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WEAK CONCENTRATION POINTS FOR MÖBIUS GROUPS

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0. Introduction

Although the limit sets of discrete groups of Möbius transformations have been studied for many decades, there are some very natural dynamical properties of limit points which have not been previously examined. In this paper, we use geometric and topological methods to study one of these, which we call the property of being a weak concentration point. Stronger concentration conditions are treated in [A-H-M] and [M1]. In [A-H-M], the limit points satisfying a certain strong concentration condition are characterized in several ways, each analogous to a classical characterization of conical limit points. This condition is strictly stronger than being a conical limit point, although for a large class of groups the points satisfying it have full measure in the conical limit set. In [M1], an intermediate concentration condition is found which is exactly equivalent, for finitely generated Fuchsian groups, to the property of being a conical limit point.

Weak concentration is the simplest and perhaps most natural of the concentration conditions. To fix notation, let Γ be a nonelementary discrete group of hyperbolic isometries acting on the Poincaré disc D^m, $m \geq 2$, and let $p \in \partial D^m$ be a limit point of Γ. By a neighborhood of p, we will always mean an open neighborhood of p in ∂D^m.

DEFINITION. An open set U in ∂D^m *can be concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ such that $p \in \gamma(U)$ and $\gamma(U) \subseteq V$.

Equivalently, U can be concentrated at p if and only if the set of translates of U contains a local basis for the topology of ∂D^m at p. As we show in Theorem 1.1, every limit point of Γ has a disconnected neighborhood which can be concentrated. Therefore we restrict attention to concentration of connected neighborhoods.

DEFINITION. The limit point p is called a *weak concentration point* for Γ if there exists a connected open set U that can be concentrated at p.

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In §2, we prove that in dimension $m = 2$, a limit point fails to be a weak concentration point if and only if it is an endpoint of a component of the domain of discontinuity (in ∂D^2) which has finite stabilizer. In higher dimensions, we do not achieve a complete characterization, but we do obtain some sufficient conditions. An easy observation, Theorem 3.1, shows that every conical limit point is a weak concentration point, and a more elaborate argument given as Theorem 3.2 shows that if the limit set approaches a point from two different tangential directions in ∂D^m (see §3 for precise definitions) then it is a weak concentration point. In particular, every parabolic fixed point in a weak concentration point. Since all limit points of a geometrically finite group are either conical limit points or parabolic fixed points, this shows that every limit point of a geometrically finite group must be a weak concentration point. For groups of the first kind, every limit point is approached from all directions by the limit set, and hence is a weak concentration point.

In §4 we prove that any subset of ∂D^m approaches all but countably of its points from some pair of antipodal directions. Consequently, any group has at most countably many limit points that are not weak concentration points.

An alternative definition of weak concentration point would be that there exists a round ball whose set of translates contains a neighborhood basis for p. Conceivably, this may be more restrictive, but the arguments of Theorems 3.1 and 3.2 provide round neighborhoods that can be concentrated. Therefore the statements of Corollaries 3.3, 3.4, 3.5, and Theorem 3.6 hold as stated even with this alternative definition, and changing the definition could affect at most a countable set of limit points.

We assume familiarity with the basic theory of Möbius groups as presented in [B] or [A2]. A Möbius group is called *elementary* when its limit set contains at most two points. We generally assume that the groups called Γ in this paper are nonelementary, although some of the statements happen to be true for elementary groups as well. Of particular importance is the following result, proven on pp. 97–98 of [B] and on p. 74 of [A2].

Double Density Theorem. *Let Γ be a nonelementary group of Möbius transformations of ∂D^m, and let U_1 and U_2 be open sets both meeting the limit of Γ. Then there exists a loxodromic element of Γ with a fixed point in U_1 and a fixed point in U_2.*

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1. Concentration of disconnected neighborhoods

Theorem 1.1. *Let Γ be a nonelementary discrete group of hyperbolic isometries acting on the Poincaré disc D^m, $m \geq 2$, and let $p \in \partial D^m$ be a limit point of Γ. Then there exists an open set which can be concentrated at p.*

Proof. Let U_0 be a neighborhood of p in ∂D^m whose closure does not contain the entire limit set. Choose a decreasing sequence of neighborhoods $U_0 \supseteq W_1 \supseteq W_2 \supseteq W_3 \supseteq \ldots$ of p whose diameters limit to 0. Let q_k, $1 \leq k < \infty$ be a sequence of distinct limit points which form a discrete subset of ∂D^m, and which are disjoint from the closure of U_0. Choose pairwise disjoint neighborhoods V_k of the q_k which are disjoint from U_0.

By the Double Density Theorem, there exists a loxodromic element of Γ whose repelling fixed point is in $V_1 - \{q_1\}$ and whose attracting fixed point is in $W_1 - \{p\}$. Since the repelling fixed point is not in the closure of the set of all q_k, we may choose a sufficiently large power γ_1 of this element so that $\gamma_1(U_0) \subseteq W_1$ and

(1) for all $k \geq 1$, $\gamma_1(q_k) \in W_1$, and
(2) $\gamma_1^{-1}(p) \in V_1$.

Now choose a neighborhood U_1 so that

(3) $\gamma_1^{-1}(p) \in U_1 \subseteq V_1$, and
(4) $\gamma_1(U_1) \subseteq W_1$.

Inductively, suppose that for all $j \leq n$, there have been selected elements $\gamma_j \in \Gamma$ so that

(1′) for $1 \leq j \leq n$ and all $k \geq 1$, $\gamma_j(q_k) \in W_j$, and
(2′) for $1 \leq j \leq n$, $\gamma_j^{-1}(p) \in V_j$,

and neighborhoods U_j so that

(3′) for all $1 \leq j \leq n$, $\gamma_j^{-1}(p) \in U_j \subseteq V_j$, and
(4′) for all $1 \leq j \leq n$ and $0 \leq k \leq n$, $\gamma_j(U_k) \subseteq W_j$.

By the Double Density Theorem, there exists a loxodromic element of Γ whose repelling fixed point is in

$$V_{n+1} \cap \gamma_1^{-1}(W_1) \cap \gamma_2^{-1}(W_2) \cap \cdots \cap \gamma_n^{-1}(W_n) - \{q_{n+1}\}$$

and whose attracting fixed point is in $W_{n+1} - \{p\}$. Let γ_{n+1} be a sufficiently large power of this element so that

(a) for all $k \geq 1$, $\gamma_{n+1}(q_k) \in W_{n+1}$, and
(b) $\gamma_{n+1}^{-1}(p) \in V_{n+1} \cap \gamma_1^{-1}(W_1) \cap \cdots \cap \gamma_n^{-1}(W_n)$.

Additionally, we may require that

(c) for $0 \leq k \leq n$, $\gamma_{n+1}(U_k) \subseteq W_{n+1}$.

Now choose a neighborhood U_{n+1} so that

(d) $\gamma_{n+1}^{-1}(p) \in U_{n+1} \subseteq V_{n+1} \cap \gamma_1^{-1}(W_1) \cap \gamma_2^{-1}(W_2) \cap \cdots \cap \gamma_n^{-1}(W_n)$, and
(e) $\gamma_{n+1}(U_{n+1}) \subseteq W_{n+1}$.

This completes the inductive construction. Let U be the union of all U_k for $k \geq 0$. Suppose V is any neighborhood of p. Then there is some W_m contained in V. By (4′), $\gamma_m(U) \subseteq W_m$, and by (3′), $p \in \gamma_m(U)$. This completes the proof.

2. Weak concentration points for Fuchsian groups

For Fuchsian groups, we can give a complete characterization of weak concentration points.

THEOREM 2.1. *Let Γ be a nonelementary discrete group of Möbius transformations of D^2, and let p be a limit point of Γ. Then p is a weak concentration point if and only if p is not an endpoint of an interval of discontinuity of Γ which has finite stabilizer.*

Proof. Suppose that U is a connected neighborhood of p that can be concentrated at p. Obviously U is not the entire circle ∂D^2, so U is an interval (z_1, z_2) which can be concentrated into arbitrarily small neighborhoods of p. It follows that on each of the two sides of p, either the orbit of z_1 or of z_2 must limit to p. Therefore p cannot be the endpoint of an interval of discontinuity with finite stabilizer.

Suppose now that p is not an endpoint of an interval of discontinuity with trivial stabilizer. If p is an endpoint of an interval of discontinuity, then p is an endpoint of the axis of a hyperbolic isometry in Γ, and it is clear that p is a weak concentration point. So we may assume that for every interval neighborhood J of p, both components of $J—\{p\}$ contain limit points; i.e., the limit set approaches p "from both directions."

We will now work in the upper half-plane model, with p equal to the point at infinity. We will find a positive real number r and loxodromic elements $\gamma_j \in \Gamma$ satisfying

(a) $\gamma_j(\infty)$ are negative real numbers with $\lim \gamma_j(\infty) = \infty$, and
(b) $\gamma_j(r)$ are positive real numbers with $\lim \gamma_j(r) = \infty$.

Then, the interval (r, ∞) can be concentrated at ∞.

Since the limit set approaches ∞ from both directions, the Double Density Theorem implies that there is a hyperbolic isometry $\gamma_1 \in \Gamma$ whose attracting fixed point is a real number less than -1 and whose repelling fixed point is greater than 1. Choose a limit point r_1 with $1 < r_1 < \gamma_1^{-1}(\infty)$; since ∞ is a limit point from both directions, we may choose r_1 close enough to $\gamma_1^{-1}(\infty)$ so that $\gamma_1(r_1) > 1$. Let $U_0 = \partial D^2$ and choose a neighborhood U_1 of r_1, with compact closure in $\mathbf{R}$, such that for any $s \in U_1$, $\gamma_1(s)$ lies within distance 1 of $\gamma_1(r_1)$.

Inductively, suppose that limit points r_k, hyperbolic elements $\gamma_k \in \Gamma, 1 \leq k \leq n$, and neighborhoods U_k of r_k have been constructed with the following properties:

(1) For $1 \leq k \leq n, \gamma_k(r_k)$ is greater than k and $\gamma_k(\infty)$ is less than $-k$.
(2) $\overline{U_k} \subseteq U_{k-1}$, and for every $s \in U_k$, $\gamma_k(s)$ lies within a distance 1 of $\gamma_k(r_k)$.

By the Double Density Theorem, there is a hyperbolic isometry in Γ whose attracting fixed point is a real number less than $-n - 1$ and whose repelling fixed point is contained in U_n. Choose a large positive power γ_{n+1} of this element so that $\gamma_{n+1}^{-1}(\infty)$ lies in U_n. Since the limit set approaches ∞ from the positive numbers, there is a limit point $r_{n+1} \in U_n$ with $r_{n+1} < \gamma_{n+1}^{-1}(\infty)$ and close enough to $\gamma_{n+1}^{-1}(\infty)$ so that $\gamma_{n+1}(r_{n+1}) > n + 1$. Finally, we choose a neighborhood U_{n+1} of r_{n+1} with $\overline{U_{n+1}} \subseteq U_n$ and so that for any $s \in U_{n+1}$, $\gamma_{n+1}(s)$ lies within distance 1 of $\gamma_{n+1}(r_{n+1})$. This completes the inductive step of the construction.

Since all $\overline{U_n} \subseteq U_{n-1}$ the intersection of the U_n is nonempty. One checks easily that the sequence γ_n together with any r in the intersection of the U_n must satisfy properties (1) and (2). This completes the proof of Theorem 2.1.

When $\Gamma$is a finitely generated nonelemenatry Fuchsian group, every interval of discontinuity in ∂D^2 has infinite stabilizer. For consider a convex finite-sided fundamental polygon P for Γ (which exists by Theorem 10.1.2 in [B]). Each point in an interval of discontinuity J must lie in a free side of a translate of P (defined on p. 223 of [B]). The endpoints of this free side are the endpoints of sides where two translates of P meet, hence lie in J. Therefore J is a union of infinitely many translates of free sides of P. Since P has only finitely many free sides, it follows that the stabilizer of J must be infinite. (More elegantly, regard Γ as a Kleinian group and use the Ahlfors Finiteness Theorem to observe that the quotient of its region of discontinuity is of finite type. If J has finite stabilizer, then its image in this surface is properly imbedded and must limit to cusps. This implies that the endpoints of J are parabolic fixed points, an impossibility since then J would contain infinitely many translates of these endpoints.) Therefore Theorem 2.1 implies:

COROLLARY 2.2. *If Γ is a nonelementary finitely generated Fuchsian group, then every limit point of Γ is a weak concentration point.*

For groups of the first kind, there are no intervals of discontinuity, so Theorem 2.1 becomes:

COROLLARY 2.3. *If Γ is a Fuchsian group of the first kind, then every limit point of Γ is a weak concentration point.*

3. Weak concentration points for high-dimensional groups

In higher dimensions, the characterization of weak concentration points appears to be more subtle. As a first observation, we check that conical approach is always a sufficient condition.

THEOREM 3.1. *Let Γ be a discrete group of isometries of the Poincaré disc D^m, $m \geq 2$, and let p be a conical limit point for Γ. Then there is a round open ball which can be concentrated at p.*

Proof. Let λ be the half-geodesic in D^m that runs from 0 to p. Since p is a conical limit point, there is some closed ball B centered at 0 for which infinitely many translates of B meet λ. Therefore there exist a sequence of elements $\gamma_j \in \Gamma$ and a sequence of points p_j on λ so the $\lim p_j = p$ and $\gamma_j(p_j) \in$ B. Let x_j denote the unit tangent vector to λ at p_j, pointing in the direction of p. The space $T(B)$ of unit tangent vectors to points of B is compact, so by passing to a subsequence we may assume that the $T(\gamma_j)(x_j)$ converge in $T(B)$. Let H denote the $(m-1)$-dimensional hyperbolic hyperplane normal to this limit vector. For sufficently large j, $\gamma_j^{-1}(H)$ meets λ near p_j and is almost perpendicular to λ. Therefore one of the two components of $\partial D^m - \partial H$ can be concentrated at p. This completes the proof of Theorem 3.1.

Next, by generalizing the proof of Theorem 2.1, we will obtain a useful sufficient condition for p to be a weak concentration point. A definition is required. Let X be a subset of S^{m-1}, and let $p \in S^{m-1}$. For convenience, choose $q \neq p$ and a conformal equivalence of $S^{m-1} - \{q\}$ with $\mathbf{R}^{m-1}$, to regard p as lying in $\mathbf{R}^{m-1}$. Denote by $A(v, w)$ the angle between two vectors in $\mathbf{R}^{m-1}$. We need not assume the vectors are based at the same point, since the tangent spaces to any two points in $\mathbf{R}^{m-1}$ can be canonically identified. Let u be a unit vector in $\mathbf{R}^{m-1}$. We say that X *approaches p from the direction u*, and that u is a *direction of approach for X at p*, if there is a sequence of points $x_j \in X$ that converges to p and satisfies $\lim A(u, x_j - p) = 0$.

THEOREM 3.2. *Let Γ be a discrete group of isometries of the Poincaré disc D^m, $m \geq 2$, and let p be a limit point of Γ. If the limit set of Γ approaches p*

from at least two directions, then there is a round open ball that can be concentrated at p.

Proof. When $m = 2$, this follows from Theorem 2.1, so assume that $m \geq 3$. We will work in the upper-half space model $\mathbf{R}^m_{\geq 0} \cup \{\infty\}$, where $(\mathbf{R}^{m-1} \times \{0\}) \cup \{\infty\}$ corresponds to ∂D^m and with p as the point at infinity. Denote $\mathbf{R}^{m-1} \times \{0\}$ by $\mathbf{R}^{m-1}$. Let α and β be two straight lines in $\mathbf{R}^{m-1}$ emanating from 0 such that the limit set of Γ approaches ∞ from the directions determined by terminal segments of α and β. Thus, for any ε, $M > 0$ there exists a limit point x of Γ in $\mathbf{R}^{m-1}$ with $|x| > M$ and whose distance from α is less than $\varepsilon|x|$, and similarly for β. In the ensuing argument, α will play the role of the negative real axis in the proof of Theorem 2.1, and β the role of the positive real axis.

Let the (positive) angle between α and β be written as 6θ. We will inductively construct a sequence of elements γ_j of Γ, $j \geq 1$, limit points $r_j, j \geq 0$, and neighborhoods U_j of the r_j satisfying the following properties for all $j \geq 1$.

(1) $\gamma_j^{-1}(\infty)$ lies in U_{j-1}.
(2) $\gamma_j(\infty)$ has distance greater than j from the origin and distance less than $\sin(\theta)|\gamma_j(\infty)|$ from α.
(3) $\gamma_j(r_j)$ has distance greater than $|\gamma_j(\infty)| + 2|\gamma_j(\infty)|/\sin(\theta)$ from the origin and distance less than $\sin(\theta)|\gamma_j(r_j)|$from β.
(4) $\overline{U_j} \subseteq U_{j-1}$.
(5) The (Euclidean) diameter of U_j is less than $1/j$, and each point x of $\gamma_j(U_j)$ has distance greater than $|\gamma_j(\infty)| + 2|\gamma_j(\infty)|/\sin(\theta)$ from the origin and distance less than $\sin(\theta)|\gamma_j(x)|$from β.

To start, choose a limit point r_0 other than ∞, and a small neighborhood U_0 of r_0. Since the limit set approaches ∞ from the direction of β, we can use double density to obtain a loxodromic transformation whose repelling fixed point lies in U_0 and whose attracting fixed point b_1 has distance at least 1 from the origin and distance less than $\sin(\theta)|b_1|$from α. Choose a large power of γ_1 of this element so that conditions (1) and (2) are satisfied with $j = 1$. Since $\gamma_j^{-1}(\infty) \in U_0$ and the limit set of Γ approaches ∞ from the direction β, we may choose a limit point $r_1 \in U_0$ so that (3) is satisfied for $j = 1$, and choose a small neighborhood U_1 of r_1 so that (4) and (5) hold. Inductively, suppose choices have been made satisfying (1) – (5) for $1 \leq j < n$. Again apply double density to find a loxodromic transformation whose repelling fixed point lies in U_{n-1} and whose attracting fixed point $|b_n|$has distance at least n from the origin and distance less than $\sin(\theta)|b_n|$ from α. Choose a large power γ_n of this element so that (1) and (2) hold, and use the approach of ∞ from the direction of β to get r_n in U_{n-1} satisfying (3). For a suitably small neighborhood U_n of r_n, (4) and (5) will hold, completing the inductive construction.

Now let a_0 be the unit vector at the origin which lies in the 2-plane spanned by α and β, is perpedicular to the bisector of the angle between α and β, and chosen so that it makes an angle of $\pi/2 - 3\theta$ with β and $\pi/2 + 3\theta$ with α. Fix an orthonormal frame for the tangent space at $\gamma_1(\infty)$. Passing to a subsequence, we may assume that the images under $\gamma_j\gamma_1^{-1}$ of this frame converge (after parallel translation to the origin and rescaling so that the lengths of the vectors are 1). Let a_1 be the vector at $\gamma_1(\infty)$ whose translates under $\gamma_j\gamma_1^{-1}$ converge (after parallel translation to the origin and rescaling) to a_0.

Let r be the unique point in the intersection of all U_j, and let S be the $(m-2)$-sphere in $\mathbf{R}^{m-1} \cup \{\infty\}$ (possibly the union of a hyperplane in $\mathbf{R}^{m-1}$ with $\{\infty\}$) which contains the points $\gamma_1(r)$ and $\gamma_1(\infty)$ and has normal vector a_1 at $\gamma_1(\infty)$. Consider the images of S under the $\gamma_j\gamma_1^{-1}$. They contain the point $\gamma_j(\infty)$ and have normal vector there limiting to a_0, and they also contain the point $\gamma_j(r)$. By condition (5), $\gamma_j(r)$ has distance greater than $|\gamma_j(\infty)| + 2|\gamma_j(\infty)|/\sin(\theta)$ from the origin and distance less than $\sin(\theta)|\gamma_j(r)|$ from β.

Claim. *If j is large enough so that $\gamma_j\gamma_1^{-1}(a_1)$ makes an angle less than θ with a_0, then $\gamma_j\gamma_1^{-1}(S)$ bounds a ball in $\mathbf{R}^{m-1}$ which has inward normal $\gamma_j\gamma_1^{-1}(a_1)$ at $\gamma_j(\infty)$ and contains all points within distance $\sin(\theta)|\gamma_j(\infty)|/2$ from the origin.*

The complement of a closed ball given by the claim can be concentrated at ∞, by the elements $\gamma_k\gamma_1^{-1} \circ (\gamma_j\gamma_1^{-1})^{-1}$ for $k > j$. Thus to prove Theorem 3.2 it remains only to verify the claim. Put $y = \gamma_j(\infty)$, $a = (\gamma_j\gamma_1^{-1})_*(a_1)$, and $x = \gamma_j(r)$. We first estimate the inner product $\langle x - y, a\rangle$. Regarded as a vector based at the origin, x meets β at an angle of less than θ, so it makes an angle between $\pi/2 - 4\theta$ and $\pi/2 - 2\theta$ with a_0 and an angle between $\pi/2 - 5\theta$ and $\pi/2 - \theta$ with a. Similarly y makes an angle between $\pi/2 + \theta$ and $\pi/2 + 5\theta$ with a. Thus

$$\langle x - y, a\rangle = \langle x, a\rangle - \langle y, a\rangle \geq |x|\sin(\theta) + |y|\sin(\theta).$$

Since this is positive, a is an inward normal to the sphere containing x and y which is perpendicular to a at y. Writing $c = y + Ra$ and equating $|y - c|$ and $|x - c|$ gives its radius to be

$$R = \frac{|x - y|^2}{2\langle x - y, a\rangle}.$$

Using the fact that $|x| > |y| + 2|y|/\sin(\theta)$, we obtain the following estimate.

$$\begin{aligned} -2R\langle y, a\rangle &\geq \frac{|x-y|^2|y|\sin(\theta)}{\langle x-y, a\rangle} \\ &\geq \frac{|x-y|^2|y|\sin(\theta)}{|x-y|} \\ &\geq 2|y|^2. \end{aligned}$$

Since $|c| = |y + Ra|$, it follows that

$$R^2 - |c|^2 = -2R\langle y, a\rangle - |y|^2 \geq |y|^2$$

and therefore $|y + Ra| < R$. Now we calculate

$$\begin{aligned} R - |c| &= \frac{R^2 - |y+Ra|^2}{R + |y+Ra|} \\ &\geq \frac{-2R\langle y, a\rangle - |y|^2}{2R} \\ &= -\langle y, a\rangle - \frac{|y|^2}{2R} \\ &\geq -\langle y, a\rangle/2 \\ &\geq |y|\sin(\theta)/2. \end{aligned}$$

verifying the claim.

COROLLARY 3.3. *If Γ is of the first kind, then every limit point of Γ is a weak concentration point.*

COROLLARY 3.4. *If Γ is nonelementary then every parabolic fixed point of Γ is a weak concentration point.*

By [B-M] and [A1], every limit point of a geometrically finite group is either a conical limit point or a parabolic fixed point. Therefore Theorem 3.1 and Corollary 3.4 give the following.

COROLLARY 3.5. *If Γ is nonelementary and geometrically finite, then every limit point of γ is a weak concentration point.*

The condition that the limit set approaches p in only one direction is very restrictive. In §4 below, we will see that any subset of S^{m-1} approaches all but countably many of its points from some pair of antipodal directions (Corallary 4.3). Thus, Theorem 3.2 implies the following result.

THEOREM 3.6. *All but countably many limit points of* Γ *are weak concentration points.*

Remark 3.7. For p to be a weak concentration point, it is not necessary for the limit set to approach p from two directions. For example, take a finitely generated group of the second kind in dimension $m = 2$. By Corollary 2.2, every limit point is a weak concentration point, but the limit set approaches some of its points from only one direction.

Remark 3.8. In dimensions $m \geq 3$, there can be components of the region of discontinuity that are round open balls and have trivial stabilizer (see Example 18 in [K-A-G]). Theorem 3.1 shows that, in contrast to the 2-dimensional case, every point in the boundary of such a component is a weak concentration point.

4. Directional approach in subsets of $\mathbf{R}^m$

For a nonzero $x \in \mathbf{R}^m$ let $U(x)$ denote a unit vector in the direction of x, and for two nonzero vectors v_1 and v_2 let $A(v_1, v_2)$ denote the angle between v_1 and v_2. Let X be a subset of $\mathbf{R}^m$, let $p \in \mathbf{R}^m$, and let u be a unit vector. Recall from §3 that *X approaches p from the direction u*, and that u is a *direction of approach for X at p*, if there is a sequence of points $x_j \in X$ that converges to p and satisfies $\lim A(u, x_j - p) = 0$. Note that p is a limit point of X if and only if X approaches p from at least one direction. Let $D(X, p)$ denote the set of directions of approach for X at p. It is a compact subset of S^{m-1}. For θ a real number, define

$$X_\theta = \{ p \in X \mid \max\{ A(v, w) \mid v, w \in D(X, p)\} \leq \theta \}.$$

If $\theta < 0$ then X_θ is the set of points in X that are not limit points, while if $\theta \geq \pi$ then $X_\theta = X$.

If $X \subseteq S^m$, where S^m is the standard round m-sphere, then X_θ is still defined, by regarding S^m as a subset of $\mathbf{R}^{m+1}$. Alternatively, given $p \in S^m$ one can choose any $q \in S^m$ with $q \neq p$ and a conformal isomorphism $S^m - \{q\} \cong \mathbf{R}^m$ to calculate the spread of the directions of approach of X at p.

PROPOSITION 4.1. *Let X be a subset of S^m. If $\theta < \pi$, then X_θ is (at most) countable.*

Proof. If $X = S^m$, then X_θ is empty for $\theta < \pi$. Otherwise, by a conformal isomorphism, we may assume that $X \subseteq \mathbf{R}^m$.

For $p \in X$ and positive numbers θ_1 and r, define

$$F(\theta_1, r) = \{p \in X \mid \sup\{A(a-p, b-p) \mid a, b \in X - \{p\}, 0 < |a-p|, |b-p| \leq r\} \leq \theta_1\}.$$

LEMMA 4.2. *If $\theta < \theta_1$, then $X_\theta \subseteq \cup_{n=1}^\infty F(\theta_1, 1/n)$.*

Proof. For each n, choose $a_n, b_n \in x$ so that $0 < |a_n - p|, |b_n - p| \leq 1/n$ and $A\,(a_n - p, b_n - p) \geq \theta_1$. Convergent subsequences of $\{U(a_n - p)\}$ and $\{U(b_n - p)\}$ provide directions of approach for X at p which meet at an angle at least θ_1.

By the lemma, for any $\theta_1 > \theta$, $X_\theta \subseteq \cup_{n=1}^\infty F(\theta_1, 1/n)$. To complete the proof of Theorem 4.1 we will show that for any $\theta_1 < \pi$ and any r, the points of $F(\theta_1, r)$ are isolated in X, and hence form a countable set.

Given $\theta_1 < \pi$, fix ε so that $\pi - 2\varepsilon > \theta_1$ and $p \in F(\theta_1, r)$. Cover the ball B of radius r centered at p by finitely many cones C_i, $1 \leq i \leq n$, with vertex at p and base on the boundary of B, so small that if $a, b \in C_i$ then $A(a-p, b-p) < \varepsilon$. Delete the cones that meet X only in p. In each remaining C_i, choose a point $a_i \in X - \{p\}$. If $x \in C_i$ and

$$0 < |x - p| \leq \tfrac{1}{2}|a_i - p|,$$

then

$$A(p - x, a_i - x) \geq \pi - 2\varepsilon > \theta_1,$$

and hence if x is any point of $X - \{p\}$ closer to p than the smallest $\frac{1}{2}|a_i - p|$, $x \notin F(\theta_1, r)$. This completes the proof.

By compactness of $D(X, p)$, we deduce:

COROLLARY 4.3. *Let X be a subset of S^m. Then X approaches all but at most countably many of its points from some pair of antipodal directions.*

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