{n-1} \sum_{i=k}^{n} \theta_{n,i} \otimes \phi_{i,k} V_k[e] \\ &\quad - \sum_{\substack{0 \leq k \leq n \\ k=p^t-1}} \sum_{i=k}^{n} \theta_{n,i}\theta_{p^t-1,0} \otimes \phi_{i,k} \\ &\quad + \sum_{n=p^k(s+1)-1} \sum_{i=0}^{s} \lambda(e,p)\theta_{n,p^k(i+1)} \otimes b_i^{p^k} V_{p^k-1}[e] \\ &\quad + \sum_{n=p^k(s+1)-1} \sum_{i=0}^{s} \theta_{n,p^k(i+1)}\theta_{p^k-1,0} \otimes \phi_{(i+1)p^k-1,p^k-1} \\ &= 1 \otimes V_n[e] + \theta_{n,0} \otimes 1 \\ &\quad + \sum_{k=1}^{n-1} \theta_{n,k} \otimes \Bigg(b_k - V_k[e] - \sum_{h=1}^{k-1} \phi_{k,h} V_h[e] \\ &\qquad\qquad + \sum_{k=p^r(u+1)-1} \lambda(e,p) b_u^{p^r} V_{p^r-1}[e] \Bigg)\end{aligned}$$

because $\theta_{n,i}\theta_{p^t-1,0} \otimes \phi_{i,k}$ with $k = p^t - 1$ is zero by Lemmas 2.2 (c) and 2.3 (b) unless $i \equiv -1 \bmod p^t$. Thus $\psi(V_n[e]) = 1 \otimes V_n[e] + \theta_{n,0} \otimes 1$, as asserted.

We now perform the analogous construction with the m_n replacing the b_n.

THEOREM 5.2. *There are elements $u_n[e] \in H_{2n}MU$ for $n \geq 1$ which are defined by the following recursive formula:*

$$u_n[e] = m_n - \sum_{n=p^t(s+1)-1} \lambda(e,p) m_{p^t-1} u_s[e]^{p^t}.$$

Define

$$u_n' = \begin{cases} p u_n[n+1] & \text{if } n = p^t - 1, p \text{ prime} \\ u_n[n+1] & \text{otherwise.} \end{cases}$$

Define

$$\bar{u}_n[e] = m_n + \sum_{n=p^t(s+1)-1} \lambda(e,p) \xi_{p,t} m_s^{p^t}$$

where

$$\xi_{p,t} = b_{p^t-1} + \sum_{k<p^t-1, k \neq p^r-1} \nu_{p^t-1,k} b_k$$

$$\text{and } \nu_{s,k} = (1 + m_{p-1} + m_{p^2-1} + \cdots + m_{p^r-1} + \cdots)_{2s-2k}^{k+1}.$$

Define

$$u_n = \begin{cases} p \bar{u}_n[n+1] & \text{if } n = p^t - 1, p \text{ prime} \\ \bar{u}_n[n+1] & \text{otherwise.} \end{cases}$$

Then

(a) $u_n[e] \equiv \bar{u}_n[f] \bmod p$ *for all primes* $p \leqslant \min(e, f)$:

(b) $Z[u_1', \ldots, u_n', \ldots] \subset PH_*MU$;

(c) $Z[u_1, \ldots, u_n, \ldots] \subset PH_*MU$.

Proof. By induction on $n \geqslant 1$, we show that $\bar{\psi}(u_n[e]) = \psi(u_n[e]) - 1 \otimes u_n[e]$ is p-torsion when $n = p^t - 1$ and $\psi(u_n[e]) = 1 \otimes u_n[e]$ otherwise. We have $u_1[e] = m_1$, so $\bar{\psi}(u_1[e])$ is 2-torsion. Assume this assertion is true in degrees less than $2n$. If $n \neq p^r - 1$ then

$$\psi(u_n[e]) = \sum_{n=p^t(s+1)-1} \alpha_{p,t} \otimes m_s^{p^t} - \sum_{n=p^t(s+1)-1} \sum_{i=1}^{t} \alpha_{p,i} \otimes m_{p^{t-i}-1}^{p^i} u_s[e]^{p^t}$$

(because $s + 1$ cannot be a power of p)

$$= 1 \otimes u_n[e] + \sum_{n=p^t(s+1)-1} \alpha_{p,t}$$

$$\otimes \left[m_s - u_s[e] - \sum_{s=p^j(k+1)-1} m_{p^j-1} u_k[e]^{p^j} \right]^{p^t}$$

(because the $\alpha_{p,t}$ are p-torsion)

$$= 1 \otimes u_n[e].$$

When $n = p^r - 1$ then $\psi(u_n[e])$ contains the above terms and in addition contains

$$\sum_{j=1}^{r-1} \sum_{i=0}^{j} \sum_{h=0}^{p^{r-j}-2} \alpha_{p,i} \beta_{p^{r-j}-1,h}[e]^{p^t} \otimes m_{p^{j-i}-1}^{p^i} \tau_h[e]^{p^t}$$

where

$$\psi(u_{p^s-1}[e]) = 1 \otimes u_{p^s-1}[e] + \sum_{h=0}^{p} \beta_{p^s-1,h}[e] \otimes \tau_h[e],$$

$$\beta_{p^s-1,h}[e] \in H_{2(p^s-h-1)} H, \quad \tau_h[e] \in H_{2h} MU \quad \text{and} \quad p\beta_{p^s-1,h}[e] = 0.$$

These additional terms are clearly p-torsion. Hence $\bar{\psi}(u_{p^r-1}[e])$ is p-torsion. Thus all the u_n' are primitive which proves (b). To prove (a) we consider the following set of simultaneous linear equations in $H_*(MU; Z_p)$ when $n = p^t(s+1) - 1$ and p does not divide $s + 1$:

$$m_{p^k(s+1)-1}^{p^{t-k}} = u_{p^k(s+1)-1}[e]^{p^{t-k}} + \sum_{j=1}^{k} m_{p^j-1}^{p^{t-k}} u_{p^{k-j}(s+1)-1}[e]^{p^{j+t-k}}, \quad 0 \leqslant k \leqslant t.$$

Consider these equations as $t + 1$ linear equations in the $t + 1$ unknowns

$$u_{p^k(s+1)-1}[e]^{p^{t-k}}, \quad 0 \leqslant k \leqslant t.$$

The coefficient matrix (a_{ij}) is lower triangular with ones on the diagonal and $a_{ij} = m_{p^{i-j}-1}^{p^{t-i}}$. Give $H_*(MU; Z_p)$ the coproduct Δ' of [3, §3]. Recall that there are

$$\zeta_n \in H_{2(p^n-1)}(MU; Z_p)$$

which have Δ'-coproduct corresponding to the coproduct of ξ_n (p odd) or ξ_n^2 ($p = 2$) in the dual of the Steenrod algebra. In addition $\chi(m_{p^n-1}) = \zeta_n$ using the Δ'-coproduct. Then

$$\begin{aligned}
\Delta'(a_{ij}) &= \Delta'\chi(\zeta_{i-j})^{p^{t-i}} \\
&= (\chi \otimes \chi) \circ T \circ \Delta'(\zeta_{i-j})^{p^{t-i}} \\
&= (\chi \otimes \chi) \circ T\left(\sum_{r=0}^{i-j} \zeta_{i-j-r}^{p^r} \otimes \zeta_r\right)^{p^{t-i}} \\
&= \sum_{r=0}^{i-j} m_{p^r-1}^{p^{t-i}} \otimes m_{p^{i-j-r}-1}^{p^{r+t-i}} \\
&= \sum_{r=0}^{i-j} a_{i,i-r} \otimes a_{i-r,j} \\
&= \sum_{h=i}^{j} a_{i,h} \otimes a_{h,j}
\end{aligned}$$

where $h = i - r$. Thus [3, Lemma 2.2] applies to this system of linear equations to give

$$\begin{aligned}
u_{p^k(s+1)-1}[e]^{p^{t-k}} &= m_{p^k(s+1)-1}^{p^{t-k}} + \sum_{j=1}^{k} \chi(m_{p^j-1}^{p^{t-k}}) m_{p^{k-j}(s+1)-1}^{p^{j+t-k}} \\
&= m_{p^k(s+1)-1}^{p^{t-k}} + \sum_{j=1}^{k} \zeta_j^{p^{t-k}} m_{p^{k-j}(s+1)-1}^{p^{j+t-k}}.
\end{aligned}$$

Taking $t = k$,

$$u_{p^t(s+1)+p^t-1} = m_{p^t(s+1)+p^t-1} + \sum_{j=1}^{t} \zeta_j m_{p^{t-j}(s+1)-1}^{p^j}.$$

By [3, Theorem 2.1],

$$\zeta_j = b_{p^j-1} + \sum_{k<p^j-1, k\neq p^r-1} \nu_{p^j-1,k} b_k \quad \text{where}$$

$$\nu_{g,k} = (1 + m_{p-1} + \cdots + m_{p^n-1} + \cdots)_{2g-2k}^{k+1}.$$

Thus $\xi_{p,t}$ reduces to ζ_t mod p which proves (a).

To prove (c) observe that

$$\bar{u}_n[e] - u_n[e] = p_1 \cdots p_{N(e)} W_{n,e}$$

where

$$\{p \mid p \text{ prime}, p \leqslant e\} = \{p_1, \dots, p_{N(e)}\}.$$

The element of lowest degree whose ψ-coproduct has a p-torsion summand is b_{p-1}. Thus, if $e \geqslant n + 1$ then $p_1 \cdots p_{N(e)} W_{n,e}$ is primitive and hence u_n is primitive by (b).

6. The Primitive Elements of H_*MU

In Section 5 we determined various polynomial subalgebras of PH_*MU. However, PH_*MU is larger than a polynomial algebra. The underlying reason is that H_*H has no p^2-torsion. For example, $\bar{\psi}(V_{p-1}[p])$ is p-torsion, so $V_{p-1} = pV_{p-1}[p]$ is primitive. However $\bar{\psi}(V_{p-1}[p]^2)$ is p-torsion, not p^2-torsion, so

$$pV_{p-1}[p]^2 = \frac{1}{p} V_{p-1}^2$$

is primitive.

The following theorems require elements $V'_n \in H_nMU$. These elements can be chosen in any one of the following ways:

(1) $V'_n = V_n[e]$ from Theorem 5.1;
(2) $V'_n = u_n[e]$ from Theorem 5.2;
(3) $V'_n = \bar{u}_n[e]$ from Theorem 5.2;
(4) $V_n = h(y_n)$ where $\pi_*MU = Z[y_1, \dots, y_n, \dots]$.

THEOREM 6.1. *Let $V'_n \in H_nMU$ for $n \geqslant 1$. Define*

$$V_n = \begin{cases} pV'_n & \text{if } n + 1 \text{ is a power of a prime } p \\ V'_n & \text{otherwise.} \end{cases}$$

Assume that $V_n \in PH_*MU$ and $V'_n \equiv b_n$ modulo decomposables for all n. Let

$$\pi: H_*MU \to H_*(MU; Z_p)$$

be the mod p *reduction. Then under the* A_**-coaction,*

$$PH_*(MU; Z_p) = Z_p[\pi(V'_n) \mid n \neq p^t - 1].$$

Proof. The following commutative diagram shows that $\pi(PH_*MU) \subset PH_*(MU; Z_p)$.

$$\begin{array}{ccccc} \pi_*(H\wedge MU) & \to & \pi_*(H\wedge S\wedge MU) & \to & \pi_*(H\wedge H\wedge MU) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_*(HZ_p\wedge MU) & \to & \pi_*(HZ_p\wedge S\wedge MU) & \to & \pi_*(HZ_p\wedge HZ_p\wedge MU). \end{array}$$

If $n + 1 \neq q^s$, q prime, then $V_n' \in PH_*MU$ while $\overline{\psi}(V_{q^s-1}')$ is q-torsion. Thus,

$$Z_p[\pi(V_n') \mid n \neq p^t - 1] \subset \pi(PH_*MU) \subset PH_*(MU; Z_p).$$

By [5], $PH_*(MU; Z_p)$ is a polynomial algebra with one generator in each degree m with $m \neq p^t - 1$. Since $V_n' \equiv b_n$ modulo decomposables, the $\pi(V_n')$ are algebraically independent. Thus

$$Z_p[\pi(V_n') \mid n \neq p^t - 1] = PH_*(MU; Z_p).$$

THEOREM 6.2. *Let V_n' and V_n be as in Theorem* 6.1. *For $I = (e_1, \ldots, e_t)$ define*

$$V_I = V_1^{e_1} \cdots V_t^{e_t} \quad \textit{and} \quad V_I' = V_1'^{e_1} \cdots V_t'^{e_t}.$$

Let $\sigma(I, p) = \sum_{s \geq 1} e_{p^s-1}$, and let $P(I)$ be the set of primes p with $\sigma(I, p) > 0$. Define

$$\hat{V}_I = \left(\prod_{p \in P(I)} p^{1-\sigma(I,p)} \right) V_I = \left(\prod_{p \in P(I)} p \right) V_I'.$$

Then:

(a) *The set of all $\hat{V}_I$ is a basis for the free abelian group PH_*MU;*

(b) *The $\hat{V}_I$ generate PH_*MU as an algebra with the relations*

$$\hat{V}_I \hat{V}_J = \left(\prod_{p \in P(I) \cap P(J)} p \right) \hat{V}_{I+J}.$$

Proof. Clearly all the $\hat{V}_I$ are in PH_*MU. Any $Y \in H_*MU$ can be written as a polynomial in the V_n': $Y = \sum \alpha_I V_I'$ with $\alpha_I \in Z$. If $p \in P(I)$ and $p \nmid \alpha_I$ then

$$\pi(Y) \notin PH_*(MU; Z_p)$$

by Theorem 6.1. Thus if $Y \in PH_*MU$ then each α_I is divisible by $\Pi_{p \in P(I)}\, p$ with quotient $\overline{\alpha}_I$. Hence $Y = \sum \overline{\alpha}_I\, \hat{V}_I$. This completes the proof of (a). Now (b) follows easily.

Observe that when we localize at a prime p, $\overline{\psi}(H_*(MU; Z_{(p)})$ and $\overline{\psi}(H_*BP)$ are contained in direct sums of copies of $\tilde{H}_*H \otimes Z_{(p)}$. However, $\tilde{H}_*H \otimes Z_{(p)}$ is a Z_p-vector space. We thus deduce the following two results.

COROLLARY 6.3. *Let V_n' and V_n be as in Theorem 6.1. Then*

$$PH_*(MU; Z_{(p)})$$

is the free abelian group with basis the set of all monomials:

(a) $V_{1e_1}' \cdots V_{te_t}'$ *where $e_{p^s-1} = 0$ for all $s \geq 1$, and*

(b) $pV_{1e_1}' \cdots V_{te_t}'$ *where $e_{p^s-1} \neq 0$ for some $s \geq 1$.*

COROLLARY 6.4. $PH_*BP = pH_*BP$.

7. The Image of the Hurewicz Homomorphism

Write $\pi_*(MU) = Z[y_1, \ldots, y_n, \ldots]$. Then the image of the Hurewicz homomorphism h is

$$Z[h(y_1), \ldots, h(y_n), \ldots] \subset PH_*MU.$$

From the description of PH_*MU in Section 6 we see that PH_*MU is strictly larger than Image h. In particular we have the following theorem.

THEOREM 7.1. (a)

$$\text{Image } h \subset PH_*MU \text{ and } \text{rank Image } h = \text{rank } PH_*MU.$$

(b) *In the notation of Section* 6,

$$PH_*MU/\text{Image } h \cong \bigoplus_I Z_{N(I)}V_I \quad \text{where } N(I) = \prod_{p \in P(I)} p^{\sigma(I,p)-1}.$$

Proof. Image $h \subset PH_*MU$ follows from the definition of h and the naturality of ψ. Consider Theorem 6.2 with $V_n = h(y_n)$. Then $V_I = h(y_1)^{e_1} \cdots h(y_t)^{e_t}$ is a basis element for Image h. If we divide V_I by $N(I)$ then we obtain the corresponding basis element of PH_*MU. This is a restatement of (b).

We show next that none of the families of primitive elements of Section 5 give a set of polynomial generators for Image h.

Example 7.2. Consider the V_n of Theorem 5.1:

$$V_1[e] = b_1 \quad \text{and} \quad V_1 = 2b_1 \in \text{Image } h,$$

$$V_2[e] = b_2 - 2b_1^2 \quad \text{and} \quad V_2 = 3b_2 - 6b_1^2 \in \text{Image } h,$$

$$V_3[e] = b_3 - 5b_1b_2 + [5 + \lambda(e, 2)]b_1^3 \quad \text{with } \lambda(e, 2) \text{ odd}$$

and

$$V_3 = 2b_3 - 10b_1b_2 + [10 + 2\lambda(e, 2)]b_1^3.$$

By [1, p. 63], $V_3 - 2V_1V_2 + h(a_{13}) - h(a_{22}) = [-12 + 2\lambda(e, 2)]b_1^3$ which is not in Image h because $-12 + 2\lambda(e, 2)$ is not divisible by 8. Thus $V_3 \notin$ Image h.

Example 7.3. Consider the u_n' of Theorem 5.2:

$$u_1[e] = m_1 = -b_1 \quad \text{and} \quad u_1' = -2b_1 \in \text{Image } h,$$

$$u_2[e] = m_2 = -b_2 + 2b_1^2 \quad \text{and} \quad u_2' = -3b_2 + 6b_1^2 \in \text{Image } h,$$

$$u_3[e] = m_3 - \lambda(e, 2)m_1^3 = -b_3 + 5b_1b_2 + [-5 + \lambda(e, 2)]b_1^3$$

and

$$u_3' = -2b_3 + 10b_1b_2 + [-10 + 2\lambda(e,2)]b_1^3 \quad \text{with } \lambda(e,2) \text{ odd.}$$

As in Example 7.2, $u_3' \notin$ Image h.

Example 7.4. Consider the u_n of Theorem 5.2:

$$\bar{u}_1[e] = m_1 \quad \text{and} \quad u_1 = u_1' \in \text{Image } h,$$

$$\bar{u}_2[e] = m_2 \quad \text{and} \quad u_2 = u_2' \in \text{Image } h,$$

$$\bar{u}_3[e] = m_3 + \lambda(e,2)\xi_{2,1}m_1^2 = m_3 + \lambda(e,2)b_1m_1^2 = u_3[e]$$
$$\text{and } u_3 = u_3' \notin \text{Image } h.$$

Observe that the generators u_n' of PH_*MU and the Hazewinkel generators H_n of Image h are defined recursively from similar formulas:

$$u_n' = \nu(n+1)m_n - \sum \frac{\nu(n+1)\lambda(n+1,p)}{\nu(s+1)^{p^t}} m_{p^t-1}u_s'^{p^t},$$

where the summation is taken so that $p^t \mid (n+1)$, $p^t \neq 1, n+1$; p prime; $n+1 = p^t(s+1)$;

$$H_n = \nu(n+1)m_n - \sum_{d+1|n+1;\ d\neq 1,n+1} \frac{\nu(n+1)\mu(n+1,d+1)}{\nu(d+1)} m_{[(n+1)/(d+1)-1]} H_d^{(n+1)/(d+1)}.$$

(See [2] for the derivation of the second formula and for an explanation of the notation.) The formulas for the u_n and H_n differ in two ways. First, to define H_n we sum over all divisors $d+1$ of $n+1$ while to define u_n we only sum over those divisors with d a prime power. Second, if p, q are primes (not necessarily distinct) then

$$\frac{\nu(p^tq^r)\lambda(p^tq^r,p)}{\nu(q^r)^{p^t}} m_{p^t-1}\, u_{q^r-1}'^{p^t}$$

is not divisible by q^2 while

$$\frac{\nu(p^tq^r)\mu(p^tq^r,q^r)}{\nu(q^r)} m_{p^t-1}H_{q^r-1}^{p^t}$$

is divisible by q^{p^t}.

If we project the u_{p^n-1}' to $\bar{u}_n$ in H_*BP then they satisfy a recursion formula similar to that of the Hazewinkel generators $\overline{H}_n$ of Image h:

$$\bar{u}_n = pm_{p^n-1} - \sum_{t=1}^{n-1} p^{1-p^t} m_{p^t-1}\,\bar{u}_{n-t}^{p^t};$$

$$\overline{H}_n = pm_{p^n-1} - \sum_{t=1}^{n-1} m_{p^t-1}\,\overline{H}_{n-t}^{p^t} \quad \text{(see [2].)}$$

These formulas differ in that $p^{1-p^t} m_{p^t-1} \bar{u}_{n-t}^{p^t}$ is not divisible by p^2 while $m_{p^t-1} \bar{H}_{n-t}^{p^t}$ is divisible by p^{p^t}.

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