ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1984

SOME REMARKS ON A PAPER OF C. DOYLE AND D. JAMES ON SUBGROUPS OF $SL(2, \mathbf{R})$

BY

GERHARD ROSENBERGER

1. Introduction

Let G be a subgroup of $SL(2, \mathbf{R})$. We say G is elementary if the commutator of any two elements of infinite order has trace 2; equivalently, G is elementary if any two elements of infinite order (regarded as linear fractional transformations) have at least one common fixed point.

The elementary subgroups of $SL(2, \mathbf{R})$ are well known and easily dealt with (cf. [2; pp. 117–147]).

C. Doyle and D. James [1] proved (in a slightly modified formulation) that a non-elementary subgroup of $SL(2, \mathbf{R})$ can be generated by hyperbolic matrices. In [1] they construct a generating system for a non-elementary group G which contains only hyperbolic matrices. But in general this generating system of G is not minimal. Here we generalize the above result of C. Doyle and D. James and prove that a non-elementary subgroup of $SL(2, \mathbf{R})$ can be generated by a minimal generating system which contains only hyperbolic matrices.

At the end of this note we give some remarks on the other results in the paper of C. Doyle and D. James [1].

2. Preliminary Remarks

Let H be any group. We call a cardinal number r the rank $r(H)$ of H if H can be generated by a generating system X with cardinal number r but not by a generating system Y with cardinal number s less than r. Let $r(H)$ be the rank of H. We call a generating system X of H minimal if X has the cardinal number $R(H)$.

Now let G be a subgroup of $SL(2, \mathbf{R})$. We use the notation $[A, B]$ for $ABA^{-1}B^{-1}$, the commutator of A, $B \in G$, and let tr A be the trace of $A \in G$. An element $A \in G$ is called hyperbolic if $|\text{tr}\, A| > 2$; it is called elliptic if $|\text{tr}\, A| < 2$.

Received December 14, 1981.

3. Minimal Generating Systems of a Subgroup of $SL(2, \mathbf{R})$

Throughout this section, let G be a non-elementary subgroup of $SL(2, \mathbf{R})$.

LEMMA 1. *If G has a minimal generating system X with* $\operatorname{tr} A = 0$ *for all $A \in X$ then G can be generated by a minimal generating system Y which contains two elements B and C with* $\operatorname{tr}[B, C] \neq 2$, $\operatorname{tr} B \neq 0$ *and* $\operatorname{tr} C \neq 0$.

Proof. The rank $r(G)$ is greater than 2 because G is non-elementary. Especially X contains two elements A and B which do not generate a cyclic group (this is trivial if $r(G) < \infty$).

We may assume (after a suitable conjugation) that

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -\rho \\ \frac{1}{\rho} & 0 \end{pmatrix}, \ |\rho| \neq 1.$$

Because G is non-elementary there exists an element

$$C = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

of X with $a \neq 0$ and $b \neq c$. Now

$$B_1 \cdot= BA = \begin{pmatrix} \rho & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \quad \text{and} \ C_1 \cdot= AC = \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}.$$

We have $\operatorname{tr} B_1 \neq 0$, $\operatorname{tr} C_1 \neq 0$ and $\operatorname{tr}[B_1, C_1] \neq 2$ because $|\rho| \neq 1$, $a \neq 0$ and $b \neq c$. Replace B by B_1 and C by C_1 to get the desired minimal generating system Y. Q.E.D.

LEMMA 2. *G can be generated by a minimal generating system which contains two elements, A and B, with* $\operatorname{tr}[A, B] \neq 2$, $\operatorname{tr} A \neq 0$ *and* $\operatorname{tr} B \neq 0$.

Proof. G is not cyclic because G is non-elementary. If G has a minimal generating system X such that $\operatorname{tr} A = 0$ for each element A of X then by Lemma 1 we may replace X by a minimal generating system Y which contains two elements B and C with $\operatorname{tr}[B, C] \neq 2$, $\operatorname{tr} B \neq 0$ and $\operatorname{tr} C \neq 0$.

Now we assume that G has no minimal generating system X such that $\operatorname{tr} A = 0$ for each $A \in X$.

Let X be a minimal generating system of G. We may assume without loss of generality that no pair of elements of X generates a cyclic group (this is trivial if G is finitely generated). Let A be an element of X. Because G is non-elementary there exists an element B in X such that $\operatorname{tr}[A, B] \neq 2$. If at most one of $\operatorname{tr} A$, $\operatorname{tr} B$ and $\operatorname{tr} AB$ equal to zero the lemma is proved.

Now let two of $\operatorname{tr} A$, $\operatorname{tr} B$ and $\operatorname{tr} AB$ be zero; without loss of generality, assume $\operatorname{tr} A = \operatorname{tr} B = 0$. By our assumptions about the minimal generating systems there exists an element C in X such that $\operatorname{tr} C \neq 0$ and $\operatorname{tr} AC \neq 0$. If $\operatorname{tr}[A, C] \neq 2$ the lemma is proved.

Now let $\operatorname{tr}[A, C] = 2$. Then A and C commute because they have the same fixed points (regarded as linear fractional transformations). In particular, C is an elliptic element of G, that is, $|\operatorname{tr} C| < 2$.

If $\operatorname{tr}[B, C] = 2$ then B and C commute and therefore A and B also commute, which contradicts $\operatorname{tr}[A, B] \neq 2$.

If $\operatorname{tr}[B, C] \neq 2$ and $\operatorname{tr} BC = 0$ then $|\operatorname{tr} BBC| = |\operatorname{tr} C| > 2$ which contradicts $|\operatorname{tr} C| < 2$.

Altogether, it follows that $\operatorname{tr}[B, C] \neq 2$ and $\operatorname{tr} BC \neq 0$ if $\operatorname{tr}[A, B] = 2$. This proves the lemma.

LEMMA 3. *Let X be a minimal generating system of G. If X contains an element*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$ then G can be generated by a minimal generating system Y with the following property: If

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

is an element of Y then $e \neq 0$ or $h \neq 0$.

Proof. Assume

$$B = \begin{pmatrix} 0 & \alpha \\ \frac{1}{\alpha} & 0 \end{pmatrix} \in X.$$

Then

$$B_1 \cdot = AB = \begin{pmatrix} \frac{b}{\alpha} & a\alpha \\ \frac{d}{\alpha} & c\alpha \end{pmatrix}$$

and $c\alpha \neq 0$. Now replace B by B_1. Q.E.D.

THEOREM. *G can be generated by a minimal generating system which contains only hyperbolic matrices.*

Proof. By lemma 2, G has a minimal generating system X which contains two elements A and B with $\operatorname{tr}[A, B] \neq 2$, $\operatorname{tr} A \neq 0$ and $\operatorname{tr} B \neq 0$. We may assume that $|\operatorname{tr} A| > 2$ and $|\operatorname{tr} B| > 2$ beause the pair $\{A, B\}$ is Nielsen-equivalent to a pair $\{C, D\}$ with $\operatorname{tr}[C, D] = \operatorname{tr}[A, B] \neq 2$, $|\operatorname{tr} C| > 2$ and $|\operatorname{tr} D| > 2$ (cf. the proof of the theorem in [4]).

Now let

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, |\alpha| > 1, \quad \text{and} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

without loss of generality. We have $c \neq 0$ and $b \neq 0$ because $\operatorname{tr}[A, B] \neq 2$. Let

$$C = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

be any element of X. By Lemma 3 we may assume that $e \neq 0$ or $h \neq 0$. Then $|\operatorname{tr} A^n C| > 2$ or $|\operatorname{tr} A^{-n} C| > 2$ for some sufficiently large n because $|\alpha| > 1$. This proves the theorem.

Remark. As their second main result, C. Doyle and D. James [1] proved some discreteness criteria for subgroups of $SL(2, \mathbf{R})$. More general discreteness criteria are given in [3] and [5].

Bibliography

1. C. Doyle and D. James, *Discreteness criteria and high order generators for subgroups of* $SL(2, \mathbf{R})$, Illinois J. Math., vol. 25 (1981), pp. 191–200.
2. J. R. Ford, *Automorphic functions*, 2nd ed., Chelsea, New York, 1951.
3. T. Jørgensen, *A note on subgroups of* $SL(2, \mathbf{C})$, Quart. J. Oxford Ser. II, vol. 28 (1977), pp. 209–212.
4. G. Rosenberger, *On discrete free subgroups of linear groups*, J. London Math. Soc. (2), vol. 17 (1978), pp. 79–85.
5. ———, *Eine Bemerkung zu einer Arbeit von T. Jørgensen*, Math. Zeitschr., vol. 165 (1979), pp. 261–265.

Universität Dortmund
Dortmund, Federal Republic of Germany