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# UPPER AND LOWER MULTIPLICITY FOR IRREDUCIBLE REPRESENTATIONS OF SIN-GROUPS

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ABSTRACT. The main purpose of the paper is to establish formulae for both the upper and the lower multiplicity of an irreducible representation of a Moore group. Moreover, we show that for a group with small invariant neighbourhoods of the identity, the set of irreducible representations with finite upper multiplicity coincides with the dual of a quotient group.

Motivated by examples in group $C^*$-algebras, Archbold [1] defined the upper and lower multiplicities $M_U(\pi)$ and $M_L(\pi)$ for an arbitrary irreducible representation $\pi$ of a $C^*$-algebra $A$. Similarly, one can define upper and lower multiplicities for $\pi$ relative to a net $\Omega$ in the dual space $\widehat{A}$ of $A$: $M_U(\pi, \Omega)$ and $M_L(\pi, \Omega)$ [2]. Since $M_U(\pi) = 1$ if and only if $\pi$ satisfies Fell's condition [1, Theorem 4.1], multiplicities may be regarded as a measure of the extent to which Fell's condition may fail for $\pi$.

Using the main result of [11], it has been shown in [3, Corollary 2.9] that if $G$ is a simply connected nilpotent Lie group and $\pi \in \widehat{G}$, then $M_U(\pi) < \infty$ if and only if the Kirillov orbit associated to $\pi$ has maximal dimension. Furthermore, Ludwig [10] has found non-trivial finite multiplicity in an explicit example for which it can be shown that $M_U(\pi) = 2$. On the other hand, in the case of the discrete space group $G = p4gm$, Raeburn [16] has described $C^*(G)$ in a way that enables one to compute multiplicities: there are three irreducible representations $\pi$ with $M_U(\pi) = 2$ and $M_L(\pi) = 1$, whereas $M_U(\pi) = 1$ for all other $\pi$. We shall see below that this example fits into a general framework in which multiplicities may be computed without recourse to a description of the group $C^*$-algebra.

In this paper we study $M_U(\pi)$ and $M_L(\pi)$ for $\pi \in \widehat{G}_r$, the reduced dual of a SIN-group $G$. To begin with, in Sections 2 and 3, we concentrate on Moore groups (groups with finite dimensional irreducible representations). In this case, $M_U(\pi)$ (and hence $M_L(\pi)$) is finite and we obtain formulae for both $M_U(\pi)$ and $M_L(\pi)$ in terms of the multiplicity of $\pi$ in certain induced representations (Theorems 2.1 and 3.6). Furthermore, the condition $M_U(\pi) = 1$ is characterized by an equation involving the dimension of $\pi$. This emphasis on Moore groups is explained by Theorem 4.3 where we show that if $M_U(\pi) < \infty$ for at least one $\pi$ in the reduced dual of a SIN-group $G$, then $G$ contains a compact normal subgroup $K$ such that $G/K$ is a Moore group and

$$\widehat{G/K} = \left\{\rho \in \widehat{G}\colon\ M_U(\rho) < \infty\right\} = \left\{\rho \in \widehat{G}\colon\ M_L(\rho) < \infty\right\}.$$

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## 1. Preliminaries

The formal definitions of multiplicity involve pure states and describe situations in which several nets of orthogonal equivalent pure states converge to a common pure limit [3, Lemma 5.2]. The following result from [3], which is a key tool in the subsequent work of this paper, reflects the relation between orthogonal vector states and the trace of an operator on Hilbert space.

THEOREM 1.1 ([3, Theorem 4.1]). *Let $A$ be a $C^*$-algebra, let $\Omega = (\pi_\alpha)_\alpha$ be a net in $\widehat{A}$ and let $F$ be a nonempty subset of $\widehat{A}$. Suppose that there exist positive integers $m_\pi$ ($\pi \in F$) and a dense self-adjoint subalgebra $B$ of $A$ such that*

$$\lim_\alpha \operatorname{tr}(\pi_\alpha(a)) = \sum_{\pi \in F} m_\pi \operatorname{tr}(\pi(a)) < \infty$$

*for all $a \in B^+$. Then*

(i) *$\Omega$ is convergent to every element of $F$ and every cluster point of $\Omega$ belongs to $F$,*
(ii) *the relative topology on $F$ is discrete,*
(iii) *$m_\pi = M_U(\pi, \Omega) = M_L(\pi, \Omega)$ for all $\pi \in F$.*

In such situations where $M_U(\pi, \Omega) = M_L(\pi, \Omega)$, we write $M(\pi, \Omega)$ for the common value. In the context of Theorem 1.1 we may say informally that $\Omega$ converges $m_\pi$ times to each $\pi \in F$.

Before applying Theorem 1.1 in Section 2, we need to show that $M_U(\pi)$ can be attained by approximating $\pi$ by a net lying in a prescribed dense subset of $\widehat{A}$. This fact is expected to have wider applications and so we organize the proof to show that a sequence can be used if $A$ is separable.

Let $\varphi$ be a pure state associated with an irreducible representation $\pi$ of a $C^*$-algebra $A$, and let $\mathcal{N}$ be the weak*-neighbourhood base at zero in the Banach dual $A^*$ consisting of all open sets of the form

$$N = \{\psi \in A^*\colon |\psi(a_i)| < \varepsilon, 1 \le i \le n\},$$

where $\varepsilon > 0$ and $a_1, \dots, a_n \in A$. We define

$$V(\varphi, N) = \left\{\sigma \in \widehat{A}\colon \langle \sigma(\cdot)\eta, \eta\rangle \in \varphi + N \text{ for some } \eta \in \mathcal{H}_\sigma, \|\eta\| = 1\right\}.$$

Let $P(A)$ denote the set of pure states of $A$. Then $V(\varphi, N)$ is the image of $(\varphi + N) \cap P(A)$ under the canonical map from $P(A)$ to $\widehat{A}$ and hence is an open neighbourhood of $\pi$ in $\widehat{A}$. For $\sigma \in V(\varphi, N)$ let

$$\operatorname{Vec}(\sigma, \varphi, N) = \{\eta \in \mathcal{H}_\sigma\colon \|\eta\| = 1, \langle \sigma(\cdot)\eta, \eta\rangle \in \varphi + N\},$$

and let $d(\sigma, \varphi, N)$ be the supremum in $\mathbb{N} \cup \{\infty\}$ of the cardinalities of finite orthonormal subsets of $\operatorname{Vec}(\sigma, \varphi, N)$.

LEMMA 1.2. *Let $A$ be a $C^*$-algebra, let $S$ be a dense subset of $\widehat{A}$ and let $\pi \in \widehat{A}$. Then there exists a net $\Omega$ in $S$ which converges to $\pi$ and satisfies $M_U(\pi) = M(\pi, \Omega)$. If $A$ is separable, then the net $\Omega$ can be chosen to be a sequence.*

*Proof.* Let $\varphi$ be a pure state of $A$ associated with $\pi$. By [1, Proposition 3.4],

$$M_U(\pi) = \inf_{N \in \mathcal{N}} \sup\{d(\sigma, \varphi, N) \colon \sigma \in V(\varphi, N) \cap S\}. \tag{1}$$

Let $R = \{k \in \mathbb{N} \colon k \leq M_U(\pi)\}$ and let $\Lambda = \mathcal{N} \times R$ with the product order. For each $(N, k) \in \Lambda$, equation (1) enables us to choose $\sigma_{(N,k)} \in V(\varphi, N) \cap S$ such that

$$d(\sigma_{(N,k)}, \varphi, N) \geq k. \tag{2}$$

Let $\Omega = (\sigma_{N,k)})_{(N,k) \in \Lambda}$. For $N_0 \in \mathcal{N}$ and $k_0 \in R$, by definition and by (2), we then have

$$\begin{aligned} M_L(\varphi, N_0, \Omega) &= \liminf_{(N,k)} d(\sigma_{(N,k)}, \varphi, N_0) \\ &\geq \liminf_{(N,k) \geq (N_0, k_0)} d(\sigma_{(N,k)}, \varphi, N) \geq k_0. \end{aligned}$$

Since $k_0$ is arbitrary in $R$, $M_L(\varphi, N_0, \Omega) \geq M_U(\pi)$. Taking the infimum over $N_0 \in \mathcal{N}$, we obtain

$$M_U(\pi) \leq M_L(\pi, \Omega) \leq M_U(\pi, \Omega) \leq M_U(\pi)$$

and hence $M_U(\pi) = M(\pi, \Omega)$.

To show that $\Omega$ is convergent to $\pi$, let $U$ be a neighbourhood of $\pi$ in $\widehat{A}$. Since the canonical map from $P(A)$ to $\widehat{A}$ is continuous, there exists $N_0 \in \mathcal{N}$ such that $V(\varphi, N_0) \subseteq U$. For $(N, k) \geq (N_0, 1)$ we then have

$$\sigma_{(N,k)} \in V(\varphi, N) \subseteq V(\varphi, N_0) \subseteq U.$$

Finally, suppose that $A$ is separable. Then there exists a decreasing sequence $(N_j)_{j \geq 1}$ in $\mathcal{N}$ such that $\{(\varphi + N_j) \cap P(A) \colon j \geq 1\}$ is a neighbourhood base for $\varphi$ in $P(A)$. For $n \geq 1$, let $\sigma_n = \sigma_{(N,k)}$ where $N = N_n$ and $k = \min\{n, M_U(\pi)\}$. Then the sequence $(\sigma_n)_n$ has the required properties. $\square$

We now have to introduce some notation and basic facts from representation theory. As is customary, we shall use the same letter, for example $\pi$, to denote a unitary representation of a locally compact group $G$ and the associated $*$-representation of the group $C^*$-algebra $C^*(G)$. Then $\ker \pi$ will denote the $C^*$-kernel of $\pi$. If $R$ and $S$ are sets of unitary representations of $G$, then $R$ is weakly contained in $S$ ($R \prec S$) if $\bigcap_{\rho \in R} \ker \rho \supseteq \bigcap_{\sigma \in S} \ker \sigma$, and $R$ and $S$ are weakly equivalent ($R \sim S$) if $R \prec S$ and $S \prec R$. Let $\rho$ and $\sigma$ be finite dimensional representations. Then $\rho \prec \sigma$ if and only if every irreducible subrepresentation $\omega$ of $\rho$ is a subrepresentation of $\sigma$: $\omega \leq \sigma$.

If, in addition, $\rho$ is irreducible, then $m(\rho, \sigma)$ will denote the multiplicity of $\rho$ as a subrepresentation of $\sigma$.

For a closed subgroup $H$ of $G$ and a representation $\tau$ of $H$, $\mathrm{ind}_H^G \tau$ is the representation of $G$ induced by $\tau$. We shall frequently use the following version of the Frobenius reciprocity theorem. Suppose that $H$ has finite index in $G$ ($[G : H] < \infty$) and let $\pi$ and $\tau$ be finite dimensional irreducible representations of $G$ and $H$, respectively. Then $m(\pi, \mathrm{ind}_H^G \tau) = m(\tau, \pi \mid H)$ (see [13] and [15, p. 135]).

Our second basic tool is the so-called Mackey machine. Let $N$ be a normal subgroup of finite index in $G$ and suppose that all the irreducible representations of $G$ are finite dimensional. $G$ acts on the dual $\widehat{N}$ of $N$ by $(a, \sigma) \to \sigma^a$, where $\sigma^a(n) = \sigma(a^{-1}na)$ $(n \in N)$. For $\sigma \in \widehat{N}$, let $G(\sigma)$ denote the orbit under this action and $S_\sigma$ the stabilizer of $\sigma$. Let $\tau \in \widehat{S_\sigma}$ such that $\tau|N \geq \sigma$; then $\mathrm{ind}_{S_\sigma}^G \tau \in \widehat{G}$ and $(\mathrm{ind}_{S_\sigma}^G \tau)|N \sim G(\sigma)$. Conversely, given $\pi \in \widehat{G}$ with $\pi|N \geq \sigma$, there exists a unique $\tau \in \widehat{S_\sigma}$ such that $\tau|N \geq \sigma$ and $\pi = \mathrm{ind}_{S_\sigma}^G \tau$. For all this see [12] (the separability hypothesis is not required when $N$ is of finite index and all irreducible representations are finite dimensional).

## 2. Upper multiplicity for Moore groups

In this section we shall establish a simple formula as well as some applications of it for the upper multiplicity of an irreducible representation of a locally compact group all of whose irreducible representations are finite dimensional. Such groups have been completely characterized by Moore [14] and are therefore usually referred to as *Moore groups*. By [14, Theorem 2 and Theorem 3] a locally compact group $G$ is a Moore group if and only if $G$ is a projective limit of groups each of which is a finite extension of a group with cocompact centre. As a result we have the following structural properties. Let $G_F$ denote the subgroup of $G$ consisting of all elements with relatively compact conjugacy classes. Then $G_F$ is an open subgroup of finite index in $G$, and the commutator subgroup of $G_F$ is relatively compact.

For $d \in \mathbb{N}$, let $\widehat{G}_d = \{\rho \in \widehat{G}\colon d_\rho = d\}$. The topology on $\widehat{G}_d$ is the weakest topology for which all the functions $\pi \to \mathrm{tr}\,\pi(f)$, $f \in C^*(G)$, are continuous [4, Proposition 3.6.4]. Now on the set $P^1(G)$ of all normalized continuous positive definite functions on $G$ the weak*-topology $\sigma(L^\infty(G), L^1(G))$ coincides with the topology of uniform convergence on compact subsets of $G$ [4, Théorème 13.5.2]. Since $\frac{1}{d}\,\mathrm{tr}\,\rho \in P^1(G)$ for all $\rho \in \widehat{G}_d$, it follows that $\mathrm{tr}\,\pi_\alpha(x) \to \mathrm{tr}\,\pi(x)$ uniformly on compact subsets of $G$ whenever $\pi_\alpha \to \pi$ in $\widehat{G}_d$.

THEOREM 2.1. *Let $G$ be a Moore group and let $\pi$ be an irreducible representation of $G$. Then, for every irreducible subrepresentation $\sigma$ of $\pi|G_F$,*

$$M_U(\pi) = m\left(\pi, \mathrm{ind}_{G_F}^G \sigma\right).$$

*Proof.* Let $\Sigma$ denote the set of all $\tau \in \widehat{G}_F$ such that $S_\tau$, the stability subgroup of $\tau$ in $G$, equals $G_F$. By [9, Lemma 2], $\Sigma$ is dense in $\widehat{G}_F$. For every $\tau \in \Sigma$, $\mathrm{ind}_{G_F}^G \tau$ is irreducible, and hence the set of all representations $\mathrm{ind}_{G_F}^G \tau$, $\tau \in \Sigma$, is dense in $\widehat{G}$. By Lemma 1.2 there exists a net $(\sigma_\alpha)_\alpha$ in $\Sigma$ such that, with $\pi_\alpha = \mathrm{ind}_{G_F}^G \sigma_\alpha$,

$$\pi_\alpha \to \pi \text{ in } \widehat{G} \quad \text{and} \quad M_U(\pi) = M(\pi, (\pi_\alpha)_\alpha).$$

Notice that $\pi_\alpha | G_F \to \pi | G_F$ and $(\mathrm{ind}_{G_F}^G \tau)|G_F \sim G(\tau)$ for every $\tau \in \widehat{G}_F$. Thus, replacing each $\sigma_\alpha$ by a suitable member of its $G$-orbit, we can assume that $\sigma_\alpha \to \sigma$ in $\widehat{G}_F$.

Let $C$ be the closure of the commutator subgroup of $G_F$. Then $C$ is compact, and since $G_F$ is type I,

$$\sigma \otimes \widehat{G_F/C} = \left\{\sigma \otimes \chi \colon \chi \in \widehat{G_F/C}\right\}$$

is open in $\widehat{G}_F$ [7, Theorem 2]. Thus $\sigma_\alpha \in \sigma \otimes \widehat{G_F/C}$ eventually, and therefore we can assume that $d_{\sigma_\alpha} = d_\sigma$ for all $\alpha$. By what we have said above about the topology of $\widehat{G}_{d_\sigma}$, it follows that

$$\mathrm{tr}\, \sigma_\alpha(x) \to \mathrm{tr}\, \sigma(x)$$

uniformly on compact subsets of $G_F$. The formula for the trace of an induced representation now shows that

$$\mathrm{tr}\, \pi_\alpha(x) \to \mathrm{tr}\left(\mathrm{ind}_{G_F}^G \sigma\right)(x)$$

uniformly on compact subsets of $G$. On the other hand,

$$\mathrm{ind}_{G_F}^G \sigma = \oplus_\rho\, m\left(\rho, \mathrm{ind}_{G_F}^G \sigma\right) \cdot \rho,$$

where the finite direct sum extends over all irreducible subrepresentations of $\mathrm{ind}_{G_F}^G \sigma$. Thus, uniformly on compact subsets of $G$,

$$\mathrm{tr}\, \pi_\alpha(x) \to \sum_\rho m\left(\rho, \mathrm{ind}_{G_F}^G \sigma\right) \mathrm{tr}\, \rho(x).$$

Theorem 1.1, with $B = C_c(G)$, now shows that $m(\rho, \mathrm{ind}_{G_F}^G \sigma) = M(\rho, (\pi_\alpha)_\alpha)$ for each such $\rho$. In particular, by the choice of the net $(\pi_\alpha)_\alpha$,

$$m\left(\pi, \mathrm{ind}_{G_F}^G \sigma\right) = M_U(\pi). \qquad \square$$

We continue with two consequences of Theorem 2.1. As before, $G$ will denote a Moore group.

COROLLARY 2.2. *$M_U(\pi) = d_\pi$ for every $\pi \in \widehat{G/G_F} \subseteq \widehat{G}$.*

*Proof.* This follows immediately from Theorem 2.1 since $\pi$ occurs with multiplicity $d_\pi$ in $\mathrm{ind}_{G_F}^G 1_{G_F}$, the pull-back to $G$ of the left regular representation of $G/G_F$. □

COROLLARY 2.3. *$M_U(\pi)^2 \leq [G : G_F]$ for every $\pi \in \widehat{G}$.*

*Proof.* Choose an irreducible subrepresentation $\sigma$ of $\pi|G_F$. Since the dimension of $\mathrm{ind}_{G_F}^G \sigma$ equals $d_\sigma[G : G_F]$, Theorem 2.1 and Frobenius reciprocity show that

$$d_\sigma[G : G_F] \geq d_\pi\, m\left(\pi, \mathrm{ind}_{G_F}^G \sigma\right) \geq d_\sigma\, m(\sigma, \pi|G_F)\, M_U(\pi) = d_\sigma\, M_U(\pi)^2,$$

whence $M_U(\pi)^2 \leq [G : G_F]$. □

In particular, it follows from Corollary 2.3 and [3, Theorem 2.6] that, for a Moore group $G$, $C^*(G)$ has bounded trace. Not surprisingly, however, this has been known before. The most comprehensive results about locally compact groups with $C^*$-algebras of bounded trace can be found in [18].

Recall that if $A$ is a $C^*$-algebra and $\pi \in \widehat{A}$, then $\pi$ is said to be a *Fell point* if it satisfies Fell's condition (that is, there exist a neighbourhood $V$ of $\pi$ in $\widehat{A}$ and a positive element $a$ in $A$ such that $\rho(a)$ is a projection of rank 1 for all $\rho \in V$). It has been shown in [1, Theorem 4.6] that $\pi$ is a Fell point if and only if $M_U(\pi) = 1$. It is therefore of interest to deduce from Theorem 2.1 a necessary and sufficient condition for $\pi \in \widehat{G}$ to have upper multiplicity 1.

COROLLARY 2.4. *Let $G$ be a Moore group and $\pi \in \widehat{G}$. Let $\sigma \in \widehat{G}_F$ be such that $\pi|G_F \geq \sigma$. Then*

$$M_U(\pi) \geq \frac{d_\pi}{d_\sigma[G : S_\sigma]}.$$

*Furthermore, $M_U(\pi) = 1$ if and only $d_\pi = d_\sigma[G : S_\sigma]$.*

*Proof.* By Mackey's theory there exists $\tau \in \widehat{S}_\sigma$ such that $\tau|G_F$ is a multiple of $\sigma$ and $\pi = \mathrm{ind}_{S_\sigma}^G \tau$. Then, by Theorem 2.1 and Frobenius reciprocity,

$$\begin{aligned} M_U(\pi) &= m\left(\pi, \mathrm{ind}_{G_F}^G \sigma\right) = m\left(\mathrm{ind}_{S_\sigma}^G \tau, \mathrm{ind}_{S_\sigma}^G (\mathrm{ind}_{G_F}^{S_\sigma} \sigma)\right) \\ &\geq m\left(\tau, \mathrm{ind}_{G_F}^{S_\sigma} \sigma\right) = m(\sigma, \tau|G_F). \end{aligned}$$

Since $d_\pi = d_\tau[G : S_\sigma]$ and $d_\tau = m(\sigma, \tau|G_F)d_\sigma$, we get

$$M_U(\pi) \geq \frac{d_\pi}{d_\sigma[G : S_\sigma]}.$$

In particular, if $M_U(\pi) = 1$, then $d_\tau = d_\sigma$ and hence $d_\pi = d_\sigma[G : S_\sigma]$.

Conversely, suppose that $d_\pi = d_\sigma[G : S_\sigma]$. Then $\tau$ is an extension of $\sigma$ and with $\Gamma = \widehat{S_\sigma/G_F}$ it follows that

$$\begin{aligned}\mathrm{ind}_{G_F}^G \sigma &= \mathrm{ind}_{S_\sigma}^G \left(\mathrm{ind}_{G_F}^{S_\sigma}(\tau|G_F)\right) = \mathrm{ind}_{S_\sigma}^G \left(\tau \otimes \mathrm{ind}_{G_F}^{S_\sigma} 1_{G_F}\right) \\ &= \mathrm{ind}_{S_\sigma}^G \left(\tau \otimes (\oplus_{\gamma\in\Gamma} d_\gamma \cdot \gamma)\right) \\ &= \oplus_{\gamma\in\Gamma} d_\gamma \cdot \mathrm{ind}_{S_\sigma}^G (\tau \otimes \gamma).\end{aligned}$$

Now, by Mackey's theory again, all the representations $\mathrm{ind}_{S_\sigma}^G(\tau \otimes \gamma)$, $\gamma \in \Gamma$, are irreducible and pairwise inequivalent (see [12] and [15, Lemma 2]). It follows that

$$M_U(\pi) = m\left(\mathrm{ind}_{S_\sigma}^G \tau, \mathrm{ind}_{G_F}^G \sigma\right) = 1. \qquad \square$$

COROLLARY 2.5. *Let $G$ be a Moore group. Then $C^*(G)$ is a Fell algebra (that is, every $\pi \in \widehat{G}$ is a Fell point) if and only if $G/G_F$ is abelian and every $\sigma \in \widehat{G}_F$ extends to some representation of its stability group.*

*Proof.* Suppose that $M_U(\pi) = 1$ for all $\pi \in \widehat{G}$. Then the dimension formula of Corollary 2.2 shows that $d_\pi = 1$ for all $\pi \in \widehat{G/G_F}$, which implies that $G/G_F$ is abelian. Also, as we have seen in the proof of Corollary 2.4, if $\tau \in \widehat{S}_\sigma$ is such that $\pi = \mathrm{ind}_{S_\sigma}^G \tau$ and $\tau|G_F$ is a multiple of $\sigma$, then $\tau|G_F = \sigma$ provided that $M_U(\pi) = 1$.

Conversely, let $G/G_F$ be abelian and suppose that every $\sigma \in \widehat{G}_F$ extends to some $\tau_\sigma \in \widehat{S}_\sigma$. Then every $\pi \in \widehat{G}$ is of the form $\pi = \mathrm{ind}_{S_\sigma}^G(\tau_\sigma \otimes \chi)$ for some $\sigma \in \widehat{G}_F$ and $\chi \in \widehat{S_\sigma/G_F}$. Since $\chi$ is 1-dimensional, we get

$$d_\pi = d_{\tau_\sigma\otimes\chi}[G : S_\sigma] = d_\sigma[G : S_\sigma].$$

Corollary 2.4 now shows that $M_U(\pi) = 1$ for all $\pi \in \widehat{G}$. $\square$

We conclude this section with two remarks illustrating the usefulness of Corollaries 2.2 and 2.5.

*Remark* 2.6. Every natural number arises as the upper multiplicity of some irreducible representation of a Moore group.

To see this, let $m \in \mathbb{N}$ and let $S_m$ be the group of permutations of $\{1, \ldots, m\}$. Let $A$ be any non-compact locally compact abelian group and form the semi-direct product $G = S_m \ltimes A^m$, where $S_m$ acts on $A^m$ by permuting the components. Then $G$ is a Moore group with $G_F = A^m$. By Corollary 2.2, $M_U(\pi) = d_\pi$ for every $\pi \in \widehat{G/G_F} = \widehat{S}_m$. Now, $S_m$ has an irreducible representation of dimension $m - 1$. In fact, the so-called Specht module associated to the partition $(m - 1, 1)$ of $m$ is irreducible of dimension $m - 1$ (see, for instance, [5, Theorem 4.12 and Example 5.1]).

*Remark* 2.7. Suppose that $G$ is a semi-direct product $G = H \ltimes N$ where $H$ is finite, $N$ is abelian and $N = G_F$. Then $C^*(G)$ is a Fell algebra if and only if $H$ is abelian. This follows from Corollary 2.5 once we have shown that if $H$ is abelian, then every $\sigma \in \widehat{N}$ extends to a character of its stability subgroup. However, this is guaranteed by the fact that this stability group is of the form $H_\sigma \ltimes N$ for some subgroup $H_\sigma$ of $H$ and that $H_\sigma$ is abelian.

## 3. Lower multiplicity for Moore groups

Let $G$ be a non-compact Moore group and $\pi \in \widehat{G}$. Notice that since $G$ is non-compact, $\pi$ cannot be open in $\widehat{G}$ and thus $M_L(\pi)$ is defined. The purpose of this section is to show that, like $M_U(\pi)$, the lower multiplicity $M_L(\pi)$ can be realized as the multiplicity of $\pi$ in a certain induced representation $\mathrm{ind}_H^G \tau$. Here $H$ is a subgroup of $G$ containing $G_F$ and $\tau$ is an irreducible representation of $H$. However, although there are only finitely many possibilities, there seems to be no canonical choice of the pair $(H, \tau)$.

LEMMA 3.1. *Let $G$ be a Moore group and $N$ a closed normal subgroup of $G$ such that $G/N$ is abelian. Let $\pi \in \widehat{G}$ and*

$$\widehat{G}_{N,\pi} = \left\{\rho \in \widehat{G}\colon \rho|N \sim \pi|N\right\}.$$

*If $(\rho_\alpha)_\alpha$ is a net in $\widehat{G}_{N,\pi}$ converging to some $\rho \in \widehat{G}_{N,\pi}$, then* $\mathrm{tr}\,\rho_\alpha(x) \to \mathrm{tr}\,\rho(x)$ *uniformly on compact subsets of $G$.*

*Proof.* By the remark preceding Theorem 2.1, it suffices to show that $\widehat{G}_{N,\pi} \subseteq \widehat{G}_{d_\pi}$. To that end, fix $\rho \in \widehat{G}_{N,\pi}$. Then, since $G/N$ is abelian,

$$\rho \prec \mathrm{ind}_N^G(\rho|N) \sim \mathrm{ind}_N^G(\pi|N) = \pi \otimes \mathrm{ind}_N^G 1_N \sim \pi \otimes \widehat{G/N}.$$

Thus there is a net $(\chi_\beta)_\beta$ in $\widehat{G/N}$ such that $\pi \otimes \chi_\beta \to \rho$ in $\widehat{G}$. It follows that $d_\rho \le d_\pi$. Similarly, $\pi \prec \rho \otimes \widehat{G/N}$, and as before this yields that $d_\pi \le d_\rho$, as required. □

Let $G$ be any locally compact group and $H$ an open subgroup of $G$. Let $\tau$ be a unitary representation of $H$ and $\pi = \mathrm{ind}_H^G \tau$. In the course of the proof of the next lemma we shall use the fact that if $\pi(C^*(G))$ is finite dimensional, then $H$ must have finite index in $G$. This conclusion is not surprising and has been shown to be true in [9, Lemma 3] whenever $H$ is a closed (not necessarily open) normal subgroup of $G$. Since the proof is very short and much less technical in the case of an open subgroup, we include it for convenience.

Let $\pi(C^*(G))$ be of dimension $d$, and suppose that $H$ has at least $d+1$ different left cosets in $G$, say $a_0H, \dots, a_dH$. Fix some $v \in \mathcal{H}_\tau$ and $f \in C_c(H) \subseteq C_c(G)$ such that $\tau(f)v \neq 0$ and define $\xi \in \mathcal{H}_\pi$ by $\xi(h) = \tau(h^{-1})v$ for $h \in H$ and

$\xi(x) = 0$ for $x \in G \setminus H$. Next, observe that for $a$ and $x$ in $G$, $\pi(L_a f)\xi(x) = 0$ if $x \notin aH$, while $\pi(L_a f)\xi(a) = \tau(f)v$. Now there exist $\lambda_0, \ldots, \lambda_d \in \mathbb{C}$ such that $\sum_{j=0}^{d} \lambda_j \, \pi(L_{a_j} f) = 0$ and $\lambda_k \neq 0$ for at least one value of $k$. It follows that

$$0 = \sum_{j=0}^{d} \lambda_j \, \pi(L_{a_j} f)\xi(a_k) = \lambda_k \, \tau(f)v,$$

a contradiction.

LEMMA 3.2. *Let $H$ be a Moore group and define subsets $S$ of $\widehat{H_F}$ and $T$ of $\widehat{H}$ by*

$$S = \left\{\sigma \in \widehat{H_F} \colon S_\sigma = H\right\} \text{ and } T = \left\{\tau \in \widehat{H} \colon \tau | H_F \sim \sigma \text{ for some } \sigma \in S\right\}.$$

*Suppose that $(\tau_\alpha)_\alpha$ is a net in $T$ converging to some $\tau \in T$, then* $\operatorname{tr} \tau_\alpha(x) \to \operatorname{tr} \tau(x)$ *uniformly on compact subsets of $H$.*

*Proof.* Let $N = H_F$ and let $C$ denote the closure of the commutator subgroup of $N$. Then $C$ is compact and hence $\omega \otimes \widehat{N/C}$ is open in $\widehat{N}$ for every $\omega \in \widehat{N}$.

Now let $\sigma_\alpha, \sigma \in S$ be such that $\tau_\alpha | N \sim \sigma_\alpha$ and $\tau | N \sim \sigma$. Since $\sigma_\alpha \to \sigma$ in $\widehat{N}$, we have $\sigma_\alpha \in \sigma \otimes \widehat{N/C}$ eventually. Thus we can assume that for every $\alpha$ there exists $\lambda_\alpha \in \widehat{N/C}$ such that $\sigma_\alpha = \sigma \otimes \lambda_\alpha$. By hypothesis, $\sigma$ and $\sigma_\alpha$ are $H$-invariant. However, $\lambda_\alpha$ need not be $H$-invariant.

Consider any $\omega \in \widehat{N}$ and $\chi \in \widehat{N/C}$ such that $\omega$ and $\omega \otimes \chi$ belong to $S$. Then, for every $a \in H$,

$$\omega \otimes \chi = (\omega \otimes \chi)^a = \omega^a \otimes \chi^a = \omega \otimes \chi^a,$$

so that $(\chi^a \bar{\chi}) \otimes \omega = \omega$. Now, let $X_\omega = \{\mu \in \widehat{N/C} \colon \omega \otimes \mu = \omega\}$. Then $X_\omega$ is a closed subgroup of $\widehat{N/C}$ and hence of the form $X_\omega = \widehat{N/M}$ for some closed subgroup $M$ of $N$ (containing $C$). By definition of $X_\omega$, $\omega \sim \operatorname{ind}_M^N(\omega|M)$. Moreover, $\operatorname{tr} \omega(x) = \mu(x) \operatorname{tr} \omega(x)$ for all $x \in N$ and $\mu \in \widehat{N/M}$. This implies that $\operatorname{tr} \omega(x) = 0$ for every $x \in N \setminus M$, and this in turn implies that $M$ is open in $N$. By the remark preceding the lemma we conclude that $M$ has finite index in $N$ and hence in $H$. Let $L$ be the largest normal subgroup of $H$ contained in $M$. Then $L$ is of finite index in $H$ and if $\mu \in \widehat{N/C}$ is such that $\omega \otimes \mu = \omega$, then $\mu | L = 1_L$.

We now apply this to $\omega = \sigma$. Thus there exists a normal subgroup $L$ of finite index in $H$ with the property that $\chi^a | L = \chi | L$ for all $a \in H$ whenever $\chi \in \widehat{N/C}$ is such that $\sigma \otimes \chi \in S$. Since $\sigma_\alpha = \sigma \otimes \lambda_\alpha \in S$, we have $\lambda_\alpha^a | L = \lambda_\alpha | L$ for every $\alpha$ and every $a \in H$. That is, all the $\lambda_\alpha | L$ are $H$-invariant.

Let $K = \{x \in L \colon \lambda_\alpha(x) = 1 \text{ for all } \alpha\}$. Then $K$ is normal in $H$ since all $\lambda_\alpha | L$ are $H$-invariant. Furthermore, since the set of all characters $\lambda_\alpha | L$ separates the points of $L/K$, it follows that $L/K$ is contained in the centre of $H/K$. Thus $H/K$ has a centre of finite index and hence a finite commutator subgroup. Let $E$ denote its pull-back to $H$. Then $E/K$ is finite and $H/E$ is abelian.

By definition of $K$, for all $\alpha$ we have,

$$\tau_\alpha|K \sim \sigma_\alpha|K = \sigma|K \otimes \lambda_\alpha|K = \sigma|K \sim \tau|K.$$

Choose an irreducible subrepresentation $\gamma$ of $\tau|E$. Since $\tau_\alpha|E \to \tau|E$, for each $\alpha$ there is an irreducible subrepresentation $\gamma_\alpha$ of $\tau_\alpha|E$ such that $\gamma_\alpha \to \gamma$ in $\widehat{E}$. Let $J = \ker(\mathrm{ind}_K^E(\tau|K))$ and $A = C^*(E)/J$. Then $A$ is a finite dimensional $C^*$-algebra since $\tau$ is finite dimensional and $E/K$ is finite. Moreover, since $\tau_\alpha|K \sim \tau|K$,

$$\ker\gamma_\alpha \supseteq \ker\tau_\alpha \supseteq \ker\left(\mathrm{ind}_K^E(\tau_\alpha|K)\right) = J.$$

Thus $\gamma_\alpha \to \gamma$ in $\widehat{A}$ and hence $\gamma_\alpha = \gamma$ eventually. It follows that

$$\tau_\alpha|E \sim G(\gamma_\alpha) = G(\gamma) \sim \tau|E$$

eventually. Therefore we may assume that $\tau_\alpha|E \sim \tau|E$, that is, $\tau_\alpha \in \widehat{H}_{E,\tau}$, for all $\alpha$. An application of Lemma 3.1 now shows that $\mathrm{tr}\,\tau_\alpha(x) \to \mathrm{tr}\,\tau(x)$ uniformly on compact subsets of $H$. $\square$

Let $G$ be a Moore group and $\sigma \in \widehat{G}_F$. A subgroup $H$ of $G$ containing $G_F$ is called *admissible for* $\sigma$ if there exists a net $(\sigma_\alpha)_\alpha$ in $\widehat{G}_F$ with the following properties: $\sigma_\alpha \to \sigma, \sigma \notin G(\sigma_\alpha)$ and $S_{\sigma_\alpha} = H$ for all $\alpha$. Then clearly $H \subseteq S_\sigma$ since $\widehat{G}_F$ is a Hausdorff space.

LEMMA 3.3. *Let $G$ be a non-compact Moore group and let $\sigma \in \widehat{G}_F$. Let $H$ be an admissible subgroup for $\sigma$ and let $\tau \in \widehat{H}$ such that $\tau|G_F$ is a multiple of $\sigma$. Then*

$$M_L(\pi) \le m\left(\pi, \mathrm{ind}_H^G\,\tau\right)$$

*for every irreducible subrepresentation $\pi$ of* $\mathrm{ind}_H^G\,\tau$.

*Proof.* Let $\pi$ be an irreducible subrepresentation of $\mathrm{ind}_H^G\,\tau$. Since $\tau$ is an irreducible subrepresentation of $\mathrm{ind}_{G_F}^H\,\sigma$ and $\mathrm{ind}_{G_F}^H\,\sigma_\alpha \to \mathrm{ind}_{G_F}^H\,\sigma$, there exist $\tau_\alpha \in \widehat{H}$ such that $\tau_\alpha|G_F \sim \sigma_\alpha$ and $\tau_\alpha \to \tau$ in $\widehat{H}$. Let $\pi_\alpha = \mathrm{ind}_H^G\,\tau_\alpha$; then $\pi_\alpha \in \widehat{G}$ and $\pi_\alpha \to \mathrm{ind}_H^G\,\tau$, and hence $\pi_\alpha \to \pi$ in $\widehat{G}$. Moreover, $\pi_\alpha \neq \pi$ for every $\alpha$ since $\pi|G_F \sim G(\sigma), \pi_\alpha|G_F \sim G(\sigma_\alpha)$ and $G(\sigma_\alpha) \cap G(\sigma) = \emptyset$ by hypothesis.

An application of Lemma 3.2 shows that $\mathrm{tr}\,\tau_\alpha(x) \to \mathrm{tr}\,\tau(x)$ uniformly on compact subsets of $H$. This implies that

$$\mathrm{tr}\,\pi_\alpha(x) = \mathrm{tr}\left(\mathrm{ind}_H^G\,\tau_\alpha\right)(x) \to \mathrm{tr}\left(\mathrm{ind}_H^G\,\tau\right)(x)$$

uniformly on compact subsets of $G$. Now

$$\mathrm{tr}\left(\mathrm{ind}_H^G\,\tau\right) = \sum_\rho m\left(\rho, \mathrm{ind}_H^G\,\tau\right)\mathrm{tr}\,\rho,$$

where the sum extends over all irreducible subrepresentations $\rho$ of $\mathrm{ind}_H^G\,\tau$. It follows from Theorem 1.1 that $m(\rho, \mathrm{ind}_H^G\,\tau) = M(\rho, (\pi_\alpha)_\alpha)$. In particular,

$$M_L(\pi) \le M(\pi, (\pi_\alpha)_\alpha) \;=\; m\left(\pi, \mathrm{ind}_H^G\,\tau\right). \qquad \square$$

COROLLARY 3.4. *Let $G$ be a non-compact Moore group, let $\sigma \in \widehat{G}_F$ and suppose that $S_\sigma$ itself is admissible for $\sigma$. Then $M_L(\pi) = 1$ for every irreducible subrepresentation of* $\mathrm{ind}_{G_F}^G\,\sigma$.

*Proof.* Given such a $\pi$, there exists $\tau \in \widehat{S_\sigma}$ so that $\pi = \mathrm{ind}_{S_\sigma}^G\,\tau$ and $\tau|G_F$ is a multiple of $\sigma$. Now Lemma 3.3 with $H = S_\sigma$ gives that $M_L(\pi) \le m(\pi, \mathrm{ind}_{S_\sigma}^G\,\tau) = 1$. $\square$

The following example is a typical application of Corollary 3.4.

*Example* 3.5. Let $G = S_m \ltimes A^m$ be as in Remark 2.6. Then, for each $\sigma \in \widehat{A}^m$, the stability group of $\sigma$ is admissible for $\sigma$. This can be seen as follows. If $\sigma = (\sigma_1, \dots, \sigma_m) \in \widehat{A}^m$ $(\sigma_j \in \widehat{A})$ and $\varphi \in S_m$, then $\varphi$ belongs to the stability group of $\sigma$ if and only if $\sigma_{\varphi(j)} = \sigma_j$ for all $j = 1, \dots, m$. Now, since $\widehat{A}$ has no isolated points, there exists a net $(\sigma^{(\alpha)})_\alpha$ in $\widehat{A}^m$ converging to $\sigma$ such that $\sigma^{(\alpha)} \neq \sigma$ for all $\alpha$ and $\sigma_j^{(\alpha)} = \sigma_i^{(\alpha)}$ if and only if $\sigma_j = \sigma_i$ for each $\alpha$ and all $i, j \in \{1, \dots, m\}$. Corollary 3.4 shows that $M_L(\pi) = 1$ for every $\pi \in \widehat{G}$.

Now we are ready to combine Lemmas 3.2 and 3.3 with Theorem 1.1 and results from [2] to obtain the formula for lower multiplicity alluded to at the beginning of this section.

THEOREM 3.6. *Let $G$ be a non-compact Moore group and let $\pi \in \widehat{G}$ and $\sigma \in \widehat{G}_F$ such that $\sigma \le \pi|G_F$. Then*

$$M_L(\pi) = \min_{(H,\tau)} m\left(\pi, \mathrm{ind}_H^G\,\tau\right),$$

*where $(H, \tau)$ runs through all pairs consisting of an admissible subgroup $H$ for $\sigma$ and an irreducible representation $\tau$ of $H$ such that $\pi \le \mathrm{ind}_H^G\,\tau$ and $\tau|G_F \sim \sigma$.*

*Proof.* In view of Lemma 3.3 it suffices to show that $m(\pi, \mathrm{ind}_H^G\,\tau) \le M_L(\pi)$ for some pair $(H, \tau)$.

By [2, Proposition 2.2] there exists a net $\Omega_1$ in $\widehat{G} \setminus \{\pi\}$ converging to $\pi$ such that $M_L(\pi) = M_L(\pi, \Omega_1)$. By Proposition 2.3 of [2], $\Omega_1$ possesses a subnet $\Omega_2$ satisfying $M(\pi, \Omega_2) = M_L(\pi, \Omega_1)$. Since $G/G_F$ is finite and by successively choosing further subnets, we find a subgroup $H$ of $G$ containing $G_F$ and a subnet $\Omega = (\pi_\alpha)_\alpha$ of $\Omega_2$ with the following properties:

(1) $\pi_\alpha|G_F \sim G(\sigma_\alpha)$ for some $\sigma_\alpha \in \widehat{G}_F$ such that $\sigma_\alpha \to \sigma$ in $\widehat{G}_F$ and $S_{\sigma_\alpha} = H$ for all $\alpha$.

(2) $\pi_\alpha = \operatorname{ind}_H^G \tau_\alpha$ where $\tau_\alpha \in \widehat{H}$ is such that $\tau_\alpha|G_F \sim \sigma_\alpha$ and $\tau_\alpha \to \tau$ in $\widehat{H}$ for some $\tau \in \widehat{H}$.

Of course, (1) and (2) imply that $\tau|G_F \sim \sigma$. It follows from Lemma 3.2 that $\operatorname{tr}\tau_\alpha \to \operatorname{tr}\tau$ uniformly on compact subsets of $H$ and therefore, uniformly on compact subsets of $G$,

$$\operatorname{tr}\pi_\alpha(x) \to \operatorname{tr}\left(\operatorname{ind}_H^G \tau\right)(x) = \sum_\rho m\left(\rho, \operatorname{ind}_H^G \tau\right)\operatorname{tr}\rho(x),$$

where the sum extends over all irreducible subrepresentations of $\operatorname{ind}_H^G \tau$. Since $\Omega$ converges to $\pi$ in $\widehat{G}$, (i) and (iii) of Theorem 1.1 now show that $\pi$ is a subrepresentation of $\operatorname{ind}_H^G \sigma$ and that

$$m\left(\pi, \operatorname{ind}_H^G \tau\right) = M(\pi, \Omega).$$

Summarizing by the choice of $\Omega_1$, $\Omega_2$ and $\Omega$ and since $\Omega$ is a subnet of $\Omega_2$, we get

$$m\left(\pi, \operatorname{ind}_H^G \tau\right) = M(\pi, \Omega) = M(\pi, \Omega_2) = M_L(\pi, \Omega_1) = M_L(\pi),$$

as required. □

COROLLARY 3.7. *Let $\pi \in \widehat{G}$ and $\sigma \in \widehat{G}_F$ be such that $\sigma \leq \pi|G_F$. If the representation $\operatorname{ind}_{G_F}^{S_\sigma} \sigma$ is multiplicity free, then $M_L(\pi) = 1$.*

*Proof.* By Theorem 3.6 there are a subgroup $H$ of $S_\sigma$ containing $G_F$ and some $\tau \in \widehat{H}$ such that $\tau|G_F \sim \sigma$, $\pi \leq \operatorname{ind}_H^G \tau$ and $M_L(\pi) = m(\pi, \operatorname{ind}_H^G \tau)$. By hypothesis,

$$\operatorname{ind}_{G_F}^{S_\sigma} \sigma = \operatorname{ind}_H^{S_\sigma}\left(\operatorname{ind}_{G_F}^H \sigma\right)$$

is multiplicity free and hence so is its subrepresentation $\operatorname{ind}_H^{S_\sigma} \tau$ (notice that $\tau \leq \operatorname{ind}_{G_F}^H \sigma$). Now, for every irreducible subrepresentation $\rho$ of $\operatorname{ind}_H^{S_\sigma} \tau$, $\operatorname{ind}_{S_\sigma}^G \rho$ is irreducible, and the mapping $\rho \to \operatorname{ind}_{S_\sigma}^G \rho$ is injective. Since $\pi = \operatorname{ind}_{S_\sigma}^G \rho$ for some such $\rho$, it follows that $\pi$ occurs only once in $\operatorname{ind}_H^G \tau$, as was to be shown. □

The hypothesis of Corollary 3.7 that $\operatorname{ind}_{G_F}^{S_\sigma} \sigma$ be multiplicity free is fulfilled, for instance, if $G_F$ splits in $G$ (that is, $G$ is a semi-direct product of some finite group $A$ with $G_F$) and $G_F$ and $A \cap S_\sigma$ are abelian.

It is conceivable that $M_L(\pi)$ might be equal to 1 for every irreducible representation $\pi$ of a Moore group. However, we incline to the opposite view and hope that the formula of Theorem 3.6 will be useful in the attempt to construct a Moore group $G$ and an irreducible representation $\pi$ of $G$ with $M_L(\pi) > 1$.

## 4. Finite multiplicities for SIN-groups

In this final section we turn to SIN-groups. Recall that a locally compact group $G$ is said to have small invariant neighbourhoods if $G$ has a neighbourhood basis of the identity consisting of sets $V$ such that $x^{-1}Vx = V$ for all $x \in G$. In particular, discrete groups are SIN-groups. The representation theory of SIN-groups, notably the left regular representation, has been studied in [6] and [19]. Moore groups are precisely those SIN-groups which are of type I.

For a SIN-group $G$, it is immediate from the definition that the subgroup $G_F$ of all elements with relatively compact conjugacy classes is open in $G$. The reduced group $C^*$-algebra $C_r^*(G)$ is the image of $C^*(G)$ under the left regular representation, and $\widehat{G}_r \subseteq \widehat{G}$ denotes the dual space of $C_r^*(G)$. Recall that $\widehat{G}_r = \widehat{G}$ if and only if $G$ is amenable.

LEMMA 4.1. *Let $G$ be a* SIN-*group and let $I$ be a non-zero closed ideal of $C_r^*(G)$. If $I$ is a type* I *$C^*$-algebra, then $G/G_F$ is finite and the commutator subgroup of $G_F$ is relatively compact.*

*Proof.* Let $\mathrm{VN}(G)$ be the von Neumann algebra generated by the left regular representation of $G$. Since $G$ is an SIN-group, $\mathrm{VN}(G)$ is a finite von Neumann algebra [4, Proposition 13.10.5]. By hypothesis on $I$, the weak closure $\bar{I}$ of $I$ in $\mathrm{VN}(G)$ is a type I von Neumann algebra. Thus there exists a non-zero central projection $E$ in $\mathrm{VN}(G)$ such that $E(\mathrm{VN}(G))$ is type I, finite. The statement of the lemma now follows from [6, Satz 2] (see also [19, Theorem 3]). □

In what follows, for $\pi \in \widehat{G}_r$, we denote by $M_U^r(\pi)$ and $M_U(\pi)$ the upper multiplicity of $\pi$ viewed as a representation of $C_r^*(G)$ and of $C^*(G)$, respectively.

THEOREM 4.2. *For a non-compact* SIN-*group $G$ the following three conditions are equivalent.*

(i) *There exist $\pi \in \widehat{G}_r$ such that $M_U^r(\pi) < \infty$.*
(ii) *There exists a non-empty open subset $V$ of $\widehat{G}_r$ such that $M_L^r(\rho) < \infty$ for all $\rho \in V$.*
(iii) *$G_F$ has finite index in $G$ and a relatively compact commutator subgroup.*

*Proof.* Suppose that (i) holds. Then the set of all $\rho \in \widehat{G}_r$ with $M_U^r(\rho) < \infty$ is non-empty and open in $\widehat{G}_r$ by [1, Proposition 2.3]. Thus (i) implies (ii).

Let $V$ be as in (ii) and let $I$ be the closed ideal of $C_r^*(G)$ with $\widehat{I} = V$. Since $M_L^r(\rho)$ is defined for every $\rho \in V$, no singleton $\{\rho\}$, $\rho \in V$, can be open in $\widehat{G}_r$. It follows from Theorem 4.4 of [1] that $\rho(C_r^*(G)) \supseteq \mathcal{K}(\mathcal{H}_\rho)$ and hence $\rho(I) \supseteq \mathcal{K}(\mathcal{H}_\rho)$ for every $\rho \in \widehat{I}$. Thus $I$ is a type I $C^*$-algebra by the Glimm-Sakai theorem and then (iii) is a consequence of Lemma 4.1.

Finally, let (iii) be satisfied and denote by $C$ the closure of the commutator subgroup of $G_F$. Then $G/C$ is almost abelian and hence every $\rho \in \widehat{G/C}$ has finite upper multiplicity relative to $C^*(G/C)$ (see Theorem 2.1). However, since $C$ is compact, $C^*(G/C)$ is an ideal of $C^*(G)$ and hence that multiplicity coincides with the upper multiplicity of $\rho$ relative to $C^*(G)$ (see [3, Lemma 2.7]). This proves (i). □

In particular, since groups as in (iii) of the preceding theorem are amenable, this shows that if $G$ is a non-amenable SIN-group then $M_U(\pi) = \infty$ for all $\pi \in \widehat{G}_r$. In this case there may or may not exist $\rho \in \widehat{G}$ with $M_U(\rho) < \infty$. For example, if $G = \mathbb{F}_2$, the free group on two generators, then $M_U(\rho) = \infty$ for all $\rho \in \widehat{G}$ because $C^*(\mathbb{F}_2)$ is antiliminal, whereas if $G$ has Kazhdan's property $(T)$ then $M_U(1_G) = 1$.

As an additional example consider an arbitrary non-compact nilpotent SIN-group $G$. For such $G$, it has recently been shown in [8] that $M_L(\pi) = \infty$ for each infinite dimensional $\pi \in \widehat{G}$. This implies that $M_U(\pi) = \infty$ for every $\pi \in \widehat{G}$. On the other hand, $M_L(\pi) = 1$ for every finite dimensional $\pi \in \widehat{G}$ [8].

We conclude this section with a precise description (in the situation of Theorem 4.2) of the set of all irreducible representations with finite upper (respectively, lower) multiplicity.

THEOREM 4.3. *Let $G$ be a non-compact* SIN-*group and suppose that $G$ satisfies one* (*and hence all*) *of the conditions of Theorem* 4.2. *Then there exists a compact normal subgroup $K$ of $G$ such that*

$$\widehat{G/K} = \left\{\pi \in \widehat{G}:\ M_U(\pi) < \infty\right\} = \left\{\pi \in \widehat{G}:\ M_L(\pi) < \infty\right\} = \left\{\pi \in \widehat{G}:\ d_\pi < \infty\right\}.$$

*Proof.* We know that $G/G_F$ is finite and $C$, the closure of the commutator subgroup of $G_F$, is compact. We define a normal subgroup $K$ of $G$ by

$$K = \{x \in G:\ \rho(x) = 1 \text{ for all } \rho \in \widehat{G} \text{ such that } d_\rho < \infty\}.$$

Then $K \subseteq C$ since $G/C$ is a Moore group. By definition of $K$, $G_F/K$ is a maximally almost periodic group with relatively compact commutator subgroup. As such, $G_F/K$ is a Moore group [17] and hence so is $G/K$. Thus

$$\widehat{G/K} = \left\{\pi \in \widehat{G}:\ d_\pi < \infty\right\}.$$

From Theorem 2.1 we know that $M_U(\pi) < \infty$ for every $\pi \in \widehat{G/K}$. Note that, as in the proof of Theorem 4.2, $M_U(\pi)$ is the same relative to $C^*(G/K)$ as relative to $C^*(G)$. To complete the proof of the theorem it remains to show that $M_L(\pi) = \infty$ for every $\pi \in \widehat{G} \setminus \widehat{G/K}$.

Suppose $\pi \in \widehat{G}$ is such that $M_L(\pi) < \infty$. Notice that $\{\pi\}$ cannot be open in $\widehat{G}$ since $G$ is non-compact. It follows that $\pi(C^*(G)) \supseteq \mathcal{K}(\mathcal{H}_\pi)$. Now, $G$ being a finite extension of a group with relatively compact commutator subgroup, $C^*(G)$ has a $T_1$ primitive ideal space [7], [15]. Hence $\pi(C^*(G))$ is simple, whence $\pi(C^*(G)) =$

$\mathcal{K}(\mathcal{H}_\pi)$. Since $G$ is a SIN-group, $L^1(G)$ (and hence $C^*(G)$) has a central approximate identity $F$. Thus, for every $f \in F, \pi(f)$ is compact and $\pi(f) \in \mathbb{C} \cdot 1$ since $\pi$ is irreducible. This forces $\pi$ to be finite dimensional, so that $\pi \in \widehat{G/K}$. □

## REFERENCES


[1] R. J. Archbold, *Upper and lower multiplicity for irreducible representations of $C^*$-algebras*, Proc. London Math. Soc.(3) **69** (1994), 121–143.

[2] R.J. Archbold and J. S. Spielberg, *Upper and lower multiplicity for irreducible representations of $C^*$-algebras. II*, J. Operator Theory **36** (1996), 201–231.

[3] R. J. Archbold, D. W. B. Somerset and J. S. Spielberg, *Upper multiplicity and bounded trace ideals in $C^*$-algebras*, J. Funct. Anal. **146** (1997), 430–463.

[4] J. Dixmier, *$C^*$-algebras*, North-Holland, 1977.

[5] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics no. 682, Springer-Verlag, 1978.

[6] E. Kaniuth, *Die Struktur der regulären Darstellung lokalkompakter Gruppen mit invarianter Umgebungsbasis der Eins*, Math. Ann. **194** (1971), 225–248.

[7] ———, *Primitive ideal spaces of groups with relatively compact conjugacy classes*, Arch. Math. **32** (1979), 16–24.

[8] ———, *Lower multiplicity for irreducible representations of nilpotent locally compact groups*, J. Funct. Anal., to appear.

[9] E. Kaniuth and K. F. Taylor, *Kazhdan constants and the dual space topology*, Math. Ann. **293** (1992), 495–508.

[10] J. Ludwig, *On the behaviour of sequences in the dual of a nilpotent Lie group*, Math. Ann. **287** (1990), 239–257.

[11] J. Ludwig, G. Rosenbaum and J. Samuel, *The elements of bounded trace in the $C^*$-algebra of a nilpotent Lie group*, Invent. Math. **83** (1986), 167–190.

[12] G. W. Mackey, *Unitary representations of group extensions*, Acta Math. **99** (1958), 265–311.

[13] C. C. Moore, *On the Frobenius reciprocity theorem for locally compact groups*, Pacific J. Math. **12** (1962), 359–365.

[14] ———, *Groups with finite dimensional irreducible representations*, Trans. Amer. Math. Soc. **166** (1972), 401–410.

[15] D. Poguntke, *Der Raum der primitiven Ideale von endlichen Erweiterungen lokalkompakter Gruppen*, Arch. Math. **28** (1977), 133–138.

[16] I. Raeburn, *On group $C^*$-algebras of bounded representation dimension*, Trans. Amer. Math. Soc. **272** (1982), 629–644.

[17] L. Robertson, *A note on the structure of Moore groups*, Bull. Amer. Math. Soc. **75** (1969), 594–599.

[18] T. Sund, *Groups with bounded trace and the trivial cortex problem*, Internat. J. Math. **5** (1994), 561–593.

[19] K. F. Taylor, *The type structure of the regular representation of a locally compact group*, Math. Ann. **222** (1976), 211–224.



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