GENERALIZED BRANCHING PROCESSES I: EXISTENCE AND UNIQUENESS THEOREMS[1]

BY

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1. Introduction

It is the purpose of this paper to study the mathematical formalism related to the evolution of a population, each of whose members has an associated numerical characteristic. It is convenient to refer to the members of the population as particles and to their characteristics as energy, since this language describes one of the important applications—namely, the cascade process. Another example is a biological population of cells, with a characteristic such as the size or weight of a cell.

It is assumed that the process originates at time $t = 0$ with a single parent particle of energy X_0, which after a time T splits into N resultant particles of energies $X_1, \cdots, X_N$ respectively. Each of the resultant particles then behaves as if it were itself a parent particle, the behavior being assumed independent of any other particles existing at the time. The quantities $T, N, X_1, \cdots, X_N$ are random variables. Let $G(t) = P\{T \leqq t\}$ be the distribution function of T; $q_j = P\{N = j\}$ the probability function of N; and $\Phi_j(x_1, \cdots, x_j \mid X_0) = P\{X_1 \leqq x_1, \cdots, X_j \leqq x_j \mid X_0\}$ the conditional joint distribution function of $X_1, \cdots, X_j$, given that a parent of energy X_0 has given rise to j offspring. It will be seen that these distributions are sufficient to describe the process. It is assumed that an offspring cannot split instantly upon birth, i.e., that $G(0) = 0$.

A variety of interesting questions can be asked about the evolution of such a population. The energy distribution of particles existing at a specified time t, i.e., the distribution function of the number of particles of energy at least x, for any $x \geqq 0$, at time t, is of particular interest. This distribution and its moments have been the subject of considerable study in the case when $G(\cdot)$ is exponential, in which case the process is Markovian. It will be discussed by the author for more general cases in a future publication.

Another quantity of interest, and the subject of this paper, is the total energy $X(t)$ of all particles existing at t. The study of this quantity was initially suggested to the author by T. E. Harris in the case of the Markovian cascade process. In the cascade process it is customary, however, to make an assumption of conservation of energy, namely that $X_1 + \cdots + X_j \leqq X_0$, while this restriction will not be made here. There will also be no restrictions on $G(\cdot)$ other than $G(0) = 0$.

If it is assumed that all particles have energy identically equal to one, then

Received January 4, 1963.

[1] Research sponsored by the Office of Naval Research.

$X(t)$ is simply the total number of particles existing at t. The process then becomes what is usually called the age-dependent branching process, which has been studied, e.g., by Bellman and Harris [1], Levinson [6], and Sevast'yanov [11]. The present population process is thus a generalization of this standard branching process.

To study the random variable $X(t)$, one may start by defining it constructively in terms of the random variables T, N, and $X_1, \cdots, X_N$. In order to do this rigorously, one must work with the rather complicated sample space on which the process is defined. One can then prove that the distribution of $X(t)$ satisfies a basic integral equation (I.E.). This somewhat lengthy task was performed by the author in [10] in the case of the cascade process. The construction of probability spaces of the above kind has been carried out in great generality by J. E. Moyal [7], [8], [9]. In [8], the state space is an abstract space (as opposed to the positive real line of the present paper), but the process is assumed Markovian. In [9] there is no such assumption, and the constructions are obtained in complete generality; but it is not the objective to study integral equations of the kind which are the subject of this paper.

An alternative approach is to start with I.E. being formally given, and then to proceed purely analytically to prove that the equation has a solution, that this solution is unique among a certain class of functions, and that it is a distribution function. From I.E. one then obtains equations for the moments of the process, and criteria for the existence and uniqueness of solutions of the moment equations. Essentially this approach was taken by Levinson in [6] for the case of the age-dependent branching process. It will be carried out for the generalized process in this paper. In Section 2 it is shown that I.E. has a unique bounded solution which is a distribution function. In Section 3 the existence, uniqueness, boundedness, and monotonicity properties of the moments of the process are studied. Section 4 gives some examples, and Section 5 briefly discusses the total energy of the process up to t, a quantity closely related to $X(t)$.

In a sequel to this paper the asymptotic properties of the process are studied for a more restricted class of Φ-functions. In particular, the convergence to a random variable of $X(t)$ divided by its mean is studied.

2. The basic integral equation

If one denotes the conditional distribution of $X(t)$, given that $X(0) = x_0$, by $P(x, t \mid x_0)$, then the law of total probability suggests that $P(x, t \mid x_0)$, and $\hat{P}(s, t \mid x_0)$, its Fourier-Stieltjes transform (characteristic function), satisfy respectively the equations

$$\begin{aligned} P(x, t \mid x_0) &= [1 - G(t)]Z(x - x_0) + q_0 G(t) Z(x) \\ &\quad + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \\ &\qquad \cdot [P(x, t - y \mid x_1) * \cdots * P(x, t - y \mid x_j)], \end{aligned} \tag{2.1}$$

and

$$\hat{P}(s, t \mid x_0) = [1 - G(t)]e^{isx_0} + q_0 G(t)$$

$$(2.2)\qquad + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \prod_{i=1}^{j} \hat{P}(s, t - y \mid x_i),$$

where $*$ is the convolution operation, and $Z(x) = 0$ for $x < 0$, $Z(x) = 1$ for $x \geqq 0$.

For purposes of the present study, the equations (2.1) and (2.2) are to be regarded as formally given, quite independently of their probabilistic origins.

THEOREM 1. *If $\sum_{n=0}^{\infty} nq_n = \nu < \infty$, then there exist unique bounded solutions $P(x, t \mid x_0)$ and $\hat{P}(s, t \mid x_0)$ of (2.1) and (2.2) respectively. P is a distribution function, and $\hat{P}$ is its characteristic function.*

Proof. Define $\hat{P}_0(s, t \mid x_0) = 0$, and for $k \geqq 0$

$$\hat{P}_{k+1}(s, t \mid x_0) = [1 - G(t)]e^{isx_0} + q_0 G(t)$$

$$(2.3)\qquad + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \prod_{i=1}^{j} \hat{P}_k(s, t - y \mid x_i).$$

Then $| \hat{P}_1(s, t \mid x_0) | \leqq [1 - G(t)] + q_0 G(t) \leqq 1$, and if $| \hat{P}_k(s, t \mid x_0) | \leqq 1$, then by (2.3),

$$| \hat{P}_{k+1}(s, t \mid x_0) | \leqq [1 - G(t)] + \sum q_i G(t) \leqq 1.$$

Therefore by induction

$$(2.4)\qquad | \hat{P}_k(s, t \mid x_0) | \leqq 1, \qquad k = 0, 1, \cdots .$$

Adopt the convention $\prod_{h=n+1}^{n} a_h \equiv 1$ for any $\{a_h\}$. Then one can show that

$$(2.5)\qquad \begin{aligned} \left| \prod_{i=1}^{j} \hat{P}_k(s, t \mid x_i) - \prod_{i=1}^{j} \hat{P}_{k-1}(s, t \mid x_i) \right| &\leqq \sum_{i=1}^{j} | \hat{P}_k(s, t \mid x_i) - \hat{P}_{k-1}(s, t \mid x_i) | \\ &\qquad \cdot \left| \prod_{h=1}^{i-1} \hat{P}_{k-1}(s, t \mid x_h) \prod_{h=i+1}^{j} \hat{P}_k(s, t \mid x_h) \right| \\ &\leqq \sum_{i=1}^{j} | \hat{P}_k(s, t \mid x_i) - \hat{P}_{k-1}(s, t \mid x_i) |. \end{aligned}$$

Hence

$$
\begin{aligned}
&| \hat{P}_{k+1}(s, t \mid x_0) - \hat{P}_k(s, t \mid x_0) | \\
&\quad \leqq \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \\
&\qquad \cdot \left| \prod_{i=1}^{j} \hat{P}_k(s, t - y \mid x_i) - \prod_{i=1}^{j} \hat{P}_{k-1}(s, t - y \mid x_i) \right| \\
&\quad \leqq \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \\
&\qquad \cdot \sum_{i=1}^{j} | \hat{P}_k(s, t - y \mid x_i) - \hat{P}_{k-1}(s, t - y \mid x_i) |.
\end{aligned} \tag{2.6}
$$

But $P_0(s, t \mid x_0) \equiv 0$, and hence by (2.4)

$$| \hat{P}_1(s, t \mid x_0) - \hat{P}_0(s, t \mid x_0) | \leqq 1.$$

By induction on (2.6) it then follows that

$$| \hat{P}_{k+1}(s, t \mid x_0) - \hat{P}_k(s, t \mid x_0) | \leqq \nu^k G_k(t), \tag{2.7}$$

where $G_k(t)$ is the k-fold convolution of $G(t)$.

Now let $H(t) = \sum_{n=1}^{\infty} \nu^n G_n(t)$. Then it is easily verified that $H(t)$ satisfies the equation

$$H(t) = \nu G(t) + \nu \int_0^t H(t - y)\, dG(y). \tag{2.8}$$

This is the well known renewal equation (see e.g., Feller [4]), and it is easy to show (and known) that $H(t) < \infty$ for any $t < \infty$. Hence $\sum_{k=1}^{\infty} \nu^k G_k(t)$ converges, and in fact since $G_k(t)$ is a nondecreasing function of t, converges uniformly for $0 \leqq t \leqq t' < \infty$. But by (2.7), we have that for any $m > 1$

$$| \hat{P}_{k+m}(s, t \mid x_0) - \hat{P}_k(s, t \mid x_0) | \leqq \sum_{i=k}^{\infty} \nu^i G_i(t).$$

Therefore there exists a function $\hat{P}(s, t \mid x_0)$ such that

$$\hat{P}_k(s, t \mid x_0) \to \hat{P}(s, t \mid x_0) \quad \text{as} \quad k \to \infty \tag{2.9}$$

uniformly for $0 \leqq t \leqq t' < \infty$. From (2.3) it also follows that $\hat{P}(s, t \mid x_0)$ satisfies (2.2).

It will now be shown that $\hat{P}(s, t \mid x_0)$ is the unique bounded solution of (2.2). To do this, suppose that $Q(s, t \mid x_0)$ is another such solution. Then by an argument similar to that which led to the inequalities (2.6), one can show that

$$
\begin{aligned}
&| \hat{P}(s, t \mid x_0) - Q(s, t \mid x_0) | \\
&\quad \leqq \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \\
&\qquad \cdot \sum_{i=1}^{j} | \hat{P}(s, t - y \mid x_i) - Q(s, t - y \mid x_i) |.
\end{aligned} \tag{2.10}
$$

Let $A(s, t \mid x_0) = e^{-\alpha t} \mid \hat{P}(s, t \mid x_0) - Q(s, t \mid x_0) \mid$, where $\alpha \geqq 0$, and is to be chosen later. Then

$$A(s, t \mid x_0) \leqq \sum_{j=1}^{\infty} q_j \int_0^t e^{-\alpha y} \, dG(y) \cdot \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \sum_{i=1}^{j} A(s, t - y \mid x_i). \tag{2.11}$$

From (2.4) and (2.9) it follows that

$$\mid \hat{P}(s, t \mid x_0) \mid \leqq 1, \tag{2.12}$$

and since by assumption $Q(s, t \mid x_0)$ is bounded, it must also be that $\sup \{A(s, t \mid x_0) : 0 \leqq t \leqq t'\} = B(s, t')$ is bounded. Hence

$$B(s,t') \leqq B(s,t') \sum jq_j \int_0^{t'} e^{-\alpha y} \, dG(y). \tag{2.13}$$

Since $G(0) = 0$, (2.13) is contradicted by taking α sufficiently large, unless $B(s, t') = 0$. Therefore $\hat{P}(s, t \mid x_0) = Q(s, t \mid x_0)$, proving uniqueness for equation (2.2).

Turning to (2.1), define the iterates $P_0(x, t \mid x_0) = 0$, and for $k \geqq 0$

$$\begin{aligned} P_{k+1}(x, t \mid x_0) = {} & [1 - G(t)]Z(x - x_0) + q_0 G(t) Z(x) \\ & + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \\ & \cdot [P_k(x, t - y \mid x_1) * \cdots * P_k(x, t - y \mid x_j)]. \end{aligned} \tag{2.14}$$

From (2.14) it follows by induction that $P_k(x, t \mid x_0)$, $k = 0, 1, 2, \cdots$, are nondecreasing and right continuous functions of x which are 0 for x negative. Comparison of (2.3) and (2.14) shows that $\hat{P}_k(s, t \mid x_0)$ is the Fourier-Stieltjes transform of $P_k(x, t \mid x_0)$ with respect to x.

Now when $s = 0$, the constant 1 is a solution of (2.2), and hence by the previously proved uniqueness of the solution of (2.2), and (2.9), $\hat{P}_k(0, t \mid x_0) \to 1$ as $k \to \infty$. Thus for sufficiently large k, $\hat{P}_k(0, t \mid x_0) > 0$, and $P_k(x, t \mid x_0)/\hat{P}_k(0, t \mid x_0)$ is a distribution function whose characteristic function is $\hat{P}_k(s, t \mid x_0)/\hat{P}_k(0, t \mid x_0)$. The latter are thus continuous in s, and hence since $\hat{P}_k(s, t \mid x_0)/\hat{P}_k(0, t \mid x_0) \to \hat{P}(s, t \mid x_0)$ uniformly for $0 \leqq t \leqq t' < \infty$, $\hat{P}(s, t \mid x_0)$ is continuous at $s = 0$. Therefore by the continuity theorem for characteristic functions (see e.g., Cramér [2]), $\lim_{k \to \infty} P_k(x, t \mid x_0)/\hat{P}_k(0, t \mid x_0) \equiv P(x, t \mid x_0)$ exists, is a distribution function, and has $\hat{P}(s, t \mid x_0)$ as its characteristic function. Since $\hat{P}_k(0, t \mid x_0) \to 1$, it follows that $P_k(x, t \mid x_0) \to P(x, t \mid x_0)$, and going to the limit in (2.14), one sees that $P(x, t \mid x_0)$ is a solution of (2.1). The uniqueness theorem for characteristic functions (see e.g., Cramér [2]), and the previously proved uniqueness of the solution of (2.2), then imply the uniqueness result for (2.1).

3. Moments

In this section the existence and properties of

$$\mu^{(n)}(t \mid x_0) = \int_0^\infty x^n P(dx, t \mid x_0) = \frac{1}{i^n} \frac{\partial^n}{\partial s^n} \hat{P}(0, t \mid x_0) \tag{3.1}$$

will be studied. (Write $\mu^{(1)}(t \mid x_0) = \mu(t \mid x_0)$.)

It is useful to define an associated discrete process which is derived from the original process by setting $G(t) = Z(t - 1)$, i.e., requiring that each particle live for exactly one time unit. Then one may speak of the k^{th} generation of the process as consisting of the particles existing at time $t = k$. This is simply the model of the standard discrete branching process (see Harris [5]), generalized to include consideration of the associated characteristic or energy.

The mean energy of the first generation, i.e., of the offspring of an initial particle of energy x_0 is clearly

$$M_1(x_0) = \int_0^\infty \Phi(dx \mid x_0)x, \tag{3.2}$$

where

$$\Phi(x \mid x_0) = \textstyle\sum_{j=1}^\infty q_j \sum_{i=1}^j \Phi_{ij}(x \mid x_0), \tag{3.3}$$

$$\Phi_{ij}(x \mid x_0) = \Phi_j(\infty, \cdots, \infty, x, \infty, \cdots, \infty \mid x_0), \tag{3.3.1}$$

the x being the i^{th} component.

The mean energy of the k^{th} generation, say $M_k(x_0)$, can be computed for $k > 1$ in terms of the iterates

$$M_k(x_0) = \int_0^\infty \Phi(dx \mid x_0) M_{k-1}(x).$$

It will be convenient to write $M_0(x_0) = x_0$. Let

$$\textstyle\sum_{k=0}^\infty M_k(x_0)[G_k(t) - G_{k+1}(t)] = m(t \mid x_0),$$

provided the series converges.

LEMMA 1. *$m(t \mid x_0)$ is a solution of the equation*

$$m(t \mid x_0) = [1 - G(t)]x_0 + \int_0^t dG(y) \int_0^\infty \Phi(dx \mid x_0) m(t - y \mid x). \tag{3.4}$$

Proof. Substitute $\sum_{k=0}^\infty M_k(x)[G_k(t - y) - G_{k+1}(t - y)]$ into the right side of (3.4), and the latter becomes

$$[1 - G(t)]x_0 + \textstyle\sum_{k=0}^\infty M_{k+1}(x)[G_{k+1}(t) - G_{k+2}(t)]$$

which is $m(t \mid x_0)$, proving the lemma.

It will also be necessary to consider the higher moments of the associated discrete process. To derive these, let $F_k(x \mid x_0)$ be the conditional distribution of the total energy of the k^{th} generation, given that the initial particle had energy x_0, and let $\hat{F}_k(s \mid x_0)$ be its characteristic function. Then

$$F_k(x \mid x_0) = q_0 Z(x) + \sum_{j=1}^{\infty} q_j \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \cdot [\hat{F}_{k-1}(x \mid x_1) * \cdots * \hat{F}_{k-1}(x \mid x_j)],$$

and

$$(3.5) \qquad \hat{F}_k(s \mid x_0) = q_0 + \sum_{j=1}^{\infty} q_j \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \prod_{i=1}^{j} \hat{F}_{k-1}(s \mid x_i).$$

Let $M_k^{(n)}(x_0)$ be the n^{th} moment of $F_k(x \mid x_0)$, provided the latter exists. Then differentiation of (3.5) yields

$$M_1^{(n)}(x_0) = \sum_{j=1}^{\infty} q_j \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0)[x_1 + \cdots + x_j]^n,$$

and for $k > 1$

$$(3.6) \qquad M_k^{(n)}(x_0) = \sum_{j=1}^{\infty} q_j \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \sum_{\substack{n_1 + \cdots n_j = n \\ n_i \geq 0}} \binom{n}{n_1 \cdots n_j} \cdot M_{k-1}^{(n_1)}(x_1) \cdots M_{k-1}^{(n_j)}(x_j)$$

where $\binom{n}{n_1 \cdots n_j}$ is the multinomial coefficient, and $M_k^{(0)}(x_0) \equiv 1$. If one adopts the convention $M_0^{(n)}(x_0) = x_0^n$, then (3.6) holds for all $k > 0$.

Rather than working directly with the moments $M_k^{(n)}(x_0)$, which are defined iteratively by a nonlinear operation, it will be easier to work with upper bounds which are defined iteratively by a linear operation. Define

$$(3.7) \qquad N_0^{(n)}(x_0) = M_0^{(n)}(x_0) = x_0^n,$$

and for $k \geq 1$

$$(3.8) \qquad N_k^{(n)}(x_0) = \int_0^{\infty} \Phi^{(n)}(dx \mid x_0) N_{k-1}^{(n)}(x),$$

where

$$(3.8.1) \qquad \Phi^{(n)}(x \mid x_0) = \textstyle\sum_{j=1}^{\infty} j^{n-1} q_j \sum_{i=1}^{j} \Phi_{ij}(x \mid x_0)$$

and $\Phi_{ij}(x \mid x_0)$ is as defined in (3.3.1).

LEMMA 2. *For any set of nonnegative, finite, real numbers $a_1, \cdots, a_k$, and any positive integer n, $\left(\sum_{i=1}^{k} a_i\right)^n \leq k^{n-1} \sum_{i=1}^{k} a_i^n$. For any set of nonnegative random variables $Z_1, \cdots, Z_k$ with finite n^{th} moments,*

$$E\left[\textstyle\sum_{i=1}^{k} Z_i\right]^n \leq k^{n-1} E\left[\textstyle\sum_{i=1}^{k} Z_i^n\right].$$

Proof. The first inequality is a consequence of Jensen's inequality (see e.g., Doob [3, p. 33]) and the fact that x^n is a convex continuous function of x. The second inequality is a trivial consequence of the first.

LEMMA 3. *For all $n \geq 1$, $k \geq 1$, $M_k^{(n)}(x_0) \leq N_k^{(n)}(x_0)$. For $n = 1$ equality holds.*

Proof. We use induction on k. For $k = 1$ the result follows from the first inequality of Lemma 2. Assume it is true for some k. Then by the second inequality of Lemma 2

$$\begin{aligned} M_{k+1}^{(n)}(x_0) &= \sum q_j \int \cdots \int \Phi_j(dx_1, \cdots, dx_j \mid x_0) \sum_{\substack{n_1+\cdots+n_j=n \\ n_i \geqq 0}} \binom{n}{n_1 \cdots n_j} \\ &\qquad \cdot M_k^{(n_1)}(x_1) \cdot \cdots \cdot M_k^{(n_j)}(x_j) \\ &\leqq \sum_{j=1}^{\infty} j^{n-1} q_j \int \cdots \int \Phi_j(dx_1, \cdots, dx_n \mid x_0)[M_k^{(n)}(x_1) + \cdots + M_k^{(n)}(x_j)] \\ &\leqq N_{k+1}^{(n)}(x_0) \qquad \text{by (3.8) and the induction hypothesis.} \end{aligned}$$

Define $\sum_{k=0}^{\infty} N_k^{(n)}(x_0)[G_k(t) - G_{k+1}(t)] = m^{(n)}(t \mid x_0)$, and let

$$N_{k+1}^{(n)}(x_0)/N_k^{(n)}(x_0) = \bar{M}_k^{(n)}(x_0).$$

LEMMA 4. *If for any particular $n \geqq 1$, $\bar{M}_k^{(n)}(x_0)$ is bounded in k, say by $\bar{M}^{(n)}(x_0) < \infty$, then there are a function $\bar{m}^{(n)}(x_0) < \infty$ and a constant c_n such that $m^{(n)}(t \mid x_0)$ is bounded by $c_n x_0^n \exp\{t\bar{m}^{(n)}(x_0)\}$. If, in addition, $\bar{M}^{(n)}(x_0)$ is bounded in x_0, then $\bar{m}^{(n)}(x_0)$ may be chosen bounded in x_0. If $\bar{M}^{(n)}(x_0) < 1$ ($\leqq 1$), then there exists a negative (nonpositive) $m^{(n)}(x_0)$.*

Proof. Let $\sum_{k=0}^{\infty} [\bar{M}^{(n)}(x_0)]^k [G_k(t) - G_{k+1}(t)] = m_{x_0}^{(n)}(t)$. Then by hypothesis $m^{(n)}(t \mid x_0) \leqq x_0^n\, m_{x_0}^{(n)}(t)$. It can be verified by direct substitution that $m_{x_0}^{(n)}(t)$ satisfies the equation

$$m_{x_0}^{(n)}(t) = [1 - G(t)] + \bar{M}^{(n)}(x_0) \int_0^t m_{x_0}^{(n)}(t - y)\, dG(y).$$

But this is again the standard renewal equation, and the conclusions of the lemma can be drawn from well known properties of the renewal function (see Feller [4]).

THEOREM 2.

(a)(i) *If the functions $\bar{M}_k^{(n)}(x_0)$ are bounded in k, say by $\bar{M}^{(n)}(x_0)$, for $n = 1, \cdots, r$, then the moments $\mu^{(n)}(t \mid x_0)$, $n = 1, \cdots, r$ exist and are bounded by functions of the form $c_n x_0^n \exp\{t\bar{m}^{(n)}(x_0)\}$, $n = 1, \cdots, r$, where c_n and $\bar{m}^{(n)}(x_0)$ are finite.*

(ii) *If $\bar{M}^{(n)}(x_0)$ is bounded in x_0, then one may take $\bar{m}^{(n)}(x_0)$ bounded. If $\bar{M}^{(n)}(x_0) < 1$ ($\leqq 1$), then one may take $\bar{m}^{(n)}(x_0) < 0$ ($\leqq 0$).*

(iii) *The moments, provided they exist, are solutions of the equations*

$$\begin{aligned} \mu^{(n)}(t \mid x_0) &= [1 - G(t)]x_0^n \\ &\quad + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \cdots, dx_j \mid x_0) \qquad (3.9) \\ &\quad \cdot \sum_{\substack{n_1+\cdots+n_j=n \\ n_i \geqq 0}} \binom{n}{n_1 \cdots n_j} \mu^{(n_1)}(t - y \mid x_1) \cdots \mu^{(n_j)}(t - y \mid x_j). \end{aligned}$$

(b) *If* $\bar{M}^{(n)}(x_0)$ *is bounded in* x_0 *for* $n = 1, \cdots, r$, *then* $\mu^{(n)}(t \mid x_0)$, $n = 1, \cdots, r$, *are the unique solutions of* (3.9) *among the class of functions of exponential order in t, i.e., functions bounded by functions of the form* $c_n e^{k_n t}$ *for some constants* c_n, k_n.

(c) *If* $\bar{M}(x_0)$ *is bounded in* x_0, *then* $\mu(t \mid x_0) = m(t \mid x_0)$.

Remark. It will be pointed out later that the condition that $\bar{M}^{(n)}(x_0)$ be bounded in x_0 will be satisfied in the important special cases of the process.

Proof. (a) By Lemma 4

$$\textstyle\sum_{i=0}^{\infty} N_i^{(n)}(x_0)[G_i(t) - G_{i+1}(t)] < \infty. \tag{3.10}$$

If $n = 1$, then by (2.3)

$$\left| \frac{\partial}{\partial s} \hat{P}_1(s, t \mid x_0) \right| \leqq [1 - G(t)]x_0.$$

Hence one may differentiate (2.3) with $k = 1$ under the integral with respect to s, and obtain

$$\left| \frac{\partial}{\partial s} \hat{P}_2(s, t \mid x_0) \right| \leqq [1 - G(t)]x_0 + [G(t) - G_2(t)]M_1(x_0).$$

Proceeding by induction on (2.3) one obtains for $k = 1, 2, \cdots$, that

$$\left| \frac{\partial}{\partial s} \hat{P}_k(s, t \mid x_0) \right| \leqq \sum_{i=0}^{k} M_i(x_0)[G_i(t) - G_{i+1}(t)].$$

Hence

$$\mu_k(t \mid x_0) = \frac{1}{i} \frac{\partial}{\partial s} \hat{P}_k(0, t \mid x_0)$$

exists, and one verifies directly from (2.3) that

$$\mu_k(t \mid x_0) = \textstyle\sum_{i=0}^{k} M_i(x_0)[G_i(t) - G_{i+1}(t)] < \infty. \tag{3.11}$$

A similar expression will be obtained for higher moments but with the equality of (3.11) replaced by inequality. By (2.3),

$$\left| \frac{\partial^n}{\partial s^n} \hat{P}_1(s, t \mid x_0) \right| \leqq [1 - G(t)]x_0^n,$$

and if for $n = 1, \cdots, n_0$,

$$\left| \frac{\partial^n}{\partial s^n} \hat{P}_k(s, t \mid x_0) \right| \leqq \sum_{i=0}^{k} N_i^{(n)}(x_0)[G_i(t) - G_{i+1}(t)] \quad (< \infty \text{ by } (3.10)), \tag{3.12}$$

then for any $n \leqq n_0$

$$
\begin{aligned}
\left| \frac{\partial^n}{\partial s^n} \hat{P}_{k+1}(s, t \mid x_0) \right| &\leqq [1 - G(t)] x_0^n \\
&+ \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \cdots, dx_j \mid x_0) \sum_{\substack{n_1 + \cdots + n_j = n \\ n_i \geqq 0}} \binom{n}{n_1 \cdots n_j} \\
&\cdot | \hat{P}_k^{(n_1)}(s, t - y \mid x_1) | \cdot \cdots \cdot | \hat{P}_k^{(n_j)}(s, t - y \mid x_j) |,
\end{aligned} \tag{3.13}
$$

where

$$\hat{P}_k^{(n)}(s, t \mid x_0) = \frac{\partial^n}{\partial s^n} P_k(s, t \mid x_0).$$

But

$$| \hat{P}_k^{(n)}(s, t \mid x_0) | \leqq \int_0^{\infty} x^n P_k(dx, t \mid x_0) = \mu_k^{(n)}(t \mid x_0),$$

and hence by Lemma 2

$$
\begin{aligned}
&\sum_{\substack{n_1 + \cdots + n_j = n \\ n_i \geqq 0}} \binom{n}{n_1 \cdots n_j} | \hat{P}_k^{(n_1)}(s, t \mid x_1) | \cdot \cdots \cdot | \hat{P}_k^{(n_j)}(s, t \mid x_j) | \\
&\leqq \int_0^{\infty} \cdots \int_0^{\infty} (u_1 + \cdots + u_j)^n P_k(du_1, t \mid x_1) \cdots P_k(du_j, t \mid x_j) \\
&\leqq j^{n-1} \int \cdots \int (u_1^n + \cdots + u_j^n) P_k(du_1, t \mid x_1) \cdots P_k(du_j, t \mid x_j) \\
&= j^{n-1} [\mu_k^{(n)}(t \mid x_1) + \cdots + \mu_k^{(n)}(t \mid x_j)].
\end{aligned}
$$

Therefore (3.13)

$$
\begin{aligned}
&\leqq [1 - G(t)] x_0^n + \sum j^{n-1} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \cdots dx_j \mid x_0) \sum_{i=0}^{k} \sum_{h=1}^{j} N_i^{(n)}(x_h) \\
&\qquad \cdot [G_i(t - y) - G_{i+1}(t - y)] \\
&\leqq [1 - G(t)]\, x_0^n + \textstyle\sum_{i=0}^{k} N_{i+1}(x_0) [G_{i+1}(t) - G_{i+2}(t)] \\
&= \textstyle\sum_{i=0}^{k+1} N_i(x_0) [G_i(t) - G_{i+1}(t)].
\end{aligned}
$$

Thus (3.12) is established by induction for all k. If we set $s = 0$ in (3.12), it follows that

$$\mu_k^{(n)}(t \mid x_0) \leqq m^{(n)}(t \mid x_0) < \infty.$$

But by Theorem 1 it follows that

$$\mu^{(n)}(t \mid x_0) = \lim_{A \to \infty} \int_0^A x^n P(dx, t \mid x_0) = \lim_{A \to \infty} \lim_{k \to \infty} \int_0^A x^n P_k(dx, t \mid x_0),$$

and hence, since

$$\lim_{k \to \infty} \int_0^A x^n P_k(dx, t \mid x_0) \leqq m^{(n)}(t \mid x_0) \quad \text{for all } A,$$

one may conclude that for any $n \geqq 1$

$$\mu^{(n)}(t \mid x_0) \leqq m^{(n)}(t \mid x_0) < \infty. \tag{3.14}$$

This proves the first part of (a)(i). The second part of (a)(i) and (a)(ii) follow at once from Lemma 4 and (3.14).

To show (a)(iii) simply note that (3.14) allows one to differentiate (2.2) with respect to s under the integral. This yields (3.9).

(b) This proof is by induction on r. First let $r = 1$. Then (3.9) reads

$$\mu(t \mid x_0) = [1 - G(t)]x_0 + \int_0^t dG(y) \int_0^\infty \Phi(dx \mid x_0)\mu(t - y \mid x), \tag{3.15}$$

where Φ is defined in (3.3). By Lemma 4 and (3.14), μ is of exponential order in t. Suppose that $\rho(t \mid x_0)$ is another such solution of (3.15). Let $\beta(t \mid x_0) = e^{-\alpha t} \mid \mu(t \mid x_0) - \rho(t \mid x_0) \mid$, where α is to be determined later. Then

$$\beta(t \mid x_0) \leqq \int_0^t e^{-\alpha y}\, dG(y) \int_0^\infty \Phi(dx \mid x_0)\beta(t - y \mid x). \tag{3.16}$$

But by the hypothesis and by Lemma 4 one may take α sufficiently large so that $\beta(t \mid x_0)$ is bounded in x_0. Let

$$\gamma(\tau) = \sup \{\beta(t \mid x_0) : 0 \leqq t \leqq \tau, 0 \leqq x_0 < \infty\}.$$

Then by (3.16),

$$\gamma(\tau) \leqq \gamma(\tau) \int_0^\tau e^{-\alpha y}\, dG(y),$$

which is impossible for large α unless $\gamma(\tau) = 0$. Since τ is arbitrary, this implies that $\beta(t \mid x_0) \equiv 0$.

Now assume the result for some $r > 1$, and suppose that $\rho^{(n)}(t \mid x_0)$, $n = 1, \cdots, r$, are functions which are all of exponential order in t, and which simultaneously satisfy (3.9) for $n = 1, \cdots, r$. Then by the induction hypothesis $\rho^{(n)}(t \mid x_0) = \mu^{(n)}(t \mid x_0)$ for $n = 1, \cdots, r$. Now suppose that $\rho^{(r+1)}(t \mid x_0)$ is of exponential order in t and satisfies (3.9). Let $\beta^{(r+1)}(t \mid x_0) = e^{-\alpha t} \mid \mu^{(r+1)}(t \mid x_0) - \rho^{(r+1)}(t \mid x_0) \mid$. Then since $\rho^{(n)} = \mu^{(n)}$ for $n = 1, \cdots, r$, it is easy to see that $\beta^{(r+1)}(t \mid x_0)$ satisfies (3.16). Hence the same argument as before leads to the conclusion that $\beta^{(r+1)}(t \mid x_0) = 0$, completing the uniqueness proof.

(c) This result follows at once from Lemmas 1 and 4, parts (a)(iii) and (b) of this theorem, and (3.14). This completes the proof of the theorem.

The monotonicity properties of the moments will now be investigated, the purpose being to show that a reasonable monotonicity assumption on the distributions Φ_j is reflected by a similar property in the moments.

DEFINITION. The distributions $\Phi_j(x_1, \cdots, x_j \mid x_0)$ will be called monotone if for any $x_0' \leqq x_0''$, and any $j, x_1, \cdots, x_j$,

$$\Phi_j(x_1, \cdots, x_j \mid x_0') \geqq \Phi_j(x_1, \cdots, x_j \mid x_0'').$$

This assumption says essentially that high energy parents tend to have higher energy offspring than low energy parents.

THEOREM 3. *If $\bar{M}^{(n)}(x_0)$ is bounded in x_0, and the distributions Φ_j are monotone, then for any $n \geqq 1$,*
(a) *$\mu^{(n)}(t \mid x_0)$ is a nondecreasing function of x_0;*
(b) *if in addition*

$$\int_0^\infty \Phi(dx \mid x_0)\left(\frac{x}{x_0}\right)^n \geqq 1 \qquad \text{for } n = 1, \cdots, r,$$

then $\mu^{(n)}(t \mid x_0)$ is nondecreasing in t for $n = 1, \cdots, r$. If

$$\int \Phi^{(n)}(dx \mid x_0)\left(\frac{x}{x_0}\right)^n \leqq 1 \qquad \text{for } n = 1, \cdots, r,$$

then $\mu^{(n)}(t \mid x_0)$ is nonincreasing in t for $n = 1, \cdots, r$.

Proof. (a) This proof is by induction on n. When $n = 1$, then by Theorem 2, $\mu(t \mid x_0) = \sum_{k=0}^\infty M_k(x_0)[G_k(t) - G_{k+1}(t)]$. But the monotonicity of the Φ_j implies that $M_k(x_0)$ is monotone in x_0, and hence so is $\mu(t \mid x_0)$. Next suppose the result proved for $n = 1, \cdots, r$. Define

$$\eta^{(n)}(t \mid x_0) = \sum q_j \int_0^t dG(y)$$

$$(3.17) \qquad \cdot \int \cdots \int \Phi_j(dx_1, \cdots dx_j \mid x_0) \sum_{\substack{n_1+\cdots+n_j=n \\ 0 \leqq n_i \leqq n-1}} \binom{n}{n_1 \cdots n_j}$$

$$\cdot \mu^{(n_1)}(t - y \mid x_1) \cdots \mu^{(n_j)}(t - y \mid x_j) + [1 - G(t)]x_0^n,$$

and let $\bar{\mu}_0^{(n)}(t \mid x_0) = \eta^{(n)}(t \mid x_0)$,

$$(3.18) \quad \bar{\mu}_{k+1}^{(n)}(t \mid x_0) = \eta^{(n)}(t \mid x_0) + \int_0^t dG(y) \int \Phi(dx \mid x_0)\bar{\mu}_k^{(n)}(t - y \mid x).$$

(Note that the right side of (3.17) differs from (3.9) in that one summation is over $0 \leqq n_i \leqq n - 1$, rather than $0 \leqq n_i \leqq n$.) Then it is easy to verify by the methods of the proof of Theorem 2 that $\lim_{k\to\infty} \bar{\mu}_k^{(n)}(t \mid x_0) = \bar{\mu}^{(n)}(t \mid x)$ exists and is a solution of (3.9); also by the uniqueness theorem that $\bar{\mu}^{(n)}(t \mid x) = \mu^{(n)}(t \mid x)$. By the original induction hypothesis $\eta^{(n)}(t \mid x_0)$ is monotone in x_0 for $n = 1, \cdots, r + 1$, and hence by using (3.18) and another induction argument on k, it follows that $\bar{\mu}_k^{(r+1)}(t \mid x_0)$ is monotone in x_0 for all k; hence so is $\mu^{(r+1)}(t \mid x_0)$.

(b) The argument is again by induction. For $n = 1$ one has

$$\begin{aligned}\mu(t \mid x) &= \textstyle\sum_{k=0}^{\infty} M_k(x_0)[G_k(t) - G_{k+1}(t)] \\ &= M_0(x_0) + \textstyle\sum_{k=0}^{\infty} [\bar{M}_k(x_0) - 1]M_k(x_0)G_{k+1}(t).\end{aligned}$$

But by hypothesis

$$\int \Phi(dx \mid x_0)x \geqq x_0 ,$$

i.e., $M_1(x_0) \geqq M_0(x_0)$. Due to the monotonicity of Φ, this implies that $M_{k+1}(x_0) \geqq M_k(x_0)$ for all $k \geqq 1$, and hence $\sum [\bar{M}_k(x_0) - 1]M_k(x_0)G_{k+1}(t)$ is nondecreasing in t. If

$$\int \Phi(dx \mid x_0)x \leqq x_0 ,$$

then $\bar{M}_k(x_0) \leqq 1$, and the other inequalities are similarly reversed.

Suppose now that for $n = 1, \cdots, r$, $\mu^{(n)}(t \mid x_0)$ are nondecreasing in t. Let

$$\xi^{(n)}(t \mid x_0) = \eta^{(n)}(t \mid x_0) - [1 - G(t)]x_0^n ,$$

and define

$$\xi_0^{(n)}(t \mid x_0) = \xi^{(n)}(t \mid x_0), \tag{3.19}$$

and for $k \geqq 0$

$$\xi_{k+1}^{(n)}(t \mid x_0) = \int_0^t dG(y) \int_0^\infty \Phi(dx \mid x_0)\xi_k^{(n)}(t - y \mid x); \tag{3.20}$$

further define

$$\hat{M}_1^{(n)}(x_0) = \int_0^\infty \Phi(dx \mid x_0)x^n, \tag{3.21}$$

and for $k > 1$

$$\hat{M}_{k+1}^{(n)}(x_0) = \int_0^\infty \Phi(dx \mid x_0)\hat{M}_k^{(n)}(x). \tag{3.22}$$

Then

$$\bar{\mu}_k^{(n)}(t \mid x_0) = \textstyle\sum_{i=0}^{k} \xi_i^{(n)}(t \mid x_0) + \sum_{i=0}^{k} [G_i(t) - G_{i+1}(t)]\hat{M}_i^{(n)}(x_0). \tag{3.23}$$

By the induction hypothesis $\xi_0^{(r+1)}(t \mid x_0)$ is nondecreasing in t, and hence by (3.20), $\xi_k^{(r+1)}(t \mid x_0)$ is also. By an argument analogous to that of the previous paragraph, it follows from the hypothesis of part (b) of the theorem that $\sum_{i=0}^{k} [G_i(t) - G_{i+1}(t)]\hat{M}_i^{(r+1)}(x_0)$ is similarly monotone in t. Hence so is $\bar{\mu}_k^{(r+1)}(t \mid x_0)$, and so is $\mu^{(r+1)}(t \mid x_0)$.

Finally, suppose that

$$\int_0^\infty \Phi(dx \mid x_0) \left(\frac{x}{x_0}\right)^n \leqq 1.$$

It has been shown that then $\mu(t \mid x)$ is a nonincreasing function of t. Suppose that $\mu^{(n)}(t \mid x_0)$ are nonincreasing in t for $n = 1, \cdots, r$. Then for such n, $\mu^{(n)}(t \mid x_0) \leqq \mu^{(n)}(0 \mid x_0) = x_0^n$, and hence for $t_2 \geqq t_1$ one has

$$(3.24) \quad \eta^{(r+1)}(t_2 \mid x_0) - \eta^{(r+1)}(t_1 \mid x_0) \leqq [G(t_1) - G(t_2)]x_0^{r+1}$$
$$+ \sum q_j \int_0^{t_1} dG(y) \int \cdots \int \Phi_j(dx_1, \cdots, dx_j \mid x_0) \sum \binom{n}{n_1 \cdots n_j}$$
$$\cdot [\mu^{(n_1)}(t_2 - y \mid x_1) \cdots \mu^{(n_j)}(t_2 - y \mid x_j) - \mu^{(n_1)}(t_1 - y \mid x_1) \cdots \mu^{(n_j)}(t_1 - y \mid x_j)]$$
$$+ \int_{t_1}^{t_2} dG(y) \int \Phi^{(r+1)}(dx \mid x_0) x^{r+1}$$
$$\leqq [G(t_2) - G(t_1)] \left[\int \Phi^{(r+1)}(dx \mid x_0) x^{r+1} - x_0^{r+1} \right] \leqq 0.$$

Thus $\bar{\mu}_0^{(r+1)}(t \mid x_0)$ is nonincreasing in t. Arguing by induction on k in (3.18), noting that by the induction hypothesis $\bar{\mu}_k^{(r+1)}(t \mid x_0) \leqq x_0^{r+1}$, and writing an expression analogous to (3.24), one easily sees that $\bar{\mu}_k^{(r+1)}(t \mid x_0)$ is nonincreasing in t for all k. Hence so is $\mu^{(r+1)}(t \mid x_0)$. This completes the induction on r and the proof.

4. Examples

(a) *The age-dependent binary branching process.* (See e.g., Bellman and Harris [1], and Harris [5].) In this case $x_0 = 1$, $q_2 = 1$, and $\Phi_2(x_1, x_2 \mid 1) = 1$ if $x_1 \geqq 1$ and $x_2 \geqq 1$, $\Phi_2 = 0$ otherwise. Then $M_n(x_0) = 2^n$, and $\bar{M}_n(x_0) \equiv 2$. Hence the condition for the existence of the mean is satisfied (as is of course known), and $\mu(t \mid x_0) = \sum_{n=0}^{\infty} 2^n[G_n(t) - G_{n+1}(t)]$. If $G(t)$ is exponential with parameter λ, then $\mu(t \mid x_0) = e^{\lambda t}$. Note that in this example, the total energy is simply the number of particles existing at t.

(b) *The age-dependent branching process.* (See e.g., Levinson [6].) Here $x_0 = 1$, and for $j = 1, 2, \cdots, \Phi_j(x_1, \cdots, x_j \mid 1) = 1$ if $x_i \geqq 1, i = 1, \cdots, j$, $\Phi_j = 0$ otherwise. Let $\sum nq_n = \nu$. Then $M_n(x_0) = \nu^n$, $\bar{M}_n(x_0) \equiv \nu$. Thus a sufficient condition for the existence of the mean is $\nu < \infty$. This result is also given in Levinson's paper, but subject to regularity conditions on $G(\cdot)$. Similar computations of course work for higher moments.

(c) *Nonexistence of the mean.* Take $G(t) = 1 - e^{-t}$ for $t \geqq 0$, $G(t) = 0$ otherwise; $x_0 = 2$; $q_1 = 1$; $\Phi_1(x \mid x_0) = 1$ when $x \geqq 2^{x_0}$, $\Phi_1 = 0$ otherwise. Then $M_n(x_0) \geqq (n!)^2$ and $\mu(t \mid x_0) \geqq e^{-t} \sum k!\, t^k$. This trivial example shows that the moments do not always exist.

(d) *Homogeneous distributions.* In a sequel to this paper attention will be limited to the important class of Φ-distributions which are homogeneous in the sense that $\Phi_j(kx_1, \cdots, kx_j \mid kx_0) = \Phi_j(x_1, \cdots, x_j \mid x_0)$. (This assumption is usually made in the study of cascade processes.) It will be shown there that all the hypotheses of the above theorems are satisfied when this homogeneity assumption is made.

5. The total energy of the process

Another random variable closely related to $X(t)$ is the total energy of the entire process *up to* time t, say $Y(t)$. Let $R(y, t \mid x_0)$ be the distribution function of $Y(t)$, given that the initial particle had energy x_0, and let $\hat{R}(s, t \mid x_0)$ be its characteristic function. These functions can be shown to satisfy the equations

$$(5.1)\qquad \begin{aligned} R(y, t \mid x_0) &= [1 - (1 - q_0)G(t)]Z(y - x_0) \\ &\quad + \sum_{j=1}^{\infty} q_j \int_0^t dG(\tau) \int \cdots \int \Phi(dy_1, \cdots, dy_j \mid x_0) \\ &\qquad \cdot R(y - x_0, t - \tau \mid y_1) * \cdots * R(y - x_0, t - \tau \mid y_j), \end{aligned}$$

and

$$(5.2)\qquad \begin{aligned} \hat{R}(s, t \mid x_0) &= [1 - (1 - q_0)G(t)]e^{isx_0} \\ &\quad + e^{isx_0} \sum_{j=1}^{\infty} q_j \int_0^t dG(\tau) \int \cdots \int \Phi(dy_1, \cdots, dy_j \mid x_0) \prod \hat{R}(s, t - \tau \mid x_j). \end{aligned}$$

Analogous to Theorem 1, there is then

THEOREM 4. *If $\sum nq_n = \nu < \infty$, then there exist unique bounded solutions $R(y, t \mid x_0)$ and $\hat{R}(s, t \mid x_0)$ of* (5.1) *and* (5.2) *respectively. R is a distribution function, and $\hat{R}$ is its characteristic function.*

Proof. The proof is a complete analogue of the proof of Theorem 1, and it is not necessary to write it out a second time.

Similarly, there exist close analogies between the moments of $X(t)$ and $Y(t)$. In order to avoid repetition, we will be content here simply to point out the result for the mean. Let

$$\eta(t \mid x_0) = \int_0^{\infty} yR(dy, t \mid x_0).$$

Then one can show

THEOREM 5. *If the functions $\bar{M}_k(x_0)$ are bounded, then*

$$\eta(t \mid x_0) = \textstyle\sum_{k=0}^{\infty} M_k(x_0)G_k(t) < \infty$$

are of exponential order in t, and among the class of the latter functions, are the unique solutions of the equation

$$(5.3)\qquad \eta(t \mid x_0) = x_0 + \int_0^t dG(\tau) \int_0^{\infty} \Phi(dx \mid x_0)\eta(t - \tau \mid x).$$

Proof. Theorem 5 is proved similarly to Theorem 2.

References

1. R. Bellman and T. Harris, *On age-dependent binary branching processes*, Ann. of Math. (2), vol. 55 (1952), pp. 280–295.
2. H. Cramér, *Mathematical methods of statistics*, Princeton, Princeton University Press, 1951.
3. J. L. Doob, *Stochastic processes*, New York, Wiley, 1953.
4. W. Feller, *On the integral equation of renewal theory*, Ann. Math. Statistics, vol. 12 (1941), pp. 243–267.
5. T. E. Harris, *Branching processes*, Ann. Math. Statistics, vol. 19 (1948), pp. 474–494.
6. N. Levinson, *Limiting theorems for age-dependent branching processes*, Illinois J. Math., vol. 4 (1960), pp. 100–118.
7. J. E. Moyal, *Multiplicative population chains*, Proc. Roy. Soc. London, Ser. A, vol. 266 (1962), pp. 518–526.
8. ———, *Multiplicative population processes*, Stanford University Technical Report, 1961.
9. ———, *The general theory of stochastic population processes*, Acta Math., vol. 108 (1962), pp. 1–31.
10. P. E. Ney, *Some contributions to the theory of cascades*, Ph.D. Thesis, Columbia University, 1961.
11. B. A. Sevast'yanov, *The theory of branching stochastic processes*, Uspehi Mat. Nauk (N.S.), vol. 6, no. 6 (1951), pp. 47–99 (in Russian).

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