ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 4, Winter 1999

THE EQUIVARIANT BRAUER GROUP AND TWISTED TRANSFORMATION GROUP C^*-ALGEBRAS

JUDITH A. PACKER

ABSTRACT. Twisted transformation group C^*-algebras associated to locally compact dynamical systems $(X = Y/N, G)$ are studied, where G is abelian, N is a closed subgroup of G, and Y is a locally trivial principal G-bundle over $Z = Y/G$. An explicit homomorphism between $H^2(G, C(X, \mathbb{T}))$ and the equivariant Brauer group of Crocker, Kumjian, Raeburn and Williams, $Br_N(Z)$, is constructed, and this homomorphism is used to give conditions under which a twisted transformation group C^*-algebra $C_0(X) \times_{\tau,\omega} G$ will be strongly Morita equivalent to another twisted transformation group C^*-algebra $C_0(Z) \times_{Id,\omega} N$. These results are applied to the study of twisted group C^*-algebras $C^*(\Gamma, \mu)$ where Γ is a finitely generated torsion free two-step nilpotent group.

Introduction

Fifteen years ago, M. Rieffel published the extremely useful observation that if the locally compact groups G and N have commuting free and proper actions on a locally compact Hausdorff space Y, then the transformation group C^*-algebras $C_0(Y/N) \times G$ and $C_0(Y/G) \times N$ are strongly Morita equivalent to one another [Ri]. This result, attributed by Rieffel to P. Green, was a motivating factor behind I. Raeburn's paper [Ra], as well as for A. Kumjian's, Raeburn's and D. Williams' recent proof that for second countable Y, G and N as above, the equivariant Brauer groups $\mathrm{Br}_G(Y/N)$, $\mathrm{Br}_N(Y/G)$ and $\mathrm{Br}_{G\times N}(Y)$ are isomorphic to each other. In this note, we investigate how the isomorphism of the equivariant Brauer groups above can be used to obtain information about twisted transformation group C^*-algebras corresponding to a dynamical system $(Y/N, G)$ in the case where G is abelian and N is a closed subgroup of G, so that N acts trivially on Y/G. In this case $\mathrm{Br}_N(Y/G)$ is known to be isomorphic to the direct sum $C(Y/G, H^2(N, \mathbb{T})) \oplus \check{H}^1(Y/G, \hat{\mathcal{N}}) \oplus \check{H}^2(Y/G, \mathcal{S})$, at least for N elementary abelian (cf. [PRW], [P2]) and our aim in this paper is to use the above structure to describe the strong Morita equivalence between twisted transformation group C^*-algebras for $(Y/N, G)$ and crossed product C^*-algebras of the form $B \times_\beta N$, where B is a stable, separable continuous trace C^*-algebra with spectrum Y/G, and the induced action $\hat{\beta}$ of N on Y/G is trivial. Along with giving precise formulas for the element in $\mathrm{Br}_N(Y/G)$ corresponding to a twisted transformation group $(Y/N, G, \omega)$ where $[\omega] \in H^2(G, C(Y/N, \mathbb{T}))$, we will determine conditions under which a twisted transformation group C^*-algebra $C_0(Y/N) \times_{\tau,\omega} G$ will be strongly

Received March 30, 1998.
1991 Mathematics Subject Classification. Primary 46L05, 46L35; Secondary 22D25, 55R10.
Research supported by the NUS Academic Research Fund Committee.

Morita equivalent to another such C^*-algebra $C_0(Y/G) \times_{Id,\omega_N} N$. This question was first raised in [P1, Section 3] and a special case of this situation has already been considered in [LP2] in order to study twisted group C^*-algebras associated to discrete Heisenberg groups. This motivates Section 3 of our paper, which gives an analysis of more general twisted group C^*-algebras $C^*(\Gamma, \mu)$ where Γ is a finitely generated, torsion free two-step nilpotent discrete group, and $[\mu] \in H^2(\Gamma, \mathbb{T})$. Under appropriate conditions on $[\mu]$ these C^*-algebras will be isomorphic to twisted transformation group C^*-algebras $C(Y/N) \times_{\tau,\omega} G$, of the form described above and the invariants of the associated C^*-dynamical system $[(B, \beta, N)] \in \mathrm{Br}_N(Y/G)$ can in many cases be explicitly computed. These results can be used to state conditions under which $C^*(\Gamma, \mu)$ will be strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$, where Γ_0 is a subgroup of Γ of finite index and $\mu_0 = \mu$ restricted to $\Gamma_0 \times \Gamma_0$. This result can be extremely useful in K-theory calculations.

I would like to thank S. Echterhoff and I. Raeburn for useful conversations on the topic of this paper and I. Raeburn for making available to me at a very early stage a preprint of [RW2].

1. Preliminaries

1.1. *The equivariant Brauer group.* Let (Y, τ, G) be a locally compact second countable topological dynamical system. The equivariant Brauer group $\mathrm{Br}_G(Y)$ is defined to be the set of all equivalence classes of C^*-dynamical systems $[(A, \alpha, G)]$, where A is a stable, separable continuous trace C^*-algebra with spectrum Y, α is a strongly continuous action of the group G on A such that the induced action $\hat{\alpha}$ of G on $\hat{A} = Y$ is given by τ, and $(A_1, \alpha_1, G_1) \sim (A_2, \alpha_2, G)$ if there exists a $*$-isomorphism Φ: $A_1 \to A_2$ preserving the spectrum Y such that α_2 is exterior equivalent to $\Phi \circ \alpha_1 \circ \Phi^{-1}$. In [CKRW] it was shown that $\mathrm{Br}_G(Y)$ was an abelian group with multiplication given by balanced tensor product over $C_0(Y)$ and $[1]_{\mathrm{Br}_G(Y)} = [(C_0(Y)\otimes\mathcal{K}, \tau\otimes\mathrm{Id}, G)]$. This group is defined very naturally in the sense that if (Y_1, τ_1, G_1) and (Y_2, τ_2, G_2) are equivalent dynamical systems, i.e., if there is a homeomorphism Φ: $Y_1 \to Y_2$ and an isomorphism A: $G_1 \to G_2$ such that $\Phi(\tau_1(g)y) = \tau_2(A(g))\Phi(y), \forall y \in Y_1$, then $\mathrm{Br}_{G_1}(Y_1) \cong \mathrm{Br}_{G_2}(Y_2)$.

A filtration involving the Moore cohomology groups $H^p(G, \check{H}^q(Y, \mathcal{S}))$, $p+q = 2$, was developed in [CKRW] to aid in the computation of $\mathrm{Br}_G(Y)$. We mention two of the homomorphisms from this filtration.

PROPOSITION 1.1 [CKRW, Theorem 5.1(3)]. *Let (Y, τ, G) be a topological dynamical system and $\mathrm{Br}_G(Y)$ the associated equivariant Brauer group. Then there are homomorphisms*

$$d_{(Y,G)}\colon [H^2(Y, \mathbb{Z})]^G \to H^2(G, (C(Y, \mathbb{T}))$$

and

$$\xi_{(Y,G)}\colon H^2(G, C(Y,\mathbb{T})) \to \mathrm{Br}_G(Y)$$

such that the sequence

$$[H^2(Y,\mathbb{Z})]^G \xrightarrow{d_{(Y,G)}} H^2(G,(C(Y,\mathbb{T})) \xrightarrow{\xi_{(Y,G)}} \mathrm{Br}_G(Y) \tag{1.1}$$

is exact.

We mention for future reference that the map $\xi_{(Y,G)}$ sends $[\sigma] \in H^2(G, (C(Y,\mathbb{T}))$ to the equivalence class of the C^*-dynamical system given by $[(C_0(Y)\otimes\mathcal{K}(L^2(G)), \alpha_\sigma, G)]$, where α_σ is the action of G on $C_0(Y)\otimes\mathcal{K}$ associated to the twisted C^*-dynamical system $(C_0(Y), \tau, \sigma, G)$ by the stabilization trick of [PR1](see [P1], Equations 2.1–2.4).

We state two more results mentioned in the introduction concerning the equivariant Brauer group which will be of use to us.

THEOREM 1.2 [KRW]. *Let P be an l.c.s.c Hausdorff space carrying commuting free and proper actions of the locally compact groups G and H. Then there are isomorphisms $\theta_G\colon \mathrm{Br}_G(P/H) \to \mathrm{Br}_{G\times H}(P)$ and $\theta_H\colon \mathrm{Br}_H(P/G) \to \mathrm{Br}_{G\times H}(P)$. Furthermore if $\theta_G([A,\alpha,G)]) = ([(C,\gamma,G\times H)]$ and $\theta_H([(B,\beta,H)]) = [(C,\gamma,G\times H)]$ then the C^*-algebras $A \times_\alpha G$, $C \times_\gamma (G\times H)$ and $B \times_\beta H$ are all strongly Morita equivalent to one another.*

The next result gives an explicit description of the group $\mathrm{Br}_N(Z)$ where N is an elementary abelian group acting trivially on the space Z (this theorem has recently been extended to compactly generated groups N by S. Echterhoff and D. Williams [EW]).

THEOREM 1.3 [P2], [PRW]. *Let N be an elementary abelian group acting trivially on the l.c.s.c Hausdorff space Z. Then there is an isomorphism*

$$\begin{aligned}\mathrm{Br}_N(Z) &\cong C(Z, H^2(N,\mathbb{T})) \oplus \check{H}^1(Z,\hat{\mathcal{N}}) \oplus \check{H}^3(Z,\mathbb{Z})\\ &= \check{H}^0(Z,\mathcal{H}^2(N,\mathbb{T})) \oplus \check{H}^1(Z,\mathcal{H}^1(N,\mathbb{T})) \oplus \check{H}^2(Z,\mathcal{H}^0(N,\mathbb{T}))\end{aligned} \tag{1.2}$$

Denoting by $\Pi_i\colon i = 0,1,2$, the projection of $\mathrm{Br}_N(Z)$ onto each summand in (1.2), we recall that Π_0 can be identified with the Mackey obstruction map $M_N\colon \mathrm{Br}_N(Z) \to C(Z, H^2(N,\mathbb{T}))$ and $\Pi_2([(B,\beta,N)]$ gives exactly the Dixmier-Douady class of B. The map Π_1 is related to the Phillips-Raeburn obstruction.

We also recall that under the hypotheses of Theorem 1.3 there is a monomorphism $E_{(Z,N)}\colon H^2_{\mathrm{pt}}(N, C(Z,\mathbb{T})) \to \check{H}^1(Z,\hat{\mathcal{N}})$ whose range is denoted by $\check{H}^1_C(Z,\hat{\mathcal{N}})$ and represents the set of equivalence classes of characteristic principal $\hat{N}$ bundles over Z[RW1, Prop 3.8]. Here $H^2_{\mathrm{pt}}(N, C(Z,\mathbb{T}))$ represents the group of equivalence classes of pointwise trivial 2-cocycles. We then have the following relationship between Proposition 1.1 and Theorem 1.3.

COROLLARY 1.4 [P2, 2.4]. *Let N be an elementary abelian group acting trivially on the l.c.s.c Hausdorff space Z, and let $[(B, \beta, N)] \in Br_N(Z)$. Then $[(B, \beta, N)] \in \xi_{(Z,N)}(H^2(N, C(Z, \mathbb{T})))$ if and only if $\Pi_2([(B, \beta, N)]) = \delta(B) = \{0\}$ and $\Pi_1([(B, \beta, N)] \in \check{H}^1_C(Z, \hat{\mathcal{N}})$.*

1.2. *The Λ-invariant.* The Λ-invariant, first defined by I. Raeburn and D. Williams in their study of continuous trace C^*-dynamical systems [RW1], built on prior work of J. Huebschmann [Hu], and at least for a discrete group G with normal subgroup N can be viewed as one way of organizing the information one obtains about $H^2(G, M)$ from the Lyndon-Hochschild-Serre spectral sequence. Let G be an l.c.s.c. group with closed normal subgroup N. Suppose that M is a Polish G/N module, with the abelian group structure on M denoted by $(a, b) \longmapsto ab$, $a, b \in M$. Let $Z(G, N; M)$ denote the set of pairs $\{(\lambda, \mu)\}$ where λ: $G \times N \to M$ and μ: $N \to M$ are Borel maps satisfying

$$\mu \in Z^2(N, M), \tag{1.3}$$

$$\lambda(1_G, n) = 1_M = \lambda(s, 1_N), (s, n) \in G \times N, \tag{1.4}$$

$$\lambda(m, n) = \mu(m, n)\mu(n, m)^{-1}, (m, n) \in N \times N, \tag{1.5}$$

$$\lambda(st, n) = \lambda(s, n)s(\lambda(s, t, n)), (s, t, n) \in G \times G \times N, \tag{1.6}$$

$$\lambda(s, mn) = s(\mu(m, n))^{-1}\mu(m, n)\lambda(s, m)\lambda(s, n), (s, m, n) \in G \times N \times N. \tag{1.7}$$

With pointwise operations, $Z(G, N; M)$ is an abelian group. Let $B(G, N; M)$ denote the subgroup of $\Lambda(G, N; M)$ consisting of all pairs of the form

$$\{\Delta_\rho = (s(\rho(n)^{-1})\rho(n), \rho(m)\rho(n)\rho(mn)^{-1})\}$$

where ρ: $N \to M$ is a Borel map. Then the Λ-invariant group $\Lambda(G, N; M)$ is defined to be the quotient group $Z(G, N; M)/B(G, N; M)$. It can be shown [RW2] that the Λ-invariant fits into the Inflation-Restriction sequence indicated,

$$\begin{array}{l} 0 \to H^1(G/N, M) \xrightarrow{\text{Inf}} H^1(G, M) \xrightarrow{\text{Res}} H^1(N, [M])^{G/N} \\ \to H^2(G/N, M) \xrightarrow{\text{Inf}} H^2(G, M) \xrightarrow{\text{r}} \Lambda(G, N; M) \xrightarrow{\delta} H^3(G/N, M) \xrightarrow{\text{Inf}} H^3(G, M), \end{array} \tag{1.8}$$

and that $\Lambda(G, N; M)$ is determined by the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(G/N, \mathrm{Hom}(N, M)) \arrow[r, "\mathrm{i}"] & \Lambda(G, N; M) \arrow[dll, "j"'] \\ {[H^2(N, M)]^G} \arrow[r, "\mathrm{k}"] & H^2(G/N, \mathrm{Hom}(N, M)) & \end{tikzcd} \qquad (1.9)$$

Formulas for the maps r, δ, i, j, k are given in [RW2]. The case of interest to us is the situation where $M = C(Y, \mathbb{T})$, where G is abelian and Y is a G-space with constant stabilizer subgroup N, and G/N acts freely and properly on Y. Then the map r: $H^2(G, C(Y, \mathbb{T})) \to \Lambda(G, N; C(Y, \mathbb{T}))$ is given by $r([\sigma]) = [(\lambda, \mu)]$, where

$$\lambda(g, n) = \sigma(g, n)\sigma(n, g)^{-1}, \quad (g, n) \in G \times N, \qquad (1.10)$$

$$\mu = \sigma_{|N\times N}. \qquad (1.11)$$

Moreover in this situation, Raeburn and Williams have defined a subgroup

$$Z_{\mathrm{pt}}(G, N; C(Y, \mathbb{T})) \subseteq Z(G, N, C(Y, \mathbb{T}))$$

by

$$[(\lambda, n)] \in Z_{\mathrm{pt}}(G, N; C(Y, \mathbb{T})) \quad \text{if } \mu \in Z^2_{\mathrm{pt}}(N, C(Y, \mathbb{T})).$$

Since clearly $B(G, N; C(Y, \mathbb{T})) \subset Z_{\mathrm{pt}}(G, N; C(Y, \mathbb{T}))$ it is possible to define the subgroup

$$\Lambda_{\mathrm{pt}}(G, N; C(Y, \mathbb{T})) = Z_{\mathrm{pt}}(G, N; C(Y, \mathbb{T}))/B(G, N; C(Y, \mathbb{T})) \subset \Lambda(G, N; C(Y, \mathbb{T})).$$

Under the above assumptions, Raeburn and Williams have proved the following:

THEOREM 1.5 [RW1, Theorem 6.5, Proposition 7.1]. *Let G be an l.c.s.c abelian group acting on the l.c.s.c Hausdorff space Y with constant stabilizer subgroup N in such a way that Y is a locally trivial principal G/N - bundle over the quotient space $Z = Y/(G/N)$. Let $I_N(Y) \subset \mathrm{Br}_G(Y)$ be defined by $I_N(Y) = \{[(A, \alpha, G)]: \alpha/N \in \mathrm{Inn}(A)\}$; i.e., the action α restricted to N is inner. There are homomorphisms*

$$d: I_N(Y) \to \Lambda_{\mathrm{pt}}(G, N; C(Y, \mathbb{T})) \qquad (1.12)$$

and

$$F_{(Y,G)}: \Lambda_{\mathrm{pt}}(G, N; C(Y, \mathbb{T})) \to \check{H}^1(Z, \hat{\mathcal{N}}) \qquad (1.13)$$

such that $F_{(Y,G)} \circ d([(A, \alpha, G)]) = [\lambda_\alpha]$, where $[\lambda_\alpha]$ is the class of the principal $\hat{N}$ bundle $\widehat{A \times_\alpha G} \to Y/G = Z$. Furthermore $F_{(Y,G)}$ is a monomorphism and the *image of $F_{(Y,G)}$ is equal to*

$$\{[F] \in \check{H}^1(Z, \hat{N}):\ p^*(F) \in \check{H}^1_C(Y, \hat{\mathcal{N}})\},$$

where $p\colon Y \to Y/G = Z$ is the quotient map.

2. Strong Morita equivalence of twisted transformation group C^*-algebras

Let Y be an l.c.s.c. Hausdorff space, let G be an l.c.s.c. abelian group with closed subgroup N, acting freely and properly on Y_1, and suppose that

(2.1) $p_1\colon Y \to Y/N = X$ is a locally trivial principal N-bundle,
(2.2) $p_2\colon X \to X/G = Y/G = Z$ is a locally trivial principal G/N – bundle,
(2.3) $p_3 = p_2 \circ p_1\colon Y \to Y/G = Z$ is a locally trivial principal G – bundle.

If G is an elementary abelian group, (2.1)–(2.3) will follow automatically from the freeness and properness of the G-action on Y. We will consider twisted transformation group C^*-algebras $C_0(X) \times_{\tau,\omega} G$, where $[\omega] \in H^2(G, C(X, \mathbb{T}))$, and our main aim will be to identify the element $[(B, \beta, N)] \in \mathrm{Br}_N(Z)$ guaranteed by Theorem 1.2 such that $C_0(X) \times_{\tau,\omega} G$ is strongly Morita equivalent to $B \times_\beta N$; along the way we will state conditions on $[\omega]$ which will guarantee that $C_0(X) \times_{\tau,\omega} G$ is strongly Morita equivalent to a twisted transformation group C^*-algebra of the form $C_0(Z) \times_{\mathrm{Id},\widetilde{\omega}} N$, $\widetilde{\omega} \in Z^2(N, C(Z, \mathbb{T}))$. As the latter type of C^*-algebras can be decomposed as the C^*-algebra of sections of a C^*-bundle whose fibers are twisted abelian group C^*-algebras, they are more easy to study.

We note that examples from [RR] and [PR3] show, first of all, that there exists a locally trivial principal G/N-bundle X over Z which is not the quotient of a G-bundle, such that the ordinary transformation group C^*-algebra $C_0(X) \times_\tau G$ is not strongly Morita equivalent to <u>any</u> crossed product of the form $B \times_\beta N$, where $[(B, \beta, N)] \in \mathrm{Br}_N(Z)$, and, second, that even if X is the quotient by the action of N of a G-bundle Y over Z, $C_0(X) \times_{\tau,\omega} G$ need not be strongly Morita equivalent to a twisted transformation group C^*-algebra $C_0(Z) \times_{\mathrm{Id},\widetilde{\omega}} N$, so that some sort of conditions on X, Z and $[\omega] \in H^2(G, C(X, \mathbb{T}))$ are necessary to obtain positive results.

For future reference we point out that if G is a countable discrete abelian group and X is a locally trivial G/N bundle over the space $Z = \mathbb{T}^k$, $k \in N$, then X is always the quotient of a principal G-bundle (X, G):

PROPOSITION 2.1. *Let G be a countable discrete abelian group and N a subgroup of G, and let X be a locally trivial principal G/N-bundle over $Z = \mathbb{T}^k$, $k \in N$. Then there is a locally trivial principal G-bundle Y over $\mathbb{T}^k$ such that $Y/N = X$.*

Proof. The bundle $(X, G/N)$ over $Z = \mathbb{T}^k$ is classified by an element $[\gamma] \in \check{H}^1(Z, \mathcal{G}/N) = \check{H}^1(Z, G/N)$, and X will be the quotient of a G-bundle (Y, G) over

$Z = \mathbb{T}^k$ if and only if $[\gamma]$ is in the range of the map $\pi_*\colon \check{H}^1(\mathbb{T}^k, G) \to \check{H}^1(\mathbb{T}^k, G/N)$, where $\pi\colon G \to G/N$ is the projection map. By a slight modification of [PR3, Lemma 2.6], there is a commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Z}^k, G) & \xrightarrow{\pi_*} & H^1(\mathbb{Z}^k, G/N) \\ \Big\downarrow (\lambda_G)^* & & \Big\downarrow (\lambda_{G/N})^* \\ \check{H}^1(\mathbb{T}^k, G) & \xrightarrow{\pi_*} & \check{H}^1(\mathbb{T}^k, G/N) \end{array} \tag{2.4}$$

where $(\lambda_G)^*$ and $(\lambda_{G/N})^*$ are the isomorphisms between group cohomology and Čech cohomology obtained by using the fact that $\mathbb{T}^k$ is a classifying space for $\mathbb{Z}^k$. Since G is discrete abelian, every homomorphism from $\mathbb{Z}^k$ to G/N can be lifted to a homomorphism from $\mathbb{Z}^k$ to G, and the result follows from the commutativity of diagram (2.4).

We now let Y, X, and Z be as in (2.1)–(2.3), fix $[\omega] \in H^2(G, C(X, \mathbb{T}))$ and consider the twisted transformation group C^*-algebra $C_0(X) \times_{\tau,\omega} G$. By Theorem 1.2, there is a C^*-dynamical system (B, β, N) such that B has continuous trace, $\hat{B} = Z$, the induced action $\hat{\beta}$ of N on Z is trivial, and $B \times_\beta N$ is strongly Morita equivalent to $C_0(X) \times_{\tau,\omega} G$, which can be constructed using the isomorphism $\theta_N^{-1} \circ \theta_G\colon \mathrm{Br}_G(X) \to \mathrm{Br}_N(Z)$ of Theorem 1.2. We shall use a slightly different isomorphism Ψ between $\mathrm{Br}_G(X)$ and $\mathrm{Br}_N(Z)$, defined as follows:

$$\Psi = K^* \circ A^* \circ \theta_G, \tag{2.5}$$

where $\theta_G\colon \mathrm{Br}_G(X) \to \mathrm{Br}_{G\times N}(Y)$ is as in Theorem 1.2,

$$A^*\colon \mathrm{Br}_{(G\times N,\tau_1)}(Y) \to \mathrm{Br}_{(G\times N,\tau_2)}(Y)$$

is obtained by taking $\Phi = \mathrm{Id}$ and defining $A\colon G\times N \to G\times N$ by $A(g, n) = (gn, n)$, where τ_1 is the action of $G\times N$ on Y defined by $\tau_1(g, n)y = gn^{-1}y$, and $\tau_2 = \tau_1 \circ A$, as in our remarks prior to the statement of Prop. 1.1, and $K^*\colon \mathrm{Br}_{(G\times N,\tau_2)}(Y) \to \mathrm{Br}_N(Z)$ is given by Theorem 5.3 of [PRW]; more precisely, we have $K^*([(C, \gamma, G \times N)]) = [(C \times_{\gamma/G} G, \gamma_{/(N)}, N)]$, where, by abuse of notation, $\gamma/_N$ denotes the action of N on the crossed product $C \times_{\gamma/G} G$ obtained from standard decomposition results for crossed product C^*-algebras, so that $C \times_\gamma (G \times N) \cong (C \times_{\gamma/G} G) \times_{\gamma/N} N$. It is then easy to check that setting $\Psi([(C_0(X) \otimes \mathcal{K}, \alpha_\omega, G)] = \Psi(\xi_{(X,G)}([\omega])) = [(B, \beta, N)] \in \mathrm{Br}_N(Z)$, $B \times_\beta N$ will be strongly Morita equivalent to $C_0(X) \times_{\tau,\omega} G$.

Before stating the main theorem, we establish some notation. For $[\omega] \in H^2(G, C(X, \mathbb{T}))$, let $[\omega_N] = \mathrm{Res}([\omega]) \in H^2(N, C(X, \mathbb{T}))$, where $\mathrm{Res}\colon H^2(G, C(X, \mathbb{T})) \to H^2(N, C(X, \mathbb{T}))$ is the restriction map. Taking $M = C(X, \mathbb{T})$ in (1.8) and (1.9), we note that Res $= j$ so that $[\omega_N] \in [H^2(N, C(X, \mathbb{T}))]^G$. For N elementary abelian acting trivially on X, by [PRW, Cor 5.2] there is a split exact sequence

$$0 \to H^2_{\mathrm{pt}}(N, C(X, \mathbb{T})) \xrightarrow{i_*} H^2(N, C(X, \mathbb{T})) \underset{j_*}{\overset{\pi_*}{\rightleftarrows}} C(X, H^2(N, \mathbb{T})) \to 0 \tag{2.6}$$

and clearly if $[\omega_N] \in [H^2(N, C(X, \mathbb{T}))]^G$, we have $\pi_*([\omega_N]) = f_\omega \circ p_2$, where f_ω: $Z \to H^2(N, \mathbb{T})$ is continuous. One easily checks that $[f_\omega] = M_G(\xi_{(X,G)}[\omega])$, where M_G: $\mathrm{Br}_G(X) \to C(Z, H^2(N, \mathbb{T}))$ is the Mackey obstruction map defined in Section 1 of [PRW]. Given $[\omega] \in H^2(G, C(X, \mathbb{T}))$, we now define $[(\lambda_\omega, \mu_\omega)] \in \Lambda(G \times N, N; C(Y, \mathbb{T}))$. Here the action of $G \times N$ on Y is defined by τ_2; i.e., $\tau_2(g, n)y = gy$, and N is identified with the subgroup $\{1_G\} \times N$ of $G \times N$.

$$\lambda_\omega((g_1, n_1), n_2)(y) = \omega(g_1, n_2)(p_1(y))\overline{\omega(n_2, g_1)(p_1(y))} \tag{2.7}$$

$$\mu_\omega(n_1, n_2) = \omega(n_1, n_2)(p_1(y))[j_*(f_\omega)]^{-1}(n_1, n_2)(p_3(y)), \tag{2.8}$$

where j_*: $C(Z, H^2(N, \mathbb{T})) \to H^2(N, (C(Z, \mathbb{T}))$ is the splitting map for the exact sequence (2.6) where the trivial N-space X is replaced by the trivial N-space Z. By construction, one checks that

$$[(\lambda_\omega, \mu_\omega)] \in \Lambda_{\mathrm{pt}}(G \times N, N; C(Y, \mathbb{T})) \subset \Lambda(G \times N, N; C(Y, \mathbb{T})).$$

We now state the main theorem.

THEOREM 2.2. *Let G be an l.c.s.c. abelian group, with closed subgroup N which is elementary abelian, and suppose that G acts freely and properly on the l.c.s.c. Hausdorff space Y, in such a way that the quotient maps p_1: $Y \to Y/N = X$, p_2: $X \to X/G = Z$ and $p_3 = p_2 \circ p_1$: $Y \to Z$ satisfy* (2.1)–(2.3), *respectively. Denote by d_i, $i = 0, 1, 2$, the homomorphisms of $H^2(G, C(X, \mathbb{T}))$ into the groups $C(Z, H^2(N, \mathbb{T}))$, $\check{H}^1(Z, \hat{\mathcal{N}})$, and $H^2(Z, \mathcal{S})$ defined by $d_i = \Pi_i \circ \Psi \circ \xi_{(X,G)}$, respectively, where the maps Π_i are defined after* (1.2). *Then*

$$d_0([\omega]) = [f_\omega], \tag{2.9}$$

$$d_1([\omega]) = F_{(Y,G\times N)}([\lambda_\omega, \mu_\omega]), \tag{2.10}$$

$$d_2([\omega]) = \xi_{(Y,G)}(p_1^*[\omega]) \in Br_G(Y) \cong \check{H}^2(Z, \mathcal{S}), \tag{2.11}$$

with $[f_\omega]$ as in (2.6), *and $F_{(Y,G\times N)}$ as in* (1.13).

Remark. If in addition ω is a continuous cocycle on $N \times N$ and not just Borel, we note that an explicit formula for $d_2([\omega])$ is given as follows: Let $\{N_i\}_i$ be a trivialing open cover for Z corresponding to the G-bundle p_3: $Y \to Z$, and let c_i: $N_i \to Y$ be local cross sections and λ_{ij}: $N_i \cap N_j \to G$ the transition functions defined by $c_j(z) = \lambda_{ij}(z)c_i(z)$. Then

$$d_2([\omega]) = [\nu_{ijk}(z)] = [\omega(\lambda_{ij}(z), \lambda_{jk}(z))(p_1(c_i(z)))]. \tag{2.12}$$

We will first establish a sequence of lemmas.

LEMMA 2.3. *Let (Y, G), (X, G) and (Z, N) be as in the statement of Theorem 2.2, and let $[\omega] \in H^2(G, C(X, \mathbb{T}))$. Then*

$$A^* \circ \theta_G \circ \xi_{(X,G)}([\omega]) = \xi_{(Y,\tau_2,G\times N)}([\sigma_{\omega,1}]),$$

where $[\sigma_{\omega,1}] \in H^2_{(\tau_2)}(G \times N, C(Y, \mathbb{T}))$ is defined by

$$\begin{aligned}\sigma_{\omega,1}&((g_1,n_1),(g_2,n_2))(y)\\ &= \omega(g_1n_1, g_2n_2)(p_1(y)),\quad (g_1,n_1),(g_2,n_2) \in G \times N, y \in Y. \quad (2.13)\end{aligned}$$

Proof. By checking the definitions of $\xi_{(X,G)}$ as explicitly given in [P1, 2.1–2.4] and $\theta_G\colon \mathrm{Br}_G(X) \to \mathrm{Br}_{(G\times N,\tau_1)}(Y)$ as given in [KRW, Prop. 7], we see that

$$\theta_G \circ \xi_{(X,G)}([\omega]) = \xi_{(Y,\tau,G\times n)}([\sigma_{\omega,2}]),$$

where $[\sigma_{\omega,2}] \in H^2_{(\tau_1)}(G \times N, C(Y, \mathbb{T}))$ is defined by

$$\sigma_{\omega,2}((g_1,n_1),(g_2,n_2))(y) = \omega(g_1,g_2)(p_1(y)), (g_1,n_1),(g_2,n_2) \in G \times N, y \in Y. \quad (2.14)$$

In what follows, for $[\sigma] \in H^2(G \times N, C(Y, \mathbb{T}))$ we denote by $\lambda_{[\sigma]}$ the action of $G \times N$ on $C_0(Y) \otimes \mathcal{K}$ given by the stabilization trick. Using [P1, 2.1–2.2], one can also check that the action $\lambda_{\sigma_{\omega,2}} \circ A$ of $G \times N$ on $C_0(Y) \otimes \mathcal{K}$ obtained from calculating

$$A^*(\xi_{(Y,\tau_1,G\times N})([\sigma_{\omega,1}])) = A^*([(C_0(Y) \otimes \mathcal{K}, \gamma_{[\sigma_{\omega,2}]}, G \times N)]) \in \mathrm{Br}_{(G\times N,\tau_2)}(Y)$$

is exterior equivalent to the action $\gamma_{[A^*(\sigma_{\omega,2})]}$ of $G \times N$ on $C_0(Y) \otimes \mathcal{K}$, and since $\xi_{(Y,\tau_2,G\times N)}([\sigma_{\omega,1}]) = [(C_0(Y) \otimes \mathcal{K}, \gamma_{[A^*[\sigma_{\omega,2}]]}, G \times N)]$, we see that $A^* \circ \theta_G \circ \xi_{(X,G)}([\omega]) = \xi_{(Y,\tau_2,G\times N)}([\sigma_{\omega,1}])$, as desired.

We now show that $\sigma_{\omega,1}$ is cohomologous to another cocycle σ_ω, in part by using the decomposition for $H^2_{(\tau_2)}(G \times N, C(Y, \mathbb{T}))$ given in (1.8) and (1.9).

LEMMA 2.4. *Let Y, X, Z, G and N be as in the statement of Theorem 2.2. Define $[\sigma_\omega] \in Z^2_{\tau_2}(G \times N, C(Y, \mathbb{T}))$ by*

$$\sigma_\omega((y_1,n_1,),(g_2,n_2))(y) = \omega(g_1,g_2)(p_1(y))\lambda(g_1,n_2)(p_1(y))\omega(n_1,n_2)(p_1(y))$$
$$\text{for } (g_1,n_1),(g_2,n_2) \in G \times N, y \in Y, \quad (2.15)$$

where $\lambda\colon G \times N \to C(X, \mathbb{T})$ is defined by

$$\lambda(g,n)(x) = \omega(g,n)(x)\overline{\omega(n,g)(x)},\quad (g,n) \in G \times N, x \in X. \quad (2.16)$$

Then σ_ω is cohomologous to $\sigma_{\omega,1}$.

Proof. We first establish for the reader's convenience that $[(\lambda, \omega_N)] \in \Lambda(G, N; C(X, \mathbb{T}))$, although this follows from Raeburn and Williams' unpublished work [RW2, Section 5]; i.e., we shall prove that

$$\lambda(g_1g_2, n)(x) = \lambda(g_1, n)(x)\lambda(g_2, n)(g_1^{-1}x) \; g_1, g_2 \in G, n \in N, x \in X \tag{2.17}$$

and

$$\lambda(g, n_1n_2)(x) = \overline{\omega(n_1, n_2)(g^{-1}x)}\omega(n_1, n_2)(x)\lambda(g, n_1)(x)\lambda(g, n_2)(x) \quad \text{for } g \in G, n_1, n_2 \in N, x \in X. \tag{2.18}$$

We have

$$\begin{aligned}
&\lambda(g_1g_2, n)(x) \\
&\quad = \omega(g_1g_2, n)(x)\overline{\omega(n, g_1g_2)(x)} \\
&\quad = [\overline{\omega(g_1, g_2)(x)}\omega(g_2, n)(g_1^{-1}x)\omega(g_1, ng_2)(x)][\omega(g_1, g_2)(x)\overline{\omega(n, g_1)(x)} \\
&\qquad\qquad \cdot \overline{\omega(ng_1, g_2)(x)}] \\
&\quad = \omega(g_2, n)(g_1^{-1}x)\omega(g_1, ng_2)(x)\overline{\omega(n, g_1)(x)\omega(ng_1, g_2)(x)} \\
&\quad = \omega(g_2, n)(g_1^{-1}x)[\omega(g_1, n)(x)\omega(g_1n, g_2)(x)\overline{\omega(n, g_2)(g_1^{-1}x)}] \\
&\qquad \cdot \overline{\omega(n, g_1)(x)\omega(ng_1, g_2)(x)} \\
&\quad = \lambda(g_1, n)(x)\lambda(g_2, n)(g_1^{-1}x),
\end{aligned}$$

establishing (2.17).

As for (2.18), we have

$$\begin{aligned}
\lambda(g, n_1n_2)(x) &= \omega(g, n_1n_2)(x)\overline{\omega(n_1n_2, g)(x)} \\
&= \overline{\omega(n_1, n_2)(g^{-1}x)}\omega(gn_1, n_2)(x)\omega(g_1, n_1)(x)\overline{\omega(n_1n_2, g)(x)} \\
&= \overline{\omega(n_1, n_2)(g^{-1}x)}\omega(gn_1, n_2)(x)\omega(g, n_1)(x)\omega(n_1, n_2)(x) \\
&\quad \cdot \overline{\omega(n_2, g)(x)\omega(n_1, n_2g)(x)} \\
&= \overline{\omega(n_1, n_2)(g^{-1}x)}\omega(n_1, n_2)(x)\omega(g, n_1)(x)\overline{\omega(n_2, g)(x)\omega(n_1, g)(x)} \\
&\quad \cdot \omega(n_1, gn_2)(x)\omega(g, n_2)(x) \circ \overline{\omega(n_1, gn_2)(x)} \\
&= \overline{\omega(n_1, n_2)(g^{-1}x)}\omega(n_1, n_2)(x)\lambda(g, n_1)(x)\lambda(g, n_2)(x).
\end{aligned}$$

With the above identities in mind, we define $b\colon G \times N \to C(Y, \mathbb{T})$ by $b(g, n)(y) =$

$\omega(n, g)(p_1(y))$ and compute

$$\begin{aligned}
&[db\, \sigma_{\omega,1}]((g_1, n_1), (g_2, n_2))(y)\\
&\quad = \omega(n_1, g_1)(p_1(y))\omega(n_2, g_2)(p_1(g_1^{-1}y))\\
&\qquad \cdot \overline{\omega(n_1n_2, g_1g_2)(p_1(y))}\omega(g_1n_1, g_2n_2)(p_1(y))\\
&\quad = \omega(n_1, g_1)(x)\omega(n_1g_1, g_2n_2)(x)\omega(n_2, g_2)(g_1^{-1}x)\overline{\omega(n_1n_2, g_1g_2)(x)}\\
&\qquad \text{(letting } x = p_1(y))\\
&\quad = \omega(n_1, g_1g_2n_2)(x)\omega(g_1, g_2n_2)(x)\omega(n_2, g_2)(g_1^{-1}x)\overline{\omega(n_1n_2, g_1g_2)(x)}\\
&\quad = \omega(n_1, g_1g_2n_2)(x)\overline{\omega(n_1n_2, g_1g_2)(x)}\omega(g_1, n_2)(x)\omega(g_1n_2, g_2)(x)\\
&\quad = \omega(g_1, g_2)(x)\omega(g_1, n_2)(x)\overline{\omega(n_2, g_1)(x)}\omega(n_1, n_2)(x)\omega(n_1, g_1g_2n_2)(x)\\
&\qquad \cdot \overline{\omega(n_1n_2, g_1g_2)(x)}\omega(g_1n_2, g_2)(x)\overline{\omega(g_1, g_2)(x)}\omega(n_2, g_1)(x)\\
&\qquad \cdot \overline{\omega(n_1, n_2)}(x)\\
&\quad = \sigma_\omega((g_1, n_1), (g_2, n_2))(y)\omega(n_1, g_1g_2n_2)(x)\omega(g_1n_2, g_2)(x)\overline{\omega(g_1, g_2)(x)}\\
&\qquad\qquad \cdot \omega(n_2, g_1)(x)\overline{\omega(n_1, n_2g_1g_2)(x)\omega(n_2, g_1g_2)(x)}\\
&\quad = \sigma_\omega((g_1, n_1), (g_2, n_2))(y)\omega(n_2, g_1)(x)\omega(n_2g_1, g_2)(x)\\
&\qquad\qquad \cdot \overline{\omega(n_2, g_1g_2)(x)\omega(g_1, g_2)(x)}\\
&\quad = \sigma_\omega((g_1, n_1), (g_2, n_2))(y),
\end{aligned}$$

so that $\sigma_{\omega,1}$, is cohomologous to σ_ω, as desired.

Proof of Theorem 2.2. By Lemma 2.3 and 2.4, we have $d_i([\omega]) = \Pi_i \circ K^*(\xi_{(Y,\tau_2,G\times N)}([\sigma_\omega]))$, $i = 0, 1, 2$. We now write $[\sigma_\omega]$ as a product $[\tau_\omega] \cdot [m_\omega]$, where $[m_\omega], [\tau_\omega] \in H^2_{\tau_2}(G \times N, C(Y, \mathbb{T}))$ are defined by

$$m_\omega((g_1, n_1), (g_2, n_2))(y) = j_*([f_\omega])(n_1, n_2)(p_3(y)) \tag{2.19}$$

for $(g_1, n_1), (g_2, n_2) \in G \times N$, $y \in Y$, $[f_\omega]$ as in Eq. (2.6),

and

$$\tau_\omega = \sigma_\omega m_\omega^{-1}. \tag{2.20}$$

We now write

$$\xi_{(Z,N)}([j_*(f_\omega)]) = [(C_0(Z) \otimes \mathcal{K}, \beta_\omega, N)] \in \mathrm{Br}_N(Z)$$

and

$$\xi_{(Y,\tau_2,G\times N)}([m_\omega]) = [(C_0(Y) \otimes \mathcal{K}, \gamma_{[m_\omega]}, G \times N)] \in \mathrm{Br}_{(G\times N,\tau_2)}(Y).$$

Through direct computation, one checks that the action $\tau_2 \otimes \beta$ of $G \times N$ on $C_0(Y) \otimes_{C_0(Z)} C_0(Z) \otimes \mathcal{K} \cong C_0(Y) \otimes \mathcal{K}$ is exterior equivalent to $\gamma_{[m_\omega]}$. Thus

$$\begin{aligned}
&\Pi_i(K^*(\xi_{(Y,\tau_2,G\times N)}([\sigma_\omega]))) \\
&\quad = \Pi_i(K^*([(C_0(Y) \otimes \mathcal{K}, \tau \otimes \beta_\omega, G \times N)])) \\
&\quad = \Pi_i([((C_0(Y) \otimes \mathcal{K}) \times_\tau G, \beta_\omega, N)]), \quad i = 0, 1, 2, \\
&\quad = \Pi_i([(C_0(Z) \otimes \mathcal{K}, \beta_\omega, N)]) \\
&\quad = \Pi_i(\xi_{(Z,N)}([j_*([f_\omega])])), \quad i = 0, 1, 2.
\end{aligned}$$

Now $\xi_{(Z,N)} \circ j_*$: $C(Z, H^2(N, \mathbb{T})) \to \mathrm{Br}_N(Z)$ is a splitting for the Mackey obstruction $\Pi_0 = M_N$: $\mathrm{Br}_N(Z) \to C(Z, H^2(N, \mathbb{T}))$, so that $d_0([\omega]) = \Pi_0 \circ \xi_{(Z,N)} \circ j_*([f_\omega]) = [f_\omega]$. We now recall from [P2] and [PRW] that for $[(B, \beta, N)] \in \mathrm{Br}_N(Z)$, the element $[(B_1, \beta_1, N)] = [(B, \beta, N)] \cdot \xi_{(Z,N)}(j_*(M_N([(B, \beta, N)]))) \in \mathrm{Br}_N(Z)$ has trivial Mackey obstruction, so is locally unitary, in the sense of J. Phillips and I. Raeburn [PhR], by Theorem 2.1 of [Ro]. $\Pi_1([(B, \beta, N)])$ is defined to be that element $[q] \in \check{H}^1(Z, \hat{\mathcal{N}})$ representing the principal $\hat{N}$ bundle $\widehat{B_1 \times_{\beta_1} N} \longrightarrow \hat{B}_1 = Z$. It is evident from this definition that $d_1([\omega]) = \Pi_1(\xi_{(Z,N)}([j_*([f_\omega])])) = 1_{\check{H}^1(Z,\hat{\mathcal{N}})}$, and clearly

$$d_2([\omega]) = \Pi_2(\xi_{(Z,N)}(j_*([f_\omega]))) = \delta([[(C_0(Y) \otimes \mathcal{K}) \times_\tau G]) = \delta([C_0(Z) \otimes \mathcal{K}]) = 1_{\check{H}^2(Z,\mathcal{S})}.$$

We now consider $K^*(\xi_{(Y,\tau_2,G\times N)}([\tau_\omega]))$, $[\tau_\omega]$ as in (2.20). We first note that upon identifying the subgroup $\{1\} \times N$ of $G \times N$ with N, $\tau_{\omega/(\{1\}\times N)\times(\{1\}\times N)}$ can be written as $p_3^*(\omega_N \cdot j_*([f_\omega])^{-1}) \in H^2_{\mathrm{pt}}(N, C(Y, \mathbb{T}))$ by the exactness of sequence (2.6). Also, denoting $\xi_{(Y,\tau_2,G\times N)}([\tau_\omega])$ by $[(C_0(Y) \otimes \mathcal{K}, \gamma_{[\tau_\omega]}, G \times N)]$, one checks that $\gamma_{[\tau_\omega]}$ is inner when restricted to the stabilizer subgroup $\{1\} \times N$ for the τ_2 action on Y, and that $d([\gamma_{[\tau_\omega]}]) = [(\lambda_\omega, \mu_\omega)]$, where d: $I_{\{1\}\times N}(Y) \to \Lambda_{\mathrm{pt}}(G \times N, N; C(Y, \mathbb{T}))$ is as in (1.12), and $(\lambda_\omega, \mu_\omega)$ are as defined in (2.7) and (2.8). By Theorem 1.5, the class of the locally trivial principal $\hat{N}$ bundle over $Y/(G \times N) = Z$, given by $((C_0(Y) \otimes K) \times_{\gamma_{[\tau_\omega]}} G \times N)^\wedge \to Z$, is exactly $F_{(Y,\tau_2,G\times N)} \circ d([\gamma_{[\tau_\omega]}]) = F_{(Y,\tau_2,G\times N)}([\lambda_\omega, \mu_\omega])$, which by results from [RW1] will lie in the subgroup $\{[F] \in \check{H}^1(Z, \hat{N}): p_3^*(F) \in \check{H}^1_C(Y, \hat{\mathcal{N}})\}$. Hence

$$\Pi_0(K^* \circ \xi_{(Y,\tau_2,G\times N)}([\tau_\omega])) = \Pi_0((C_0(Y) \otimes \mathcal{K} \times_{\gamma_{[\tau_\omega]}} G, \gamma_{[\tau_\omega]/N}, N)) = 1_{C(Z,\mathbb{T})}$$

(as $\gamma_{[\tau_\omega]}$ restricted to $\{1\} \times N$ is locally unitary), and by the definition of Π_1 given above, $\Pi_1(K^* \circ \xi_{(Y,\tau_2,G\times N)}([\tau_\omega])) = F_{(Y,\tau_2,G\times N)}([(\lambda_\omega, \mu_\omega)])$. We now calculate

$$\begin{aligned}
&\Pi_2(K^* \circ \xi_{(Y,\tau_2,G\times N)}([\tau_\omega])) \\
&\quad = \Pi_2(K^*[(C_0(Y) \otimes \mathcal{K}, \gamma_{[\tau_\omega]}, G \times N)] \\
&\quad = \delta([C_0(Y) \otimes \mathcal{K}] \times_{\gamma_{[\tau_\omega]/G\times\{1\}}} (G \times \{1\})) \in \check{H}^2(Z, \mathcal{S}).
\end{aligned}$$

From the formula for τ_ω, one calculates that

$$\begin{aligned}[(C_0(Y) \otimes \mathcal{K}, \gamma_{[\tau_{\omega 1}/G\times\{1\}}, G \times \{1\})] &= \xi_{(Y,G)}(p_1^*([\omega])\\ &= [(C_0(Y) \otimes \mathcal{K}, \alpha_{p_1^*([\omega])}, G)] \in \mathrm{Br}_G(Y).\end{aligned}$$

It follows that

$$\delta((C_0(Y)\otimes\mathcal{K})\times_{\gamma_{\tau_{\omega 1}}} G\times\{1\}) = \delta((C_0(Y)\otimes\mathcal{K})\times_{\alpha_{p_1^*(\omega)}} G)) = \delta((C_0(Y)\times_{\tau,p_1^*(\omega)} G)\otimes\mathcal{K}),$$

the last equality given by the stabilization trick of [PR1]. But formulas for the Dixmier-Douady classes of twisted transformation group C^*-algebras where G acts freely and properly on Y have been given in [PR2, Cor 3.4 and (**) on p. 604], and if the cocycle ω is continuous on $G \times G$ and not just Borel, so is $p_1^*(\omega)$, and Corollary 3.4 in [PR2] gives $\delta((C_0(Y) \times_{\tau,p_1^*(\omega)} G) \otimes \mathcal{K})$ as a 2-cocycle with representative given by the right-hand side of (2.12).

Finally, using the fact that Π_i, $i = 0, 1, 2$, K^*, and $\xi_{(Y,\tau_2,G\times N)}$ are homomorphisms, we have $d_i([\omega]) = \Pi_i \circ K^*[\xi_{(Y,\tau_2,G\times N)}([\sigma_\omega])] = \Pi_i \circ K^* \circ \xi_{(Y,\tau_2,G\times N)}([\tau_\omega]) \cdot \pi_i \circ K^* \circ \xi_{(Y,\tau_2,G\times N)}([m_\omega])$, $i = 0, 1, 2$, which combined with our previous calculations completes the proof of Theorem 2.2 and establishes (2.12).

We now can determine conditions on $[\omega]$ under which a twisted transformation group C^*-algebra will be strongly Morita equivalent to a twisted transformation group C^*-algebra of the form $C_0(Z) \times_{\mathrm{Id},\tilde{\omega}} N$ for $[\tilde{\omega}] \in H^2(N, C(Z, \mathbb{T}))$.

COROLLARY 2.5. *Let Y, X, Z, G, N, and $[\omega] \in H^2(G, C(X, \mathbb{T}))$ be as in the statement of Theorem* 2.2. *Suppose that $d_2([\omega]) = [1]_{\check{H}^2(Z,\mathcal{S})}$ and $d_1([\omega]) \in \check{H}^1_C(Z, \hat{\mathcal{N}})$, where d_2 and d_1 are the maps defined in Theorem* 2.2. *Then there exists $\widetilde{\omega} \in H^2(N, C(Z, \mathbb{T}))$ such that $C_0(X) \times_{\tau,\omega} G$ is strongly Morita equivalent to $C_0(Z) \times_{\mathrm{Id},\tilde{\omega}} N$.*

Proof. This follows directly from Theorem 2.2 together with Corollary 1.4.

For the next result, we make the additional assumption that ω restricted to $N \times N$ takes on its values in $p_2^*(C(Z, \mathbb{T})) \subseteq C(X, \mathbb{T})$.

THEOREM 2.6. *Let Y, X, Z, G, N, and $[\omega] \in H^2(G, C(X, \mathbb{T}))$ be as in the statement of Theorem* 2. *Suppose in addition that ω is (cohomologous to) a cocycle which when restricted to $N \times N$ takes on its values in $p_2^*(C(Z, \mathbb{T})) \subseteq C(X, \mathbb{T})$. Then in order that $d_2([\omega]) = [1]_{\check{H}^2(Z,\mathcal{S})}$ and $d_1([\omega]) \in \check{H}^1_C(Z, \hat{N})$, it is necessary and sufficient that the following conditions hold*:

(i) $[p_1^*([\omega])] \in \mathrm{Im}\, d_{(Y,G)} \subseteq H^2(G, C(Y, \mathbb{T}))$.

(ii) The map $p_1^*(\lambda)$: $G \to C(Y, \hat{N})$ defined by

$$p_1^*(\lambda)(g)(y)(n) = \omega(g, n)(p_1(y))\overline{\omega(n, g)(p_1(y))}$$

is trivial in $H^1(G, C(Y, \hat{N}))$; $(g, n) \in G \times N$, $y \in Y$.

If these conditions are satisfied, $C_0(X) \times_{\tau,\omega} G$ is strongly Morita equivalent to $C_0(Z) \times_{\mathrm{Id},\omega_N} N$, where $\omega_N \in H^2(N, C(Z, \mathbb{T}))$ is obtained from $\omega_{|N\times N}$ by identifying $p_2^*(C(Z, \mathbb{T}))$ with $C(Z, \mathbb{T})$.

Proof. The proof of Theorem 2.2 combined with the results of Corollary 1.4 show that $C_0(X) \times_{\tau,\omega} G$ will be strongly Morita equivalent to $C_0(Z) \times_{\mathrm{Id},\omega} N$ if $d_2([\omega]) = [1]_{H^2(Z,\mathcal{S})}$ and $d_1([\omega]) \in \check{H}^1_C(Z, \hat{\mathcal{N}})$. Since G acts freely and properly on Y, for any $[(\mathcal{C}, \gamma, G)] \in \mathrm{Br}_G(Y)$, $\mathcal{C} \times_\gamma G$ will be a continuous trace C^*- algebra with spectrum Z, and the map $\widetilde{\delta}$: $\mathrm{Br}_G(Y) \to \check{H}^2(Z, \mathcal{S})$ given by $\widetilde{\delta}([(\mathcal{C}, \gamma, G)]) = \delta(\mathcal{C} \times_\gamma G)$ is an isomorphism. Hence by Prop 1.1 and (2.12), $d_2([\omega]) = [\gamma_{ijk}(z)] = \delta((C_0(Y) \otimes \mathcal{K}) \times_{\gamma_{[p_1^*([\omega])]}} G)) = [1]_{\check{H}^2(Z,\mathcal{S})}$ if and only if $p_1^*([\omega]) \in \mathrm{Im}\ \mathrm{d}_{(Y,G)}$, establishing (i). Since $\omega_{|N\times N}$ takes on its values in $p_2^*(C(Z, \mathbb{T})) \subseteq C(X, \mathbb{T})$, the element μ_ω: $N \times N \to C(Y, \mathbb{T})$ defined in (2.8) takes on its values in $p_3^*(C(Z, \mathbb{T}))$ and is an element of $Z^2_{\mathrm{pt}}(N, C(Y, \mathbb{T}))$ so that $[(1, \mu_\omega)] \in \Lambda_{\mathrm{pt}}(G \times N, N; C(Y, \mathbb{T}))$; and writing $\mu_\omega = p_3^*(\widetilde{\mu}_\omega)$ where $\widetilde{\mu}_\omega \in Z^2_{\mathrm{pt}}(N, C(Z, \mathbb{T}))$, we have $F_{(Y,G\times N)}([(1, \mu_\omega)]) = E_{(Z,N)}([\widetilde{\mu}_\omega]) \in \check{H}^1_C(Z, \hat{N})$. Since $d_1([\omega]) = F_{(Y,G\times N)}([(\lambda_\omega, \mu_\omega)] \in \check{H}^1_C(Z, \hat{\mathcal{N}})$ by hypothesis, it follows that $F_{(Y,G\times N)}([(\lambda_\omega, 1)]) = F_{(Y,G\times N)}([(\lambda_\omega, \mu_\omega)])\cdot F_{(Y,G\times N)}([(1, \mu_\omega)]^{-1}) \in \check{H}^1_C(Z, \hat{\mathcal{N}})$. Let $[\gamma] = F_{(Y,G\times N)}([(\lambda_\omega, 1)]) \in \check{H}^1_C(Z, \hat{\mathcal{N}})$, and find $[\rho] \in H^2_{\mathrm{pt}}(N, C(Z, \mathbb{T}))$ with $E_{(Z,N)}([\rho]) = [\gamma]$. Then $[(1, p_3^*(\rho)] \in \Lambda_{\mathrm{pt}}(G \times N, N; C(Y, \mathbb{T}))$ and $F_{(Y,G\times N)}([(1, p_3^*(\rho))]) = [\gamma] = F_{(Y,G\times N)}([(\lambda_\omega, 1)])$. Since $F_{(Y,G\times N)}$ is injective, this implies that $[(\lambda_\omega, 1)] = [(1, p_3^*(\rho))] \in \Lambda_{\mathrm{pt}}(G\times N, N; C(Y, \mathbb{T})$. Now we adapt sequence (1.9) to obtain an exact sequence for $\Lambda(G \times N, N; C(Y, \mathbb{T}))$:

$$\begin{tikzcd} 0 \arrow[r] & H^1(G, C(Y, \hat{N})) \arrow[r, "i"] & \Lambda(G \times N, N; C(Y, \mathbb{T})) \arrow[r, "j"] & {[H^2(N, C(Y, \mathbb{T}))]^G} \arrow[dl, "k"] \\ & & H^2(G, C(Y, \hat{N})) & \end{tikzcd} \tag{2.21}$$

Hence $[p_3^*(\rho)] = j([(1, p_3^*(\rho))]) = j([(\lambda_\omega, 1)]) = j \circ i(p_1^*(\lambda)] = [1]_{[H^2(N,C(Y,\mathbb{T})]^G}$. Therefore

$$[p_3^*(\rho)] = [1] \in H^2(N, C(Y, \mathbb{T}))$$

and consequently

$$[(1, p_3^*(\rho))] = [1] \in \Lambda(G \times N, N; C(Y, \mathbb{T})),$$

$$i([p_1^*(\lambda)]) = [(\lambda_\omega, 1)] = [(1, p_3^*(\rho))] = [1]_{\Lambda(G\times N, N; C(Y,\mathbb{T}))}.$$

Since i is an injection, this implies that $[p_1^*(\lambda)] = 1 \in H^1(G, C(Y, \hat{N}))$, as we desired to show .

Remark 2.7. Although the assumption that $\omega/N \times N$ takes on its values in $p_2^*(C(Z, \mathbb{T}))$ may seem very strong, there are subgroups N for which this will always happen, regardless of Y, X, Z, and G. In particular, if $N \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$, the sequences (1.8), (1.9) with $M = C(X, \mathbb{T})$ and (2.5) show first that $\omega/_{\mathbb{Z}^n\times\mathbb{Z}^n} \in [H^2(\mathbb{Z}^n, C(X, \mathbb{T}))]^G$, and second that $H^2(\mathbb{Z}^n, C(X, \mathbb{T})) \cong C(X, H^2(\mathbb{Z}^n, \mathbb{T}))$, since $H^2_{\mathrm{pt}}(\mathbb{Z}^n, C(X, \mathbb{T}))$ is trivial. Consequently, $[H^2(\mathbb{Z}^n, C(X, \mathbb{T}))]^G \cong C(X/G, H^2(\mathbb{Z}^n, \mathbb{T})) \cong C(Z, H^2(\mathbb{Z}^n, \mathbb{T}))$, and thus $\omega|_{\mathbb{Z}^n\times\mathbb{Z}^n}$ is cohomologous to a cocycle taking on its values in $p_2^*(C(Z, \mathbb{T}))$.

3. Applications to twisted two-step nilpotent group C^*-algebras

In this section, we consider twisted group C^*-algebras $C^*(\Gamma, \mu)$, where Γ is a torsion free finitely generated two-step nilpotent group, i.e., where Γ is a central extension of $\mathbb{Z}^\ell$ by $\mathbb{Z}^n$, for $\ell, n \in \mathbb{N}$, and we establish conditions on $[\mu] \in H^2(\Gamma, \mathbb{T})$ analogous to conditions (i) and (ii) of Theorem 2.6 which will imply that $C^*(\Gamma, \mu)$ is strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$ where Γ_0 is a subgroup of Γ of finite index and $\mu_0 = \mu|_{\Gamma_0\times\Gamma_0}$. Though the conditions as stated may appear somewhat specialized, they frequently arise when one is considering examples of multipliers $[\mu] \in H^2(\Gamma, \mathbb{T})$ which are not homotopic to the identity in $H^2(\Gamma, \mathbb{T})$. Let (Γ, μ) be as above, and suppose that Γ contains a central subgroup D which itself contains the commutator subgroup $C = [\Gamma, \Gamma]$, and suppose in addition the following conditions are satisfied:

(3.1) $\mu_{|D\times D}$ is trivial.

(3.2) The homomorphism $\phi_D(\mu)\colon \Gamma \to \hat{D}$ defined in [PR3] by

$$\phi_D(\mu)(\gamma)(d) = \mu(d, \gamma)\overline{\mu(\gamma, d)}, \gamma \in \Gamma, d \in D$$

has closed (i.e., finite) range R in $\hat{D}$, so that $R = D_0^\perp$ for some subgroup $D_0 \subseteq D$ of finite index.

(3.3) Setting $M = \ker\phi_D(\mu)$, the quotient group M/D_0 (which we shall prove is abelian) splits as

$$M/D_0 \equiv D/D_0 \oplus M/D.$$

(This final condition of course automatically happens if $M/D \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$.)

The commutator subgroup C of Γ will automatically satisfy (3.1) [LP1, Prop. 1], so if C also satisfies (3.2) and (3.3) for μ, we can take $D = C$.

Assuming that (3.1) and (3.2) hold, we obtain the following decomposition result.

PROPOSITION 3.1. *Let Γ be a finitely generated torsion free two-step nilpotent group, let $[\mu] \in H^2(\Gamma, \mathbb{T})$ and suppose there is a central subgroup D of Γ containing C such that* (3.1) *and* (3.2) *are satisfied. Then the twisted group C^*-algebra $C^*(\Gamma, \mu)$ is $*$-isomorphic to a twisted transformation group C^*-algebra $C(\mathbb{T}^\ell) \times_{\tau,\omega} G$, where ℓ is the rank of D, $G = \Gamma/D$, the action τ of G on $\mathbb{T}^\ell = \hat{D}$ is given by translation corresponding to the homomorphism $\phi_D(\mu)$: $\Gamma \to \hat{D}$ of* (3.2), *and $[\omega] \in H^2(G, C(X, \mathbb{T}))$, where $X = \mathbb{T}^\ell$. Moreover, letting $N = \ker \tau \cong \ker \phi_D(\mu)|_D$, there is an l.c.s.c. free and proper G space (Y, G) such that the spaces Y, $X = Y/N$, $Z = Y/G = X/G$ satisfy the conditions of* (2.1)–(2.3). *If in addition we assume that* (3.3) *holds, then we can assume without loss of generality that $\omega|_{N\times N}$ takes on values in $p_2^*(C(Z, \mathbb{T})) \subseteq C(X, \mathbb{T})$.*

Proof. By definition, $C^*(\Gamma, \mu)$ is the twisted crossed product $\mathbb{C} \times_{Id,\mu} \Gamma$, so that by the decomposition theory for twisted crossed products [PR1, Theorem 4.1]) we have $C^*(\Gamma, \mu) = \mathbb{C} \times_{\mathrm{Id},\mu} \Gamma \cong C(\hat{D}) \times_{\tau,\omega} G$, where $G = \Gamma/D$ is abelian since $C \subseteq D$, and since $\mu|_{D\times D}$ is trivial, $\mathbb{C} \times_{\mathrm{Id},\mu} D \cong C^*(D) \cong C(\hat{D}) \cong C(\mathbb{T}^\ell)$ for some $\ell \in \mathbb{N}$ by the Fourier transform. The formulas given in [PR 1], Theorem 4.1, for τ and ω show that the action τ of G on $C(\hat{D})$ is translation corresponding to the homomorphism $\phi_D(\mu)$: $\Gamma \to \hat{D}$, which, since $\mu|_{D\times D}$ is trivial, factors through $\Gamma/D = G$. Choosing a cross-section c: $G \to \Gamma$ with $c(1_G) = 1_\Gamma$, and writing

$$\eta(g_1, g_2) = c(g_1)c(g_2)c(g_1g_2)^{-1} \in D \cong \hat{\hat{D}}, \quad g_1, g_2 \in G, \tag{3.4}$$

we can compute $\omega \in Z^2(G, C(\hat{D}, \mathbb{T}))$ as

$$\begin{aligned}\omega(g_1, g_2)(x) &= \mu(c(g_1), c(g_2))\overline{\mu(\eta(g_1, g_2), c(g_1g_2))}\eta(g_1, g_2)(x)\\ &\qquad g_1, g_2 \in G, x \in \hat{D}.\end{aligned} \tag{3.5}$$

By assumption (3.2), the range R of $\phi_D(\mu)$ is a finite group which is isomorphic to G/N, for $N = \mathrm{Ker}\ \tau$. Hence N is of finite index in G and G/N acts freely and properly on $X = \hat{D}$ which is a locally trivial principal G/N-bundle over $\hat{D}/G/N = Z$. Setting $D_0 = R^\perp \subseteq D$, by the Pontryagin theory $Z = \hat{D}_0$, so that Z also has the structure of an ℓ-torus. By Proposition 2.1, there is a locally trivial principal G-bundle Y over Z such that $Y/N = X$, so that (2.1)–(2.3) are satisfied. To establish the last statement of the proposition, we let $\mu = \mathrm{Ker}\ \phi_D(\mu)$, $C_0 = \{c \in C\colon \mu(c, \gamma)\overline{\mu(\gamma, c)} = 1, \forall \gamma \in \Gamma\}$, and $D_0 = \{d \in D\colon \mu(d, \gamma)\overline{\mu(\gamma, d)} = 1, \forall\gamma \in \Gamma\}$. Since $C \subseteq D \subseteq M$, $C_0 \subseteq D_0 \subseteq M$. The argument of Theorem 1.2 in [PR3] shows that $\mu = \mathrm{Inf}\ \widetilde{\mu}$ for

$\widetilde{\mu} \in Z^2(\Gamma/D_0, \mathbb{T})$. Furthermore there is an exact sequence

$$1 \to M/D_0 \to \Gamma/D_0 \to \Gamma/D_0/M/D_0 \cong \Gamma/M \cong R \cong D_0^{\perp} \to 1. \tag{3.6}$$

Since $D_0^{\perp} = [\widehat{D/D_0}]$ and is finite, by the theory of finite abelian groups we know that $D/D_0 \cong \widehat{D/D_0}$. We now establish that M/D_0 is abelian. Let $\Gamma_1 = \Gamma/C_0$. Then μ, being the inflation of a multiplier on Γ/D_0, can also be viewed as a lift of a multiplier μ_1 on the intermediate quotient group Γ_1. If $C_0 \neq C$, Γ_1 is again a two-step nilpotent group with commutator subgroup $C_1 = C/C_0$, and letting $\phi_{C_1}(\mu_1)$: $\Gamma_1 \to \hat{C}_1$, by construction $\phi_1(\mu)$ will be surjective, and by [PR3, Cor 1.3], $K_1 = \ker \phi_{C_1}(\mu_1)$ is a normal abelian subgroup of Γ_1 containing C_1. Now set $K = \{\gamma \in \Gamma\colon \gamma \cdot C_0 \in K_1 \subseteq \Gamma_1 = \Gamma/C_0\}$. Then $K = \ker \phi_C(\mu)\colon \Gamma \to \hat{C}$, and since $K_1 = K/C_0$ is abelian , $[K, K] \subseteq C_0$. Since $M = \ker \phi_D(\mu) \subseteq \ker \phi_C(\mu) = K$, it follows that $[M, M] \subseteq [K, K] \subseteq C_0$, and since $C_0 \subseteq D_0$, we see that $M/D_0 = M/C_0/D_0/C_0$ is abelian. Recalling that $N = M/D$, upon restricting $\eta \in Z^2(G, D)$ defined in (3.4) to $N \times N$, we obtain a cocycle $\eta_N \in Z^2(N, D)$. By the Bockstein exact sequence

$$H^2(N, D_0) \xrightarrow{i_*} H^2(N, D) \xrightarrow{\pi_*} H^2(N, D/D_0) \xrightarrow{\partial} H^3(N, D_0) \tag{3.7}$$

it follows that $[\eta_N] = i_*([\kappa])$ for $[\kappa] \in H^2(N, D_0)$ if and only if $\pi_*([\eta_N]) = [1]_{H^2(N,D/D_0)}$. It follows that η_N is cohomologous to a cocycle taking values in D_0 if and only if the central extension M/D_0 of $N = M/D$ by D/D_0 corresponding to the cocycle $\pi_*(\eta_N)$: $N \times N \to D/D_0$ splits, i.e., if and only if the group extension

$$1 \to D/D_0 \to M/D_0 \to M/D \to 1 \tag{3.8}$$

splits. Consequently if M/D_0 (which we know is abelian) is isomorphic to $D/D_0 \oplus M/D$, then, upon changing η by a coboundary if necessary, we can choose η so that η_N takes on values in $D_0 \cong \hat{\hat{D}}_0 \subseteq C(\hat{D}_0, \mathbb{T}) \cong p_2^*(C(Z, \mathbb{T}))$, where p_2: $X = \hat{D} \to \hat{D}/G = \hat{D}/G = \hat{D}/R \cong \widehat{R^{\perp}} \cong \hat{D}_0 \cong Z$. It follows that for ω as defined in (3.5), $\omega|_{N\times N}$ will also be (cohomologous to) a cocycle taking on its values in $p_2^*(C(Z, \mathbb{T}))$, as we desired to show. Of course if $N = M/D$ is torsion free (hence by assumption isomorphic to $\mathbb{Z}^m$ for some $m \in \mathbb{Z}^+$) then it will always be true that M/D_0 will split as $D/D_0 \oplus M/D$, so that it will always be true that ω can be chosen so that $\omega|_{N\times N}$ takes on its values in $p_2^*(C(Z, \mathbb{T}))$. This is consistent with the results of Remark 2.7.

Remark 3.2. It follows from the above proposition that if (Γ, μ) is a group-multiplier pair where Γ is a finitely generated nilpotent torsion free two-step nilpotent group and if there exists a central subgroup D containing the commutator subgroup for Γ such that D satisfies (3.1) and (3.2), then we can apply Theorem 2.2 to construct a continuous trace C^*-algebra B with spectrum $\hat{D}_0$ and an action β of $M/D = N$ on B such that the induced action $\hat{\beta}$ is trivial on the spectrum Z and $B \times_\beta N$ is strongly Morita equivalent to $C^*(\Gamma, \mu)$. In general, the Dixmier-Douady class of

B in $\hat{H}^2(Z, \mathcal{S})$ and the Phillips-Raeburn obstruction $[\gamma] \in \hat{H}(Z, \hat{\mathcal{N}})$ associated to $[(B, \beta, N)] \in \mathrm{Br}_N(Z)$ can be non-trivial, as we will see in upcoming examples. For the next few results, however, we concentrate on finding conditions under which $C^*(\Gamma, \mu)$ will be strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$, where Γ_0 is a subgroup Γ of finite index and $\mu_0 = \mu|_{\Gamma_0 \times \Gamma_0}$.

COROLLARY 3.3. *Let Γ be a finitely generated torsion free two-step nilpotent group, let $[\mu] \in H^2(\Gamma, \mathbb{T})$, and suppose there is a central subgroup D of Γ containing the commutator subgroup satisfying* (3.1) *and* (3.2), *and such that M/D is free abelian* (*so that* (3.3) *is also satisfied*). *Then there is a subgroup Γ_0 of finite index in Γ such that, defining $\mu_0 = \mu|_{\Gamma_0 \times \Gamma_0}$, $[\mu_0]$ is in the path component of the identity in $H^2(\Gamma_0, \mathbb{T})$.*

Proof. The proof of Proposition 3.2 shows that we can define Γ_0 to be the central extension of $M/D = N$ by D_0 defined by $\eta_N \colon N \times N \to D_0$ which fits into the exact sequence

$$1 \to D_0 \to \Gamma_0 \to M/D \to 1.$$

Note $\Gamma_0 \subseteq M$. Since $\mu(d, m)\overline{\mu(m, d)} = 1 \ \forall m \in M, \forall d \in D$ by definition of M, it follows that $\mu_0(d, \gamma)\overline{\mu_0(\gamma, d)} = 1, \forall d \in D_0, \forall \gamma \in \Gamma_0$ so that $\mu_0 = \mathrm{Inf}(\widetilde{\mu}_0)$ for $\widetilde{\mu}_0 \in Z^2(M/D, \mathbb{T})$. But if M/D is free abelian, $H^2(M/D, \mathbb{T})$ is path-connected. Consequently $[\widetilde{\mu}_0]$ is in the path component of the identity in $H^2(M/D, \mathbb{T})$, so that $[\mu_0]$ is in the path component of the identity in $H^2(\Gamma_0, \mathbb{T})$.

COROLLARY 3.4. *Let Γ be a finitely generated torsion free two-step nilpotent group and $[\mu] \in H^2(\Gamma, \mathbb{T})$. Suppose there exists a central subgroup $D \subseteq \Gamma$ containing the commutator subgroup of Γ such that* (3.1)–(3.3) *are satisfied, and suppose the multiplier $[\omega] \in H^2(G, C(X, \mathbb{T}))$ defined in* (3.5) *satisfies conditions* (*i*) *and* (*ii*) *of Theorem* 2.6. *Then $C^*(\Gamma, \mu)$ is strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$, where Γ_0 is a subgroup of Γ of finite index and $\mu_0 = \mu|_{\Gamma_0 \times \Gamma_0}$. If in addition, using the notation of Corollary* 3.3, *M/D is free abelian* , *$C^*(\Gamma, \mu)$ is KK-equivalent to $C^*(\Gamma_0)$.*

Proof. By Proposition 3.1 we can write $C^*(\Gamma, \mu)$ as $C(X) \times_{\tau,\omega} G$, where $G = \Gamma/D$, $X = \hat{D}$, and $[\omega]$ is as defined in (3.5). Furthermore from the results of Prop 3.1 and by hypothesis, this transformation group C^*-algebra satisfies all the conditions of Theorem 2.6, so that $C_0(X) \times_{\tau,\omega} G$ is strongly Morita equivalent to $C_0(Z) \times_{\mathrm{Id},\omega_N} N \cong C_0(\hat{D}_0) \times_{\mathrm{Id},\omega_N} N$. But as in the proof of Proposition 3.1, $C_0(\hat{D}_0) \times_{\mathrm{Id},\omega_N} N$ is isomorphic to $C^*(\Gamma_0, \mu_0)$, where Γ_0 is the central extension of $N = M/D$ by D_0 corresponding to the two-cocycle $\eta_N \in Z^2(N, D_0)$. Hence we have shown that $C^*(\Gamma, \mu)$ is strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$. Finally, if we assume N is free abelian, then by Corollary 3.3, $[\mu_0]$ is in the same path component as $[1]_{H^2(\Gamma_0,\mathbb{T})} \in H^2(\Gamma_0, \mathbb{T})$, so that by [PR 3, Cor 2.8], $C^*(\Gamma_0, \mu_0)$ is KK-equivalent to $C^*(\Gamma_0)$. Hence $C^*(\Gamma, \mu)$ is KK-equivalent to $C^*(\Gamma_0)$.

Example 3.5. Let Γ be a lattice in the $(2n+1)$-dimensional simply connected Heisenberg Lie group for $n \geq 2$ and let $[\mu]$ be any multiplier of Γ (the structure of the lattice subgroups Γ and multipliers $[\mu]$ were discussed in [LP1]). The center $\mathcal{Z}$ of Γ is isomorphic to $\mathbb{Z}$ so that $\Gamma/\mathcal{Z} \cong \mathbb{Z}^{2n}$. Since $H^2(\mathcal{Z}, \mathbb{T}) = H^2(\mathbb{Z}, \mathbb{T})$ is trivial, without loss of generality we can assume that $\mu|_{\mathcal{Z}\times\mathcal{Z}} = 1$, so that as in Proposition 3.1, $C^*(\Gamma, \mu)$ decomposes as $C(\mathbb{T}) \times_{\tau,\omega} \mathbb{Z}^{2n}$, where the action of τ corresponds to translation coming from the homomorphism $\Phi_Z(\mu)$: $\Gamma \to \hat{\mathcal{Z}} = \mathbb{T}$ which factors through $\Gamma/\mathcal{Z} = \mathbb{Z}^{2n}$. Let $M = \ker \phi_{\mathcal{Z}}(\mu)$ and set $N = M/\mathcal{Z}$. The range R of $\phi_{\mathcal{Z}}(\mu)$ is finite for dimension $n \geq 2$ [LP1], i.e., $R = Z_0^{\perp}$ where Z_0 is a finite index subgroup of $\mathcal{Z}$. Since $M/\mathcal{Z}$ is torsion free, we have a splitting

$$M/Z_0 = \mathcal{Z}/Z_0 \oplus M/\mathcal{Z}$$

so that conditions (3.2) and (3.3) listed in the first part of this section are also satisfied. Thus, we can apply Proposition 3.1 and Theorem 2.6 to deduce that $C^*(\Gamma, \mu)$ is strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$ if we can show that the multiplier $[\omega] \in H^2(\mathbb{Z}^{2n}, C(\mathbb{T}, \mathbb{T}))$ defined in (3.5) satisfies conditions (i) and (ii) of Theorem 2.6. Now (i) is satisfied automatically, since $\check{H}^2(Z, \mathcal{S}) = \check{H}^3(\mathbb{T}, \mathbb{Z}) = \{0\}$. We thus consider whether (ii) is satisfied, i.e., whether or not $[p_1^*(\lambda)] \in H^1(G, C(Y, \hat{N}))$ defined by

$$[p_1^*(\lambda)](g)(y)(n) = \omega(g,n)(p_1(y))\overline{\omega(n,g)(p_1(y))}, \quad g \in G, n \in N, y \in Y,$$

is trivial. Now from the exact sequence (2.21), the map i: $H^1(G, C(Y, \hat{N})) \to \Lambda_{\text{pt}}(G \times N, N; C(Y, \mathbb{T}))$ is injective, and by Theorem 1.5, $F_{(Y, G\times N)}$: $\Lambda_{\text{pt}}(G \times N, N; C(Y, \mathbb{T})) \to \check{H}^1(Z, \hat{\mathcal{N}})$ is injective. Thus, $F_{(Y,G\times N)} \circ i$: $H^1(G, C(Y, \hat{N})) \to \check{H}^1(Z, \hat{\mathcal{N}})$ is injective. But as $Z \cong \mathbb{T}$ and $\hat{N} \cong \hat{\mathcal{Z}}^{2n} = \mathbb{T}^{2n}$, we have $\check{H}^1(Z, \hat{N}) \cong \check{H}^1(\mathbb{T}, \mathcal{S}^{2n}) \cong \check{H}^2(\mathbb{T}, \mathbb{Z}^{2n}) \cong \{0\}$, and it follows from the vanishing of cohomology groups that $[p_1^*(\lambda)]$ must be trivial. We thus obtain another proof of the following result, which was first proved in [LP2]:

COROLLARY 3.6. *Let Γ be a lattice in the $(2n+1)$ dimensional simply connected Heisenberg Lie group and let $[\mu] \in H^2(\Gamma, \mathbb{T})$. Then there is a subgroup Γ_0 of Γ of finite index such that the twisted group C^*-algebra $C^*(\Gamma, \mu)$ is strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$, where $[\mu_0] = [\mu/_{\Gamma_0\times\Gamma_0}]$ is in the path component of the identity in $H^2(\Gamma_0, \mathbb{T})$.*

Proof. If $n = 1$, it follows from [LP1, Theorem 3.6] that every multiplier on Γ is homotopic to the identity, so that there is nothing to prove. If $n \geq 2$, we have shown in the above analysis that $C^*(\Gamma, \mu)$ is strongly Morita equivalent to $C^*(\Gamma_0, \mu_0)$, where Γ_0 is the central extension of $M/\mathcal{Z}$ by Z_0, for $M = \ker \phi_{\mathcal{Z}}(\mu)$. Since $M/\mathcal{Z}$ is free abelian, by Corollary 3.3, $[\mu_0]$ is in the path component of the identity in $H^2(\Gamma_0, \mathbb{T})$.

Next, we give several examples where the invariants do not all vanish, yet are computable.

Example 3.7. Let Γ be the following two step nilpotent group of rank 5: Γ is a central extension of $\mathbb{Z}^3$ by $\mathbb{Z}^2$ corresponding to the cohomology class $[\eta] \in H^2(\mathbb{Z}^3, \mathbb{Z}^2)$ given by

$$\eta((x_1, x_2, x_3), (x_1', x_2', x_3')) = (ax_3x_1', dx_3x_2'), \text{ where } a, d \in \mathbb{N}, a/d.$$

As a set, Γ can be identified with $\mathbb{Z}^2 \times \mathbb{Z}^3$. Define $\mu \in Z^2(\Gamma, \mathbb{T})$ by

$$\mu((m_1, m_2, x_1, x_2, x_3), (m_1', m_2', x_1', x_2', x_3')) = e^{\frac{2\pi i}{a} x_2 m_1'},$$
$$(m_1, m_2, x_1, x_2, x_3), (m_1', m_2', x_1', x_2', x_3') \in \Gamma.$$

For $a > 1$ it can be verified that $[\mu]$ is not in the path component of the identity element in $H^2(\Gamma, \mathbb{T})$. By Proposition 3.1, we can write $C^*(\Gamma, \mu) \cong C(\mathbb{T}^2) \times_{\tau,\omega} \mathbb{Z}^3$, where $\tau\colon \mathbb{Z}^3 \to \mathbb{T}^2$ is given by

$$\tau(x_1, x_2, x_3) = (e^{2\pi \frac{i}{a} x_2}, 1),$$

and $\omega\colon \mathbb{Z}^3 \times \mathbb{Z}^3 \to C(\mathbb{T}^2, \mathbb{T})$ is defined by

$$\omega((x_1, x_2, x_3), (x_1', x_2', x_3'))(z_1, z_2) = (z_1)^{ax_3x_1'}(z_2)^{dx_3x_2'}.$$

Through direct calculation we check that condition (3.3) is satisfied, so that by Proposition 3.1 again, with $G = \mathbb{Z}^3$ and $N = \mathbb{Z} \oplus a\mathbb{Z} \oplus \mathbb{Z}$, we can find a principal G-bundle $Y = \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \times \mathbb{Z}$ over $\mathbb{Z} = \mathbb{T}^2/G$ (which can also be identified with $\mathbb{T}^2$), where the action of $G = \mathbb{Z}^3$ on Y is given by

$$(r, z, n_1, n_3) \cdot (x_1, x_2, x_3) = (r + x_2, z, n_1 + x_1, n_3 + x_3),$$
$$(r, z, n_1, n_3) \in Y, (x_1, x_2, x_3) \in G.$$

The map $p_1\colon Y \to X = Y/N = \mathbb{T}^2$ is given by

$$p_1(r, z, n_1, n_3) = (e^{2\pi i \frac{r}{a}}, z).$$

The map $p_2\colon X = \mathbb{T}^2 \to Z = \mathbb{T}^2$ is given by $p_2(z_1, z_2) = (z_1^a, z_2)$, so that $p_3 = p_2 \circ p_1\colon Y \to Z$ is given by $p_3(r, z, n_1, n_3) = (e^{2\pi i r}, z)$. Using the notation of Theorem 2.6, one computes that $[p_1^*(\omega)] \in H^2(G, C(Y, \mathbb{T}))$ is defined by

$$[p_1^*(\omega)]((x_1, x_2, x_3), (x_1', x_2', x_3'))(r, z, n_1, n_3) = e^{2\pi i r a x_3 x_1'} z^{dx_3x_2'},$$
$$(x_1, x_2, x_3), (x_1', x_2', x_3') \in G, (r, z, n_1, n_2) \in Y.$$

and $[p_1^*(\lambda)] \in H^1(G, H^1(N, C(Y, \mathbb{T}))$ is defined by

$$[p_1^*(\lambda)]((x_1, x_2, x_3), (j_1, j_2, j_3))((r, z, n_1, n_3)) = (e^{2\pi i r x_3})^{j_1} (z^{dax_3})^{j_2} (e^{-2\pi i r x_1} z^{-dx_2})^{j_3}.$$

Identifying $\hat{N} \cong \hat{\mathbb{Z}}^3$ with $\mathbb{T}^3$, we can write $H^1(G, H^1(N, C(Y, \mathbb{T}))) \cong H^1(G, C(Y, \hat{N})) \cong H^1(G, C(Y, \mathbb{T}^3))$ and with respect to this identification, we can view $p_1^*(\lambda)$ as being defined by

$$p_1^*(\lambda)((x_1, x_2, x_3), (r, z, n_1, n_3)) = (e^{2\pi i r x_3}, z^{d a x_3}, e^{-2\pi i r x_1} z^{-d x_2}),$$
$$(x_1, x_2, x_3) \in G,\ (r, z, n_1, n_3) \in Y.$$

We note now that as in Example 3.5, condition (i) of Theorem 2.6 is automatically satisfied since $\check{H}^2(Z, \mathcal{S}) \cong \check{H}^2(\mathbb{T}^2, \mathcal{S}) = \check{H}^3(\mathbb{T}^2, \mathbb{Z}) = \{0\}$. However condition (ii) does not hold. We can check that if $f\colon Y \to \mathbb{T}^3$ is defined by $f = (f_1, f_2, f_3)$ where the maps $f_i\colon Y \to \mathbb{T}$, $i = 1, 2, 3$ are given by the formulas

$$\begin{aligned} f_1(r, z, n_1, n_3) &= e^{2\pi i r n_3}, \\ f_2(r, z, n_1, n_3) &= z^{d a n_3}, \\ f_3(r, z, n_1, n_3) &= e^{-2\pi i r n_1}, \end{aligned}$$

then

$$p_1^*(\lambda) df((x_1, x_2, x_3), (r, z, n_1, n_3)) = (1, 1, z^{-d x_2}),$$
$$(x_1, x_2, x_3) \in G,\ (r, z, n_1, n_3) \in Y.$$

Let $[\rho] = [p_1^*(\lambda) df] \in H^1(G, C(Y, \mathbb{T}^3))$.

By using the definition of the bundle $F_{(Y, G\times N)}(i^*[\rho])$ given in [RW1] and the method outlined in [PR2, Lemma 3.2], we verify that if $\{N_i\}_{i=1} \subseteq Z = \mathbb{T}^2$ is a local trivialization of $p_3\colon Y \to Z$ and $c_i\colon N_i \to Y$ are sections, with $c_i(z) = c_j(z)\nu_{ij}(z)$ where $\nu_{ij}\colon N_{ij} \to G$, then the transition functions for the $\hat{N}$-bundle over Z represented by $F_{(Y, G\times N)}(i^*([\rho]))$ are

$$\lambda_{ij}(z) = [\rho(\nu_{ij}(z), c_i(z))]^{-1}, z \in N_{ij}.$$

It is evident that the $G = \mathbb{Z}^3$-bundle Y over $Z = \mathbb{T}^2$ is the product of the non-trivial $\mathbb{Z}$-bundle $\mathbb{R}$ over $\mathbb{T}$ and a trivial $\mathbb{Z}^2$-bundle over the second factor of $\mathbb{T}$, so that we can write $N_{ij} = N'_{ij} \times \mathbb{T}$ where $\{N_{ij}\} \subseteq \mathbb{T}$ is a local trivialization of the projection $p\colon \mathbb{R} \to \mathbb{T}$. Using this notation we have

$$\nu_{ij}((z_1, z_2)) = (0, \nu'_{ij}(z_1), 0), (z_1, z_2) \in N_{ij}$$

where $\nu'_{ij}\colon N'_{ij} \to \mathbb{Z}$ are the transition functions associated to the bundle $p\colon \mathbb{R} \to \mathbb{T}$ via the formula $c'_i(z_1) = \nu'_{ij}(z_1)c'_j(z_1)$, and where, as usual, $c_i\colon N'_i \to \mathbb{R}$ are local sections. Using this notation, we see that for $(z_1, z_2) \in N_{ij} = N'_{ij} \times \mathbb{T}$ we have

$$\begin{aligned} \lambda_{ij}((z_1, z_2)) &= [\rho((0, \nu'_{ij}(z_1), 0), (c'_i(z_1), z_2, 0, 0))]^{-1} \\ &= (1, 1, z_2^{d\nu'_{ij}(z_1)}). \end{aligned}$$

By the method outlined in Lemmas 3.2 and 3.3 of [LP2], the cocycle $\{\lambda_{ij}^{(l)}\} \in \check{H}^1(\mathbb{T}^2, \mathcal{S}) = \check{H}^2(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}$ defined by the formulas

$$\{\lambda_{ij}^{(l)}((z_1, z_2))\} = z_2^{\nu'_{ij}(z_1)}, \ (z_1, z_2) \in N_{ij},$$

corresponds to the element $e_2 \wedge e_1 = -e_1 \wedge e_2$ in $H^2(\mathbb{T}^2, \mathbb{Z}) \cong \Lambda^2(\mathbb{Z}^2, \mathbb{Z})$ (recall $e_1 \wedge e_2$ is the standard generator of $H^2(\mathbb{T}^2, \mathbb{Z})$), hence $\{\lambda_{ij}\} \in \check{H}^1(\mathbb{T}^2, \mathcal{S}^3) \cong \check{H}^2(\mathbb{T}^2, \mathbb{Z}^3) \cong \mathbb{Z}^3$ can be represented by the element $(0, 0, -de_1 \wedge e_2) = (0, 0, -d)$ (upon viewing $e_1 \wedge e_2$ as a standard generator in the third coordinate). Using the notation of Theorem 2.6, it follows that $d_1([\omega]) = (0, 0, -d) \in \check{H}^1(\mathbb{T}^2, \mathcal{S}^3) \cong H^2(\mathbb{T}^2, \mathbb{Z}^3)$. Finally, one calculates that $d_0([\omega]) = [c_\omega]$ where c_ω: $\mathbb{T}^2 \to Z^2(N, \mathbb{T})$ is defined by

$$c_\omega(z_1, z_2)((j_1, aj_2, j_3), (j'_1, aj'_2, j'_3)) = (z_1)^{j_3 j'_1}(z_2)^{a j_3 j'_2}$$
$$(z_1, z_2) \in Z = \mathbb{T}^2, (j_1, aj_2, j_3), (j'_1, aj'_2, j'_3) \in N.$$

Thus $C^*(\Gamma, \mu)$ is strongly Morita equivalent to a crossed product $(C(\mathbb{T}^2) \otimes \mathcal{K}) \times_\beta N = (C(\mathbb{T}^2) \otimes \mathcal{K}) \times_\beta \mathbb{Z}^3$, where $[(C(\mathbb{T}^2) \otimes \mathcal{K}, \beta, \mathbb{Z}^3)] \in Br_N(Z)$ has the invariants computed above.

Example 3.9. We end the paper by considering a twisted group C^*-algebra associated to a rank six nilpotent discrete group where all of the cohomological invariants coming from Theorem 2.6 are non-trivial. Let Γ be the central extension of $\mathbb{Z}^3$ by $\mathbb{Z}^3$ corresponding to the two-cocycle η: $\mathbb{Z}^3 \times \mathbb{Z}^3 \to \mathbb{Z}^3$ defined by

$$\eta((x_1, x_2, x_3), (x'_1, x'_2, x'_3)) = (2x_3x'_2, 2x_3x'_1, 2x_2x'_1),$$
$$(x_1, x_2, x_3), (x'_1, x'_2, x'_3) \in \mathbb{Z}^3,$$

so that setwise, Γ is identified with $\mathbb{Z}^3 \times \mathbb{Z}^3$:

$$\Gamma = \{(m_1, m_2, m_3, x_1, x_2, x_3): m_i, x_i \in \mathbb{Z}, i = 1, 2, 3\}.$$

Define the multiplier μ: $\Gamma \times \Gamma \to \mathbb{T}$ by

$$\mu((m_1, m_2, m_3, x_1, x_2, x_3), (m'_1, m'_2, m'_3, x'_1, x'_1, x'_2, x'_3)) = (-1)^{x_1 m'_1 + x_2 m'_2 + x_3 m'_3}.$$

Again, one can check that $[\mu]$ is not in the path component of the identity in $H^2(\Gamma, \mathbb{T})$, and it follows from Prop. 3.1 that $C^*(\Gamma, \mu)$ is $*$-isomorphic to the twisted transformation group C^*-algebra $C(\mathbb{T}^3) \times_{\tau,\omega} G$, where $G = \mathbb{Z}^3$, the action τ of G on $C(\mathbb{T}^3)$ corresponds to the homomorphism τ: $G \to \mathbb{T}^3$ given by $\tau(x_1, x_2, x_3) = (e^{\pi i x_1}, e^{\pi i x_2}, e^{\pi i x_3})$, $(x_1, x_2, x_3) \in G$, and the two-cocycle ω: $G \times G \to C(\mathbb{T}^3, \mathbb{T})$ is given by

$$\omega((x_1, x_2, x_3), (x'_1, x'_2, x'_3))(z_1, z_2, z_3) = z_1^{2x_3x'_2} z_2^{2x_3x'_1} z_3^{2x_2x'_1},$$
$$(z_1, z_2, z_3) \in \mathbb{T}^3 = X, (x_1, x_2, x_3), (x'_1, x'_2, x'_3) \in \mathbb{Z}^3 = G.$$

Using the notation of Proposition 3.1, we have $D = \mathbb{Z}^3 = \{(n_1, n_2, n_3), n_i \in \mathbb{Z}, i = 1, 2, 3\}$, and $D_0 = 2\mathbb{Z} \oplus 2\mathbb{Z} \oplus 2\mathbb{Z} = \{(2n_1, 2n_2, 2n_3), n_i \in \mathbb{Z}, i = 1, 2, 3\}$, $Y = \mathbb{R}^3$, with the $G = \mathbb{Z}^3$ action on Y defined by

$$(r_1, r_2, r_3)(x_1, x_2, x_3) = (r_1+x_1, r_2+x_2, r_3+x_3), \quad (r_1, r_2, r_3) \in \mathbb{R}^3, (x_1, x_2, x_3) \in \mathbb{Z}^3.$$

Letting $N = 2\mathbb{Z} \oplus 2\mathbb{Z} \oplus 2\mathbb{Z}$, the maps p_1, p_2 and p_3 are given by

$$p_1\colon Y \to X = Y/N = \mathbb{T}^3,$$

$$p_1(r_1, r_2, r_3) = (e^{\pi i r_1}, e^{\pi i r_2}, e^{\pi i r_3}), \quad (r_1, r_2, r_3) \in \mathbb{R}^3,$$

$$p_2\colon X \to Z = Y/G = \mathbb{T}^3,$$

$$p_2(z_1, z_2, z_3) = (z_1^2, z_2^2, z_3^2), \quad (z_1, z_2, z_3) \in \mathbb{T}^3 = X,$$

so that

$$p_3(r_1, r_2, r_3) = (e^{2\pi_i r_1}, e^{2\pi i r_2}, e^{2\pi_i r_3}), \quad (r_1, r_2, r_3) \in \mathbb{R}^3.$$

Applying the notation of Theorem 2.2 and its subsequent remark, we see that $[p_1^*([\omega])] \in H^2(G, C(Y, \mathbb{T}))$ is given by

$$p_1^*([\omega])((x_1, x_2, x_3), (x_1', x_2', x_3'))(r_1, r_2, r_3) = e^{2\pi i r_1 x_3 x_2'} e^{2\pi i r_2 x_3 x_1'} e^{2\pi i r_3 x_2 x_1'},$$
$$(x_1, x_2, x_3), (x_1', x_2', x_3') \in G, (r_1, r_2, r_3) \in Y.$$

and $d_2([\omega]) \in \check{H}^2(Z, \mathcal{S})$ is defined by the formulas

$$(d_2([\omega]))_{ijk}(z) = \{p_1^*([\omega])(\lambda_{ij}(z), \lambda_{jk}(z))(c_i(z))\}_{ijk}$$

where $\{N_i\} \subseteq \mathbb{T}^3$ is a local trivialization for the bundle $p_3\colon \mathbb{R}^3 \to \mathbb{T}^3$ decribed above, the maps $c_i\colon N_i \to \mathbb{R}^3$ are local sections, and $c_i(z) = \lambda_{ij}(z)c_j(z), z \in N_{ij}$. Now define $\sigma\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ by

$$\sigma((m, n), (m', n')) = nm',$$

let $\{V_i\}$ be a local trivialization for the principal $\mathbb{Z}^2$ bundle $\mathbb{R}^2 \to \mathbb{T}^2$, and let $\epsilon_i\colon V_i \to \mathbb{R}^2$ and $\rho_{ij}\colon V_i \cap V_j = V_{ij} \to \mathbb{Z}^2$ be the corresponding local sections and transition functions. Define $\theta_i\colon \mathbb{T}^3 \to \mathbb{T}^2$ $i = 1, 2, 3$ by

$$\theta_i(z_1, z_2, z_3) = \begin{cases} (z_2, z_3), & i = 1, \\ (z_1, z_3), & i = 2, \\ (z_1, z_2), & i = 3. \end{cases}$$

Let

$$\{N_{ijk}^{(1)} = \theta_1^{-1}(V_{ijk}) \subseteq \mathbb{T}^3\},$$
$$\{N_{ijk}^{(2)} = \theta_2^{-1}(V_{ijk}) \subseteq \mathbb{T}^3\},$$
$$\{N_{ijk}^{(3)} = \theta_3^{-1}(V_{ijk}) \subseteq \mathbb{T}^3\}.$$

Define cocycles $\{\eta_{ijk}^{(1)}\}$, $\{\eta_{ijk}^{(2)}\}$, $\{\eta_{ijk}^{(3)}\} \in \check{H}^2(\mathbb{T}^3, \mathcal{S})$ with respect to the open covers $\{N_{ijk}^{(1)}\}$, $\{N_{ijk}^{(2)}\}$, $\{N_{ijk}^{(3)}\}$ by

$$\eta_{ijk}^{(1)}((z_1, z_2, z_3)) = z_1^{\sigma(\rho_{ij}((z_2,z_3)),\rho_{jk}((z_2,z_3)))}, \quad (z_1, z_2, z_3) \in N_{ijk}^{(1)},$$
$$\eta_{ijk}^{(2)}((z_1, z_2, z_3)) = z_2^{\sigma(\rho_{ij}((z_1,z_3)),\rho_{jk}((z_1,z_3)))}, \quad (z_1, z_2, z_3) \in N_{ijk}^{(2)},$$
$$\eta_{ijk}^{(3)}((z_1, z_2, z_3)) = z_3^{\sigma(\rho_{ij}((z_1,z_2)),\rho_{jk}((z_1,z_2)))}, \quad (z_1, z_2, z_3) \in N_{ijk}^{(3)}.$$

By [LP2, Lemmas 3.2 and 3.3], the cocycles $\{\eta_{ijk}^{(1)}\}$, $\{\eta_{ijk}^{(2)}\}$, $\{\eta_{ijk}^{(3)}\}$ correspond to the elements $e_1 \wedge e_2 \wedge e_3$, $e_2 \wedge e_1 \wedge e_3$ and $e_3 \wedge e_1 \wedge e_2$ in $\Lambda^3(\mathbb{Z}^3, \mathbb{Z}) \cong \check{H}^3(\mathbb{T}^3, \mathbb{Z}) \cong \check{H}^2(\mathbb{T}^3, \mathcal{S})$, and by passing to refinements, one can verify that the product $[\{\eta_{ijk}^{(1)}\}] \cdot [\{\eta_{ijk}^{(2)}\}] \cdot [\{\eta_{ijk}^{(3)}\}]$ is cohomologous to $[(d_2([\omega]))_{ijk}] \in \check{H}^2(\mathbb{T}^3, \mathcal{S})$. Hence $[(d_2([\omega]))_{ijk}]$ can be represented by the invariant $e_1 \wedge e_2 \wedge e_3 + e_2 \wedge e_1 \wedge_3 + e_3 \wedge e_1 \wedge e_2 = e_1 \wedge e_2 \wedge e_3 \in \check{H}^3(\mathbb{T}^3, \mathbb{Z}) \cong \mathbb{Z}$ (recall that $e_1 \wedge e_2 \wedge e_3$ is the standard generator for $\check{H}^3(\mathbb{T}^3, \mathbb{Z})$). We now compute $d_1([\omega])$ and show that it also is non-trivial. We calculate $[p_1^*(\lambda)] = H^1(G, H^1(N, C(Y, \mathbb{T}))$ as follows:

$$p_1^*(\lambda)((x_1, x_2, x_3), (2n_1, 2n_2, 2n_3))(r_1, r_2, r_3)$$
$$= e^{2\pi i \cdot 2(r_2x_3+r_3x_2)n_1} e^{2\pi i \cdot 2(r_1x_3-r_3x_1)n_2} e^{-2\pi i \cdot 2(r_1x_2+r_2x_1)n_3},$$
$$(x_1, x_2, x_3) \in G, \ (2n_1, 2n_2, 2n_3) \in N, \ (r_1, r_2, r_3) \in Y.$$

As in Example 3.5 we identify $H^1(G, H^1(N, C(Y, \mathbb{T}))$ with $H^1(G, C(Y, \hat{N})) = H^1(\mathbb{Z}^3, C(\mathbb{R}^3, \mathbb{T}^3))$ to get

$$p_1^*(\lambda)((x_1, x_2, x_3), (r_1, r_2, r_3))$$
$$= (e^{2\pi i \cdot 2(r_2x_3+r_3x_2)}, e^{2\pi i 2(r_1x_3-r_3x_1)}, e^{-2\pi i \cdot 2(r_1x_2+r_2x_1)},$$
$$(x_1, x_2, x_3) \in \mathbb{Z}^3, \ (r_1, r_2, r_3) \in \mathbb{R}^3 = Y.$$

Now define $f: \mathbb{R}^3 \to \mathbb{T}^3$ by

$$f(r_1, r_2, r_3) = (e^{2\pi i 2r_2r_3}, 1, e^{-2\pi i 2r_1r_2}).$$

Again, one checks that

$$p_1^*(\lambda) \cdot df((x_1, x_2, x_3), (r_1, r_2, r_3)) = (1, e^{2\pi_i 2(r_1x_3-r_3x_1)}, 1).$$

Then methods similar to those outlined in Example 3.5 allow one to calculate that $F_{(Y,G\times N)}(i_*([p_1^*(\lambda)])) \in \check{H}^1(\mathbb{T}^3, \mathcal{S}^3) \cong \check{H}^2(\mathbb{T}^3, \mathbb{Z}^3)$ is identified with the element $(0, -4e_1 \wedge e_3, 0)$ upon using the identification of $\check{H}^2(\mathbb{T}^3, \mathbb{Z}^3)$ with $\Lambda^2(\mathbb{Z}^3, \mathbb{Z}) \oplus \Lambda^2(\mathbb{Z}^3, \mathbb{Z}) \oplus \Lambda^2(\mathbb{Z}^3, \mathbb{Z})$. (Recall that $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_2 \wedge e_3$ are the standard generators for $\Lambda^2(\mathbb{Z}^3, \mathbb{Z})$.) Hence $d_1([\omega]) = (0, -4e_1 \wedge e_3, 0)$ and is also nontrivial. Finally, one calculates $d_0([\omega]) = [c_\omega]$ where $c_\omega\colon \mathbb{T}^3 \to Z^2(N, \mathbb{T})$ is defined by

$$c_\omega(z_1, z_2, z_3)((2n_1, 2n_2, 2n_3), (2n_1', 2n_2', 2n_3')) = z_1^{4n_3n_2'} z_2^{4n_3n_1'} z_3^{4n_2n_1'}$$

$$(2n_1, 2n_2, 2n_3), (2n_1', 2n_2', 2n_3') \in N, (z_1, z_2, z_3) \in Z \cong \mathbb{T}^3.$$

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Department of Mathematics, National University of Singapore, Singapore 119260, Republic of Singapore
matjpj@leonis.nus.edu.sg