ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 4, Winter 1987

A GENERALIZATION OF HALÁSZ'S THEOREM TO BEURLING'S GENERALIZED INTEGERS AND ITS APPLICATION

BY

WEN-BIN ZHANG[1]

0. Introduction

In 1968 Halász [6] proved the following important result:

THEOREM. *Let $f(n)$ be a completely multiplicative function such that $|f(n)| \leq 1$ holds for all $n \in \mathbf{N}$. Suppose that*

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{c}{s-1} + o\left(\frac{1}{\sigma - 1}\right)$$

holds with constant c as $\sigma = \operatorname{Re} s \to 1+$ uniformly for $-K \leq t \leq K$ for each fixed $K > 0$. Then

$$F(x) := \sum_{n \leq x} f(n) = cx + o(x).$$

This theorem is generalized here in Theorem 1.1 to Beurling's generalized integers [1], [2]. We then apply Theorem 1.1 to prove Theorem 2.1 which is a generalization of Halász-Wirsing's theorem [4], [9]. From Theorem 2.1, we deduce Theorem 2.3 on the estimate $M(x) = o(x)$. The latter combined with a theorem of Beurling [2] and an example of Diamond [3] shows that the prime number theorem and the estimate $M(x) = o(x)$ are not completely equivalent.

1. A generalization of Halász's theorem

Let $\mathscr{P} = \{p_i\}_{i=1}^{\infty}$ be a sequence of real numbers subject to the following three conditions but otherwise arbitrary:

$$\text{(i) } p_1 > 1, \quad \text{(ii) } p_{n+1} \geq p_n, \quad \text{(iii) } p_n \to \infty.$$

Received March 7, 1986.

[1] This article is, with minor changes, a chapter of the author's Ph.D. dissertation, written at the University of Illinois at Urbana-Champaign under the direction of Professor Harold G. Diamond.

Following Beurling, we shall call such a sequence $\mathscr{P}$ a set of generalized (henceforth $g-$) primes. Let $\mathscr{N}^*$ be the set of all sequences $\nu = (\nu_1, \dots, \nu_m, \dots)$ of non-negative integers all but a finite number of which are zeros. Then, under the addition of sequences, $\mathscr{N}^*$ is an additive semi-group. For each $\nu = (\nu_1, \dots, \nu_m, \dots) \in \mathscr{N}^*$ we set

$$n(\nu) = \prod_{j=1}^{\infty} p_j^{\nu_j}.$$

Then for $\nu', \nu'' \in \mathscr{N}^*$, we have $n(\nu' + \nu'') = n(\nu')n(\nu'')$. In this sense, the set of all $n(\nu)$ is a multiplicative semi-group which we consider to be generated by $\mathscr{P}$. Moreover, this set is countable and may be arranged in a non-decreasing sequence $\mathscr{N} = \{n_i\}_{i=0}^{\infty}$ (where $n_0 = 1$, $n_1 = p_1$, etc.). We shall call $\mathscr{N}$ the set of g-integers associated with $\mathscr{P}$.

Let $f(\nu)$ be a complex-valued function defined on $\mathscr{N}^*$. We define

$$F(x) = \sum_{\substack{\nu \\ n(\nu) \leq x}} f(\nu).$$

In particular,

$$N(x) = N_{\mathscr{P}}(x) = \sum_{\substack{\nu \\ n(\nu) \leq x}} 1$$

denotes the distribution function of the g-integers associated with $\mathscr{P}$. A function f is said to be completely multiplicative if

$$f(\nu' + \nu'') = f(\nu')f(\nu'')$$

holds for all $\nu', \nu'' \in \mathscr{N}^*$. For convenience, we write $f(\nu)$ as $f(n_i)$ for $n_i = n(\nu)$. If $f(\nu)$ is completely multiplicative on $\mathscr{N}^*$ then we have

$$f(n_i n_j) = f(n_i)f(n_j)$$

for all $n_i, n_j \in \mathscr{N}$ and in this case we will call f a completely multiplicative function on $\mathscr{N}$. Suppose that $F(x) = O(x)$. Then we have

$$\hat{F}(s) := \int_{1-}^{\infty} x^{-s}\, dF(x) = \sum_{i=0}^{\infty} \frac{f(n_i)}{n_i^s}$$

for $\sigma > 1$.

THEOREM 1.1. *Let $f(n_i)$ be a completely multiplicative function on $\mathscr{N}$ such that $|f(n_i)| \leq 1$ holds for all $n_i \in \mathscr{N}$. Suppose that, for some constant $A > 0$,*

$$\int_1^{\infty} x^{-2}|N(x) - Ax|\,dx < \infty \tag{1.1}$$

and either

$$\int_1^x t^{-1}\{N(t) - At\}\log t\,dt \ll x \tag{1.2}$$

or

$$\int_1^\infty x^{-3}|N(x) - Ax|^2 \log x\,dx < \infty \tag{1.3}$$

holds. Furthermore, suppose that

$$\hat{F}(s) = \frac{c}{s-1} + o\left(\frac{1}{\sigma - 1}\right) \tag{1.4}$$

holds as $\sigma = \operatorname{Re} s \to 1+$ *uniformly for* $-K \leq t \leq K$ *for each fixed* $K > 0$. *Then we have*

$$F(x) = cx + o(x). \tag{1.5}$$

Remark. (1.4) is Halász's condition. (1.2) is an average form of the condition

$$N(x) = Ax + O(x/\log x).$$

To prove Theorem 1.1, we need several lemmas.

LEMMA 1.2. *Let* $N(x)$ *be a real-valued nondecreasing function. If, for some constant* A,

$$\int_1^\infty \frac{N(x) - Ax}{x^2}\,dx$$

converges then, as $x \to \infty$, $N(x) = Ax + o(x)$.

Proof. Let $0 < \varepsilon < 1$. We have

$$\frac{N(x)}{x} \leq \frac{1+\varepsilon}{\varepsilon}\left(\int_x^{(1+\varepsilon)x} \frac{N(t) - At}{t^2}\,dt + A\log(1+\varepsilon)\right).$$

It follows that

$$\limsup_{x\to\infty} \frac{N(x)}{x} \leq \frac{1+\varepsilon}{\varepsilon} A\log(1+\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_{x\to\infty} \frac{N(x)}{x} \leq A.$$

In the same way, from

$$\frac{N(x)}{x} \geq \frac{1-\varepsilon}{\varepsilon}\left(\int_{(1-\varepsilon)x}^{x} \frac{N(t)-At}{t^2}\,dt + A\log\frac{1}{1-\varepsilon}\right),$$

we can deduce

$$\liminf_{x\to\infty} \frac{N(x)}{x} \geq A. \qquad \blacksquare$$

LEMMA 1.3. *Assume* (1.2). *Given* $\eta > 0$, *we have, for* $1 < \sigma \leq 2$,

$$\int_{-\eta}^{\eta}\left|\int_{1}^{\infty} x^{-(\sigma+it)-1}\{N(x)-Ax\}\log x\,dx\right|^2 dt = O\big((\sigma-1)^{-1}\big). \tag{1.6}$$

Proof. Set

$$\hat{\Phi}(s) = \int_{1}^{\infty} x^{-s-1}\{N(x)-Ax\}\log x\,dx.$$

Then we have

$$\frac{\hat{\Phi}(s)}{s} = \int_{1}^{\infty} x^{-s-1}\Phi(x)\,dx = \int_{0}^{\infty} e^{-itu-\sigma u}\Phi(e^u)\,du,$$

where

$$\Phi(x) = \int_{1}^{x} t^{-1}\{N(t)-At\}\log t\,dt.$$

By Plancherel's formula for Fourier transforms [5, Chapter 3, 13], we have

$$\int_{-\infty}^{\infty}\left|\frac{\hat{\Phi}(\sigma+it)}{\sigma+it}\right|^2 dt = 2\pi\int_{0}^{\infty} e^{-2\sigma u}\Phi^2(e^u)\,du.$$

We note that, by (1.2), $\Phi(e^u) \ll e^u$ holds. It follows that

$$\int_{-\infty}^{\infty}\left|\frac{\hat{\Phi}(\sigma+it)}{\sigma+it}\right|^2 dt \ll \int_{0}^{\infty} e^{-2(\sigma-1)u}\,du \ll (\sigma-1)^{-1}$$

and hence

$$\int_{-\eta}^{\eta}|\hat{\Phi}(\sigma+it)|^2\,dt \ll_\eta \int_{-\infty}^{\infty}\left|\frac{\hat{\Phi}(\sigma+it)}{\sigma+it}\right|^2 dt \ll (\sigma-1)^{-1}. \qquad \blacksquare$$

LEMMA 1.4. *Assume* (1.3). *Then we have*

$$\int_{-\infty}^{\infty}\left|\int_{1}^{\infty}x^{-(\sigma+it)-1}\{N(x)-Ax\}\log x\,dx\right|^2 dt = o\left((\sigma-1)^{-1}\right). \tag{1.7}$$

Proof. Let I denote the integral on the left-hand side of (1.7). Then by Plancherel's formula for Fourier transforms, we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\left|\int_{0}^{\infty}e^{-itu-\sigma u}u\{N(e^u)-Ae^u\}\,du\right|^2 dt \\ &= 2\pi\int_{0}^{\infty}e^{-2\sigma u}u^2\{N(e^u)-Ae^u\}^2\,du. \end{aligned}$$

By (1.3),

$$\int_{0}^{\infty}e^{-2v}v\{N(e^v)-Ae^v\}^2\,dv$$

is convergent. Define

$$\phi(u)=\int_{u}^{\infty}e^{-2v}v\{N(e^v)-Ae^v\}^2\,dv.$$

Then $\phi(u)=o(1)$. By integration by parts, we have

$$\begin{aligned} I &= 2\pi\int_{0}^{\infty}\phi(u)e^{-2(\sigma-1)u}(1-2(\sigma-1)u)\,du \\ &\le 2\pi\int_{0}^{\infty}\phi(u)e^{-2(\sigma-1)u}\,du \\ &= o\left((\sigma-1)^{-1}\right). \end{aligned}$$

■

LEMMA 1.5 [8]. *Let* $\hat{G}_k(s)=\int_{1-}^{\infty}x^{-s}\,dG_k(x)$, $k=1,2$, *converge for* $\sigma>1$. *Suppose that* $|dG_1|\le dG_2$. *Then for all* $T\in\mathbf{R}$, $\eta>0$ *and* $\sigma>1$ *we have*

$$\int_{T}^{T+\eta}|\hat{G}_1(\sigma+it)|^2\,dt\le 2\int_{-\eta}^{\eta}|\hat{G}_2(\sigma+it)|^2\,dt.$$

Proof. We have

$$0\le\frac{1}{\eta}\int_{-\eta}^{\eta}\left(1-\frac{|t|}{\eta}\right)e^{ixt}\,dt=\begin{cases}\left(\dfrac{\sin\frac{1}{2}\eta x}{\frac{1}{2}\eta x}\right)^2 & \text{for } x\ne 0,\\ 1 & \text{for } x=0.\end{cases}$$

Therefore, for $\sigma > 1$, we have

$$\int_T^{T+\eta} |\hat{G}_1(\sigma + it)|^2 \, dt$$

$$\leq 2\int_{-\eta}^{\eta} \left(1 - \frac{|t|}{\eta}\right) \left|\hat{G}_1\left(\sigma + i\left(T + \frac{1}{2}\eta + t\right)\right)\right|^2 dt$$

$$= 2\int_{1-}^{\infty}\int_{1-}^{\infty} x^{-(\sigma + i(T+\eta/2))} y^{-(\sigma - i(T+\eta/2))}$$

$$\times \left(\int_{-\eta}^{\eta} \left(1 - \frac{|t|}{\eta}\right) x^{-it} y^{it} \, dt\right) dG_1(x) \, dG_1(y)$$

$$\leq 2\int_{1-}^{\infty}\int_{1-}^{\infty} x^{-\sigma} y^{-\sigma} \left(\int_{-\eta}^{\eta} \left(1 - \frac{|t|}{\eta}\right) x^{-it} y^{it} \, dt\right) dG_2(x) \, dG_2(y)$$

$$= 2\int_{-\eta}^{\eta} \left(1 - \frac{|t|}{\eta}\right) |\hat{G}_2(\sigma + it)|^2 \, dt$$

$$\leq 2\int_{-\eta}^{\eta} |\hat{G}_2(\sigma + it)|^2 \, dt. \qquad \blacksquare$$

Proof of Theorem 1.1. We follow the proof of Halász's theorem. We consider

$$H(x) = \int_1^x t^{-1} \left(\int_1^t \log u \, dF(u)\right) dt \tag{1.8}$$

and shall show

$$H(x) = cx \log x + o(x \log x),$$

from which the desired estimate of $F(x)$ will be obtained by a tauberian argument. We have

$$\int_1^{\infty} x^{-s} \, dH(x) = -\frac{\hat{F}'(s)}{s}$$

and, by Perron's inversion formula,

$$H(x) = \frac{1}{2\pi i} \int_{\sigma = \sigma_0} -x^s \frac{\hat{F}'(s)}{s^2} \, ds \tag{1.9}$$

$$= \frac{x}{2\pi i} \int_{\sigma = \sigma_0} -x^{s-1} \frac{\hat{F}'(s)}{s^2} \, ds$$

where $\sigma_0 = 1 + 1/\log x$. Let K be a large number, fixed for the moment, and let x be so large that $\log x > 2K$. Hence we have $|x^{s-1}| = x^{\sigma_0 - 1} = e$ for $\sigma = \sigma_0$ and $K(\sigma_0 - 1) < \frac{1}{2}$. We break the integration contour $\sigma = \sigma_0$ into the following parts:

$$\begin{aligned} I_0 &= \{s = \sigma_0 + it\colon -K(\sigma_0 - 1) \le t \le K(\sigma_0 - 1)\}, \\ I_1 &= \{s = \sigma_0 + it\colon K(\sigma_0 - 1) \le t \le K\}, \\ I_2 &= \{s = \sigma_0 + it\colon -K \le t \le -K(\sigma_0 - 1)\}, \\ I_3 &= \{s = \sigma_0 + it\colon K \le t < \infty\}, \\ I_4 &= \{s = \sigma_0 + it\colon -\infty < t \le -K\} \end{aligned}$$

and estimate the last integral in (1.9) on each part separately.

(i) Estimate of $\int_{I_0}$. For $s \in I_0$, s fixed for the moment, consider the disk

$$D_s = \left\{z\colon |z - s| \le \frac{1}{2}(\sigma_0 - 1)\right\}.$$

For $z \in D_s$, $\operatorname{Re} z - 1 \ge \frac{1}{2}(\sigma_0 - 1)$. Therefore, by the hypothesis (1.4),

$$\hat{F}(z) - \frac{c}{z - 1} = o\left(\frac{1}{\operatorname{Re} z - 1}\right) = o\left(\frac{1}{\sigma_0 - 1}\right)$$

holds uniformly for all $z \in D_s$ and all $s \in I_0$. It follows, by Cauchy's inequality for derivatives of analytic functions, that

$$\hat{F}'(s) + \frac{c}{(s - 1)^2} = o\left(\frac{1}{\sigma_0 - 1}\right)\frac{2}{\sigma_0 - 1} = o\left(\frac{1}{(\sigma_0 - 1)^2}\right)$$

holds uniformly for $s \in I_0$. Hence, we have

$$\begin{aligned} (1.10)\qquad & -\frac{1}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2}\hat{F}'(s)\,ds \\ &= \frac{1}{2\pi i}\left(\int_{I_0} c\frac{x^{s-1}}{s^2}\frac{ds}{(s-1)^2} + \int_{I_0} o\left(\frac{1}{(\sigma_0 - 1)^2}\right)\frac{x^{s-1}}{s^2}\,ds\right) \\ &= \frac{c}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2(s-1)^2}\,ds + Ko(\log x) \end{aligned}$$

since

$$\begin{aligned} \int_{I_0} o\left(\frac{1}{(\sigma_0 - 1)^2}\right)\frac{x^{s-1}}{s^2}\,ds &= 2K(\sigma_0 - 1)o\left(\frac{1}{(\sigma_0 - 1)^2}\right) \\ &= Ko\left(\frac{1}{\sigma_0 - 1}\right) \\ &= Ko(\log x). \end{aligned}$$

The last integral in (1.10) can be evaluated by using Cauchy's integral theorem. Define the semi-circle Γ by

$$\Gamma = \{s\colon \operatorname{Re} s \le \sigma_0,\, |s - \sigma_0| = K(\sigma_0 - 1)\}.$$

Note that $K(\sigma_0 - 1) < \frac{1}{2}$, $\sigma_0 > 1$ and hence $s = 0$ is not within the contour $\Gamma \cup I_0$. Therefore, the integrand has only one pole at $s = 1$, with residue $\log x - 2$ within the contour. Hence, we have

$$\frac{1}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2(s-1)^2}\, ds = (\log x - 2) + \frac{1}{2\pi i}\int_{\Gamma} \frac{x^{s-1}}{s^2(s-1)^2}\, ds.$$

On Γ, $|x^{s-1}| = x^{\sigma - 1} \le x^{\sigma_0 - 1} = e$, $|s| > \frac{1}{2}$ since $K(\sigma_0 - 1) < \frac{1}{2}$, and

$$|s - 1| \ge (K - 1)(\sigma_0 - 1),$$

hence we have

$$\left|\frac{1}{2\pi i}\int_{\Gamma} \frac{x^{s-1}}{s^2(s-1)^2}\, ds\right| \ll \frac{1}{K^2(\sigma_0 - 1)^2} K(\sigma_0 - 1) \ll K^{-1}\log x.$$

It follows that

$$\frac{c}{2\pi i}\int_{I_0} \frac{x^{s-1}}{s^2(s-1)^2}\, ds = c\log x + \frac{1}{K} O(\log x). \tag{1.11}$$

(ii) Estimates of $\int_{I_3}$ and $\int_{I_4}$. For $\sigma > 1$, we have

$$\hat{F}(s) = \prod_{i=1}^{\infty}\left(1 - \frac{f(p_i)}{p_i^s}\right)^{-1} \ne 0.$$

Define $\Lambda(\nu)$ on $\mathscr{N}^*$ by setting

$$\Lambda(\nu) = \begin{cases} \log p_i, & \text{if } \nu = (\nu_1, \dots, \nu_m, \dots) \text{ with} \\ & \quad \nu_i > 0 \text{ and } \nu_m = 0 \text{ for } m \ne i, \\ 0, & \text{otherwise,} \end{cases}$$

the analogue of the classical von Mongoldt function, and set

$$G(x) = \sum_{\substack{\nu \\ n(\nu) \le x}} \Lambda(\nu) f(\nu), \quad \psi(x) = \sum_{\substack{\nu \\ n(\nu) \le x}} \Lambda(\nu).$$

As before, we write $\Lambda(\nu)$ as $\Lambda(n_i)$ for $n_i = n(\nu)$. Then we have

$$-\frac{\hat{F}'(s)}{\hat{F}(s)} = \int_{1-}^{\infty} x^{-s}\, dG(x) = \sum_{i=0}^{\infty} \Lambda(n_i) f(n_i) n_i^{-s}$$

for $\sigma > 1$.

To estimate $\int_{I_{3,4}}$, we have

$$\left|\int_{I_{3,4}} x^{s-1}\hat{F}'(s)s^{-2}\, ds\right| \le e\int_{I_{3,4}} |\hat{F}'(s)|\,|s|^{-2}|ds|$$
$$\le e\left(\int_{I_{3,4}}\left|\frac{\hat{F}'(s)}{\hat{F}(s)}\right|^2 \frac{|ds|}{|s|^2}\right)^{1/2}\left(\int_{I_{3,4}} \frac{|\hat{F}(s)|^2}{|s|^2}|ds|\right)^{1/2},$$

by the Cauchy-Schwarz inequality. We first apply Lemmas 1.3, 1.4 and 1.5 to estimate

$$\int_{\sigma=\sigma_0}\left|\frac{\hat{F}'(s)}{\hat{F}(s)}\right|^2 \frac{|ds|}{|s|^2}.$$

Note that

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_{1-}^{\infty} x^{-s}\, d\psi(x),$$

where $\zeta(s)$ is the zeta function associated with $\mathscr{N}$, and that $|dG| \le d\psi$. Therefore,

$$\int_T^{T+\eta}\left|\frac{\hat{F}'(\sigma_0+it)}{\hat{F}(\sigma_0+it)}\right|^2 dt \le 2\int_{-\eta}^{\eta}\left|\frac{\zeta'(\sigma_0+it)}{\zeta(\sigma_0+it)}\right|^2 dt.$$

We need now a suitable choice of η. Consider

$$\zeta(s) = \frac{A}{s-1} + A + sg(s),$$

where the function $g(s)$ is defined by

$$g(s) = \int_1^{\infty} x^{-s-1}\{N(x) - Ax\}\, dx.$$

The function g is analytic on $\sigma > 1$ and continuous on $\sigma \ge 1$. Therefore, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - h(s)$$

where

$$h(s) = \frac{1}{s} + \frac{sg(s)}{(s-1)\zeta(s)} + \frac{sg'(s)}{\zeta(s)}.$$

We note that

$$(s-1)\zeta(s) = As + s(s-1)g(s)$$

is continuous on $\sigma \geq 1$ and may be extended to a continuous function on $\sigma > 1$. Hence there exists a number $\eta > 0$ such that $(s-1)\zeta(s) \neq 0$ for $|t| \leq \eta$, $1 \leq \sigma \leq 2$ since $A > 0$. We now fix $\eta > 0$. It follows that

$$|h(s)| \ll 1 + |g'(s)|$$

for $|t| \leq \eta$, $1 < \sigma \leq 2$. Therefore, by Lemma 1.3 or Lemma 1.4, we have

$$\begin{aligned}
2\int_{-\eta}^{\eta}\left|\frac{\zeta'(\sigma_0+it)}{\zeta(\sigma_0+it)}\right|^2 dt &= 2\int_{-\eta}^{\eta}\left|\frac{1}{\sigma_0-1+it} + O\big(1+|g'(\sigma_0+it)|\big)\right|^2 dt \\
&\ll 1 + \int_{-\eta}^{\eta}\frac{dt}{(\sigma_0-1)^2+t^2} \\
&\quad + \int_{-\eta}^{\eta}\left|\int_1^{\infty} x^{-(\sigma_0+it)-1}\{N(x)-Ax\}\log x\, dx\right|^2 dt \\
&= 1 + \frac{2}{\sigma_0-1}\int_0^{\eta/(\sigma_0-1)}\frac{du}{1+u^2} + O\big((\sigma_0-1)^{-1}\big) \\
&\ll (\sigma_0-1)^{-1} \\
&= \log x.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1.12)\qquad &\int_{\sigma=\sigma_0}\left|\frac{\hat{F}'(s)}{\hat{F}(s)}\right|^2\frac{|ds|}{|s|^2} \\
&= \sum_{m=0}^{\infty}\left(\int_{\sigma_0+im\eta}^{\sigma_0+i(m+1)\eta} + \int_{\sigma_0-i(m+1)\eta}^{\sigma_0-im\eta}\right)\left|\frac{\hat{F}'(s)}{\hat{F}(s)}\right|^2\frac{|ds|}{|s|^2} \\
&\ll \sum_{m=0}^{\infty}\frac{1}{1+m^2\eta^2}\log x \ll \log x.
\end{aligned}$$

We then use the same method to estimate

$$\int_{I_{3,4}}\frac{|\hat{F}(s)|^2}{|s|^2}|ds|.$$

Again, we have

$$\hat{F}(s) = \int_{1-}^{\infty} x^{-s}\,dF(x), \quad \zeta(s) = \int_{1-}^{\infty} x^{-s}\,dN(x)$$

and $|dF| \le dN$. Hence

$$\begin{aligned}\int_T^{T+1} |\hat{F}(\sigma_0 + it)|^2\, dt &\le 2\int_{-1}^{1} |\zeta(\sigma_0 + it)|^2\, dt \\ &= 2\int_{-1}^{1} \left| \frac{A}{\sigma_0 - 1 + it} + O(1) \right|^2 dt \\ &\ll \log x\end{aligned}$$

and

$$\begin{aligned}\int_{I_3} \frac{|\hat{F}(s)|^2}{|s|^2} |ds| &= \sum_{m=0}^{\infty} \int_{\sigma_0 + i(K+m)}^{\sigma_0 + i(K+m+1)} \frac{|\hat{F}(s)|^2}{|s|^2} |ds| \\ &\ll \sum_{m=0}^{\infty} \frac{1}{1 + (K+m)^2} \log x \\ &\ll \frac{\log x}{K}.\end{aligned}$$

Similarly estimate the integral

$$\int_{I_4} \frac{|\hat{F}(s)|^2}{|s|^2} |ds|.$$

Hence we deduce that

$$\left| \int_{I_{3,4}} x^{s-1} \frac{\hat{F}'(s)}{s^2}\, ds \right| \ll \frac{1}{K^{1/2}} \log x. \tag{1.13}$$

(iii) Estimates of $\int_{I_1}$ and $\int_{I_2}$. We have

$$\int_{I_{1,2}} |\hat{F}(s)|^2 |s|^{-2} |ds| \le \max_{s \in I_{1,2}} |\hat{F}(s)|^{1/2} \int_{I_{1,2}} |\hat{F}(s)|^{3/2} |s|^{-2} |ds|.$$

By (1.4),

$$\begin{aligned}\max_{s \in I_{1,2}} |\hat{F}(s)|^{1/2} &\le \max_{s \in I_{1,2}} \left| \frac{c}{s-1} + o\left(\frac{1}{\sigma_0 - 1} \right) \right|^{1/2} \\ &\ll \frac{1}{(\sigma_0 - 1)^{1/2} (1 + K^2)^{1/4}} + o\left(\frac{1}{(\sigma_0 - 1)^{1/2}} \right) \\ &\ll K^{-1/2} \log^{1/2} x + o\left(\log^{1/2} x\right).\end{aligned}$$

We next consider $|\hat{F}(s)|^{3/4}$. Since f is completely multiplicative we have

$$\left(\hat{F}(s)\right)^{3/4} = \exp\left\{ \frac{3}{4} \sum_{i=0}^{\infty} \kappa(n_i) f(n_i) n_i^{-s} \right\}$$

where $\kappa(n_i)$ denotes $\kappa(\nu)$ for $n_i = n(\nu)$ and

$$\kappa(\nu) = \begin{cases} 1/\nu_j, & \text{if } \nu = (\nu_1, \dots, \nu_m, \dots) \text{ with} \\ & \quad \nu_j > 0 \text{ and } \nu_m = 0 \text{ for } m \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned}(\hat{F}(s))^{3/4} &= \exp\left\{\int_{1-}^{\infty} x^{-s} d\left(\frac{3}{4} \sum_{\substack{\nu \\ n(\nu) \le x}} \kappa(\nu) f(\nu)\right)\right\} \\ &= \int_{1-}^{\infty} x^{-s} d\left(\exp\left\{\frac{3}{4} \sum_{\substack{\nu \\ n(\nu) \le x}} \kappa(\nu) f(\nu)\right\}\right).\end{aligned}$$

We also have

$$(\zeta(s))^{3/4} = \int_{1-}^{\infty} x^{-s} d\left(\exp\left\{\frac{3}{4} \sum_{\substack{\nu \\ n(\nu) \le x}} \kappa(\nu)\right\}\right).$$

Note that

$$\left| d\left(\exp\left\{\frac{3}{4} \sum_{\substack{\nu \\ n(\nu) \le x}} \kappa(\nu) f(\nu)\right\}\right)\right| \le d\left(\exp\left\{\frac{3}{4} \sum_{\substack{\nu \\ n(\nu) \le x}} \kappa(\nu)\right\}\right).$$

Hence, by Lemma 1.5, we have

$$\begin{aligned}\int_T^{T+1} |\hat{F}(\sigma_0 + it)|^{3/2} \, dt &\le 4\int_0^1 |\zeta(\sigma_0 + it)|^{3/2} \, dt \\ &= 4\int_0^1 \left| \frac{A}{\sigma_0 - 1 + it} + O(1) \right|^{3/2} dt \\ &\ll 1 + \int_0^1 \left(\frac{1}{(\sigma_0 - 1)^2 + t^2} \right)^{3/4} dt \\ &\le 1 + \frac{1}{(\sigma_0 - 1)^{1/2}} \int_0^{\infty} \frac{du}{(1 + u^2)^{3/4}} \\ &\ll \frac{1}{(\sigma_0 - 1)^{1/2}} \\ &= \log^{1/2} x.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{I_1} |\hat{F}(s)|^{3/2}|s|^{-2}|ds| &\le \sum_{m=0}^{[K]} \int_{\sigma_0+i(K(\sigma_0-1)+m)}^{\sigma_0+i(K(\sigma_0-1)+m+1)} |\hat{F}(s)|^{3/2}|s|^{-2}|ds| \\ &\ll \log^{1/2}x \sum_{m=0}^{[K]} \frac{1}{1+m^2} \\ &\ll \log^{1/2}x\end{aligned}$$

and hence

$$\int_{I_1} |\hat{F}(s)|^2|s|^{-2}|ds| \ll \frac{\log x}{K^{1/2}} + o(\log x).$$

Similarly estimate the integral

$$\int_{I_2} |\hat{F}(s)|\,|s|^{-2}|ds|.$$

Hence we deduce, by applying (1.12) once more, that

$$\begin{aligned}(1.14)\qquad &\left|\int_{I_{1,2}} x^{s-1}\hat{F}'(s)s^{-2}ds\right| \\ &\le e\left(\int_{I_{1,2}}\left|\frac{\hat{F}'(s)}{\hat{F}(s)}\right|^2\frac{|ds|}{|s|^2}\right)^{1/2}\left(\int_{I_{1,2}}\frac{|\hat{F}(s)|^2}{|s|^2}|ds|\right)^{1/2} \\ &\ll K^{-1/4}\log x + o(\log x).\end{aligned}$$

Combining (1.10), (1.11), (1.13) and (1.14) with (1.9), we arrive at

$$H(x) = cx\log x + K^{-1/4}O(x\log x) + Ko(x\log x).$$

Given $\varepsilon > 0$, we have

$$|K^{-1/4}O(x\log x)| < \frac{1}{2}\varepsilon x\log x$$

for $K \ge K_0$ sufficiently large. Fixing $K \ge K_0$, for $x \ge x_0$ sufficiently large, we have

$$|Ko(x\log x)| < \frac{1}{2}\varepsilon x\log x.$$

This implies

$$|H(x) - cx\log x| < \varepsilon x\log x$$

for $x \geq x_0$, i.e.,

$$H(x) = cx\log x + o(x\log x). \tag{1.15}$$

It remains to deduce (1.5) from (1.15) by a tauberian argument. Set

$$\Phi(x) = \int_1^x \log t\, dF(t).$$

Then we have

$$H(x) = \int_1^x t^{-1}\Phi(t)\, dt = cx\log x + o(x\log x).$$

For $0 < \varepsilon < \frac{1}{2}$, on the one hand we have

$$\begin{aligned}\int_x^{x+\varepsilon x} t^{-1}\Phi(t)\, dt &= \left(\int_1^{x+\varepsilon x} - \int_1^x\right) t^{-1}\Phi(t)\, dt \\ &= c\varepsilon x\log x + c(1+\varepsilon)x\log(1+\varepsilon) + o(x\log x).\end{aligned}$$

On the other hand,

$$\int_x^{x+\varepsilon x} t^{-1}\Phi(t)\, dt = \Phi(x)\log(1+\varepsilon) + \int_x^{x+\varepsilon x} t^{-1}(\Phi(t) - \Phi(x))\, dt.$$

It follows that we have

$$\begin{aligned}\Phi(x) = c\frac{\varepsilon}{\log(1+\varepsilon)}x\log x + c(1+\varepsilon)x + \frac{o(x\log x)}{\log(1+\varepsilon)} \\ - \frac{1}{\log(1+\varepsilon)}\int_x^{x+\varepsilon x} t^{-1}(\Phi(t) - \Phi(x))\, dt\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\Phi(x) - cx\log x}{x\log x}\right| \leq |c|\left|\frac{\varepsilon}{\log(1+\varepsilon)} - 1\right| + \frac{|c|(1+\varepsilon)}{\log x} + \frac{o(1)}{\log(1+\varepsilon)} \\ + \frac{1}{x\log x\log(1+\varepsilon)}\left|\int_x^{x+\varepsilon x} t^{-1}(\Phi(t) - \Phi(x))\, dt\right|.\end{aligned}$$

We note that, for $x < t \leq x + \varepsilon x$,

$$\begin{aligned}|\Phi(t) - \Phi(x)| &= \left|\int_{x+}^t \log u\, dF(u)\right| \\ &\leq \log t \int_{x+}^t dN(u) \\ &\leq (\log x + \log(1+\varepsilon))(N(x+\varepsilon x) - N(x)).\end{aligned}$$

Therefore, we have

$$\left|\frac{\Phi(x) - cx\log x}{x\log x}\right| \le |c|\left|\frac{\varepsilon}{\log(1+\varepsilon)} - 1\right| + \frac{|c|(1+\varepsilon)}{\log x} + \frac{o(1)}{\log(1+\varepsilon)}$$
$$+ \frac{1}{x\log x}(\log x + \log(1+\varepsilon))|N(x+\varepsilon x) - N(x)|$$

and hence

$$\limsup_{x\to\infty}\left|\frac{\Phi(x) - cx\log x}{x\log x}\right| \le |c|\left|\frac{\varepsilon}{\log(1+\varepsilon)} - 1\right| + A\varepsilon$$

holds for any fixed $\varepsilon > 0$ since

$$\frac{N(x+\varepsilon x) - N(x)}{x} = A\varepsilon + \frac{N(x+\varepsilon x) - A(x+\varepsilon x)}{x} - \frac{N(x) - Ax}{x}$$
$$\to A\varepsilon \quad \text{as } x\to\infty$$

by Lemma 1.2. Letting $\varepsilon \to 0$, we arrive at

$$\limsup_{x\to\infty}\left|\frac{\Phi(x) - cx\log x}{x\log x}\right| = 0,$$

i.e.,

$$\Phi(x) = cx\log x + o(x\log x).$$

Finally, by integration by parts, we have

$$F(x) = 1 + \int_\alpha^x \frac{d\Phi(t)}{\log t} = cx + o(x)$$

where $1 < \alpha < n_1$. This completes the proof of the theorem. ■

The following two corollaries are immediate.

COROLLARY 1.6. *If we replace* (1.2) *in Theorem* 1.1 *by*

$$N(x) = Ax + O(x\log^{-1}x), \quad x > 1.$$

then (1.5) *is true.*

COROLLARY 1.7. *If we replace* (1.1), (1.2) *and* (1.3) *in Theorem* 1.1 *by*

$$N(x) = Ax + O(x\log^{-\gamma}x), \quad x > 1$$

with constant $\gamma > 1$ *then* (1.5) *is true.*

2. A generalization of Halász-Wirsing's theorem

The following theorem is a generalization of Halász-Wirsing's theorem [4], [9] to g-integers.

THEOREM 2.1. *Suppose that* (1.1) *and one of* (1.2) *and* (1.3) *hold. Let f be a completely multiplicative function on $\mathcal{N}$ such that $|f(n_i)| \le 1$ for all $n_i \in \mathcal{N}$. Then*

$$F(x) = o(x) \tag{2.1}$$

holds if and only if

$$\sum_{k=1}^{\infty} \frac{1}{p_k} \operatorname{Re}\left(1 - f(p_k) p_k^{-it}\right) = \infty \tag{2.2}$$

holds for all real t.

To prove Theorem 2.1, we need the following:

LEMMA 2.2. *Assume* (1.1). *Let f be a completely multiplicative function on $\mathcal{N}$ satisfying $|f(n_i)| \le 1$ for all $n_i \in \mathcal{N}$. Let I be a compact interval in $\mathbf{R}$. Then* (2.2) *holds for all $t \in I$ if and only if*

$$\hat{F}(s) = o\left(\frac{1}{\sigma - 1}\right) \tag{2.3}$$

holds uniformly for $t \in I$ as $\sigma \to 1+$.

Proof. We first note that

$$\begin{aligned} -\operatorname{Re} \sum_{k=1}^{\infty} \log\left(1 - \frac{f(p_k)}{p_k^s}\right) &+ \sum_{k=1}^{\infty} \log\left(1 - \frac{1}{p_k^{\sigma}}\right) \\ &= - \sum_{k=1}^{\infty} \frac{1}{p_k^{\sigma}} \left(1 - \operatorname{Re} f(p_k) p_k^{-it}\right) + O(1) \end{aligned}$$

holds for $\sigma > 1$ since, by Lemma 1.2, $\pi(x) \le N(x) \ll x$. Hence

$$\frac{|\hat{F}(s)|}{\zeta(\sigma)} = \exp\left\{ - \sum_{k=1}^{\infty} \frac{1}{p_k^{\sigma}} \left(1 - \operatorname{Re} f(p_k) p_k^{-it}\right) + O(1) \right\}.$$

We then note that

$$\zeta(\sigma) = \frac{A}{\sigma - 1} + A + \sigma g(\sigma)$$

where $g(\sigma)$ is continuous on $\sigma \geq 1$. From these two facts, it follows that if (2.2) holds for all $t \in I$ then, by Dini's theorem, (2.3) holds uniformly for $t \in I$ as $\sigma \to 1+$. The inverse implication is trivial. ∎

Proof of Theorem 2.1. If (2.2) holds for all real t then, by Lemma 2.2, (2.3) holds uniformly for $-K \leq t \leq K$ for each fixed $K > 0$ and hence, by Theorem 1.1, (2.1) holds. The inverse implication is trivial. ∎

Application. Define

$$\Omega(\nu) = \nu_1 + \cdots + \nu_m + \cdots, \quad \lambda(\nu) = (-1)^{\Omega(\nu)}$$

for $\nu = (\nu_1, \ldots, \nu_m, \ldots) \in \mathcal{N}^*$, the respective generalizations of the classical functions $\Omega(n)$ and $\lambda(n)$ (Liouville function). Suppose that (1.1) and one of (1.2) and (1.3) hold. For $\sigma > 1$ and all $t \in \mathbf{R}$, we have

$$\begin{aligned}
&\zeta^3(\sigma)|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)| \\
&\quad = \exp\left\{ \sum_{k=1}^{\infty} \sum_{\alpha \geq 1} \frac{1}{\alpha p_k^{\alpha\sigma}} (3 + 4\cos(\alpha t \log p_k) + \cos(2\alpha t \log p_k)) \right\} \\
&\quad \geq 1
\end{aligned}$$

and hence

$$\zeta(\sigma)|\zeta(\sigma + it)| \to \infty \quad \text{or} \quad \log(\zeta(\sigma)|\zeta(\sigma + it)|) \to \infty$$

as $\sigma \to 1+$. On the other hand, we have

$$\begin{aligned}
\log(\zeta(\sigma)|\zeta(\sigma + it)|) &= \sum_{k=1}^{\infty} \sum_{\alpha \geq 1} \frac{1}{\alpha p_k^{\alpha\sigma}} (1 + \operatorname{Re} p_k^{-i\alpha t}) \\
&= \sum_{k=1}^{\infty} \frac{1}{p_k^{\sigma}} (1 + \operatorname{Re} p_k^{-it}) + O(1).
\end{aligned}$$

Therefore, we have

$$\sum_{k=1}^{\infty} \frac{1}{p_k^{\sigma}} (1 + \operatorname{Re} p_k^{-it}) \to \infty$$

as $\sigma \to 1+$ for all $t \in \mathbf{R}$. If we now take $f(\nu) = \lambda(\nu)$ and write $f(\nu)$ as $f(n_i)$ for $n_i = n(\nu)$ then we find that

$$\sum_{k=1}^{\infty} \frac{1}{p_k} (1 - \operatorname{Re} f(p_k) p_k^{-it}) = \sum_{k=1}^{\infty} \frac{1}{p_k} (1 + \operatorname{Re} p_k^{-it}) = \infty$$

holds for all $t \in \mathbf{R}$ because $f(p_k) = -1$. By Theorem 2.1, we have

$$\sum_{\substack{\nu \\ n(\nu) \le x}} \lambda(\nu) = o(x). \tag{2.4}$$

From this fact, we can deduce the following:

THEOREM 2.3. *Suppose that* (1.1) *and one of* (1.2) *and* (1.3) *hold. Then we have*

$$M(x) = \sum_{\substack{\nu \\ n(\nu) \le x}} \mu(\nu) = o(x),$$

where $\mu(\nu)$, the analogue of the classical Möbius function, is defined on $\mathscr{N}^*$ by setting

$$\mu(\nu) = \begin{cases} (-1)^k, & \text{if } \nu = (\nu_1, \ldots, \nu_m, \ldots), \\ & \quad 0 \le \nu_1, \ldots, \nu_m, \ldots \le 1, \nu_1 + \cdots + \nu_m + \cdots = k, \\ 0, & \text{otherwise}. \end{cases}$$

This theorem follows from (2.4) and the following:

LEMMA 2.4. *Assume $N(x) = O(x)$. Then $\sum_{\nu, n(\nu) \le x} \lambda(\nu) = o(x)$ if and only if $M(x) = o(x)$.*

Remark. From the proof below, we can see that the hypothesis $N(x) = O(x)$ can be relaxed.

Proof. We have

$$\begin{aligned} \frac{1}{\zeta(s)} &= \sum_{i=0}^{\infty} \frac{\mu(n_i)}{n_i^s} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) \\ &= \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i^s}\right)^{-1} \left(1 - \frac{1}{p_i^{2s}}\right) \\ &= \sum_{i=0}^{\infty} \frac{\lambda(n_i)}{n_i^s} \sum_{i=0}^{\infty} \frac{\mu_2(n_i)}{n_i^s}, \end{aligned}$$

where $\lambda(n_i)$ and $\mu_2(n_i)$ denote $\lambda(\nu)$ and $\mu_2(\nu)$ for $n_i = n(\nu)$ respectively and

$$\mu_2(\nu) = \begin{cases} \mu(\nu'), & \text{if } \nu = (\nu_1, \ldots, \nu_m, \ldots), \nu' = (\nu_1', \ldots, \nu_m', \ldots) \in \mathscr{N}^* \\ & \quad \text{with } \nu_m = 2\nu_m', \forall m \in \mathbf{N}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence

$$M(x) = \sum_{\substack{\nu \\ n(\nu) \le x}} \mu(\nu) = \sum_{\substack{\nu \\ n(\nu) \le x}} \left(\sum_{\substack{\nu' \\ n(\nu') \le x/n(\nu)}} \lambda(\nu') \right) \mu_2(\nu).$$

Assume $\sum_{\nu,\, n(\nu) \le x} \lambda(\nu) = o(x)$. Then we have

$$M(x) = \sum_{\substack{\nu \\ n(\nu) \le x}} o\left(\frac{x}{n(\nu)} \right) \mu_2(\nu) = o\left(x \sum_{\substack{\nu \\ n(\nu) \le x}} \frac{|\mu_2(\nu)|}{n(\nu)} \right) = o(x)$$

since

$$\sum_{\substack{\nu \\ n(\nu) \le x}} \frac{|\mu_2(\nu)|}{n(\nu)} \le \sum_{\substack{\nu \\ n(\nu) \le \sqrt{x}}} \frac{1}{(n(\nu))^2} < \infty.$$

Also, we have

$$\sum_{i=0}^{\infty} \frac{\lambda(n_i)}{n_i^s} = \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i^s} \right)^{-1} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^{2s}} \right)^{-1} \left(1 - \frac{1}{p_i^s} \right)$$
$$= \sum_{i=0}^{\infty} \frac{\mu(n_i)}{n_i^s} \sum_{i=0}^{\infty} \frac{1_2(n_i)}{n_i^s}$$

where $1_2(n_i)$ denotes $1_2(\nu)$ for $n_i = n(\nu)$ and

$$1_2(\nu) = \begin{cases} 1, & \text{if } \nu = (\nu_1, \dots, \nu_m, \dots) \in \mathcal{N}^* \text{ with } 2 | \nu_m, \forall m \in \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases}$$

In the same way, we can show that $M(x) = o(x)$ implies $\sum_{\nu,\, n(\nu) \le x} \lambda(\nu) = o(x)$. ∎

COROLLARY 2.5. *If*

$$N(x) = Ax + O(x \log^{-\gamma} x), \quad x > 1 \tag{2.5}$$

holds with constants $A > 0$ and $\gamma > 1$ then $M(x) = o(x)$.

We know that, in classical prime number theory, the prime number theorem is "equivalent" to the assertion that $M(x) = o(x)$ in the sense that each is deducible from the other by an "elementary" argument. It is interesting that this equivalence does not hold in some *g*-prime systems. Actually, we know, from Beurling's theorem [2], that the hypothesis (2.5) with $\gamma > 3/2$ implies the prime number theorem and, from Diamond's example [3], that there exists a g-prime number system which satisfies (2.5) with $\gamma = 3/2$ and for which the prime number theorem does not hold. Corollary 2.5 shows, however, that $M(x) = o(x)$ still holds for Diamond's example. Therefore, the conjecture in [7], which says that the prime number theorem and the estimate $M(x) = o(x)$ are equivalent for all g-prime number systems satisfying (2.5) with $\gamma > 1$, is not true.

References

1. P.T. Bateman and H.G. Diamond, *Asymptotic distribution of Beurling's generalized prime numbers*, Studies in Number Theory, vol. 6, Math. Assoc. Amer., Prentice-Hall, Englewood Cliffs, N.J., 1969, pp. 152–210.
2. A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés, I*, Acta Math., vol. 68 (1937), pp. 225–291.
3. H.G. Diamond, *A set of generalized numbers showing Beurling's theorem to be sharp*, Illinois J. Math., vol. 14 (1970), pp. 29–34.
4. P.D.T.A. Elliott, *Probabilistic number theory*, Springer-Verlag, New York, 1979.
5. R.R. Goldberg, *Fourier transform*, Cambridge Univ. Press, London, 1961.
6. G. Halász, *Über die Mittewerte multiplikativer zahlentheoretischer Funktionen*, Acta Math. Acad. Sci. Hung., vol. 19 (1968), pp. 365–403.
7. R.S. Hall, *Theorems about Beurling's generalized primes and the associated zeta function*, Ph.D. Thesis, Univ. of Illinois, Urbana, Illinois, 1967.
8. H.L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics, vol. 227, Springer-Verlag, New York, 1971.
9. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen, II*, Acta Math. Acad. Sci. Hung., vol. 18 (1967), pp. 411–467.

University of Illinois
Urbana, Illinois