THE HYPERCENTER OF FUNCTORIALLY DEFINED SUBGROUPS

BY

REINHOLD BAER[1]

It is not too difficult to prove the following theorem:

I. The commutator subgroup G' of the finite group G is nilpotent if, and only if, U' is nilpotent whenever the subgroup U of G is generated by 3 elements.

Considerably deeper appears to be the following improvement on this result:

I*. The commutator subgroup G' of the finite group G is nilpotent if, and only if, S' is nilpotent whenever the subgroup S of G is generated by 2 elements.

On the basis of I* it is possible to obtain the following considerable generalisation of I:

II. If N is a normal subgroup of the finite group G, then $N \cap G'$ is part of the hypercenter $\mathfrak{h}G'$ of G' if, and only if, $N \cap S' \subseteq \mathfrak{h}S'$ whenever the subgroup S of G is generated by 3 elements one of which belongs to N.

But these theorems are not isolated instances: thus one obtains again true theorems, if one substitutes everywhere for the commutator subgroup the terminal member of the descending central chain; and likewise one obtains true theorems, if one substitutes everywhere at the same time for the commutator subgroup the n^{th} term of the descending central chain and for 2 and 3 the integers $n + 1$ and $n + 2$ respectively. Hence one wishes for a unified treatment of this infinite family of theorems; and this will be provided within the framework of functorially defined characteristic (even fully invariant) subgroups.

To sketch our principal concepts and more fundamental results it will prove convenient to restrict ourselves to finite groups. Then a functor $\mathfrak{f}$ assigns to each finite group G a subgroup $\mathfrak{f}G$ subject to the following—somewhat more restrictive than might be expected—requirements:

$(\mathfrak{f}G)^\sigma \subseteq \mathfrak{f}H$ for every homomorphism σ of the finite group G into the finite group H with $(\mathfrak{f}G)^\sigma = \mathfrak{f}H$ in case σ is an epimorphism.

If $\mathfrak{f}$ is such a functor and n is a positive integer, then $\mathfrak{f}_n G$ is the set of all the elements c in G with the property:

There exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$.

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If we always have $\mathfrak{f}G = \{\mathfrak{f}_n G\}$, then we term the functor $\mathfrak{f}$ an n-functor. Examples: commutator subgroup and terminal member of the descending central chain are 2-functors; the k^{th} member of the descending central chain is a $(k+1)$-functor. A slightly more general and detailed discussion of these and related concepts may be found in §1; the connection with group-theoretical properties, particularly useful for the construction of such functors, is given in §2.

We may now state our principal results: Suppose that $\mathfrak{f}$ is an n-functor with $1 < n$.

A. The following properties of the finite group G are equivalent:

(i) $\mathfrak{f}G$ is nilpotent.

(ii) $\mathfrak{f}S$ is nilpotent whenever the subgroup S of G is generated by $n+1$ elements.

(iii) $\mathfrak{f}S$ is nilpotent whenever the subgroup S of G is generated by n elements and $T/\mathfrak{f}T$ is soluble whenever $T \neq G$ is a subgroup of $\mathfrak{f}G$.

Again the proof of the equivalence of (i) and (ii) is not too difficult [Theorem 4.1,(a)] whereas the proof of the equivalence of (i) and (iii) is considerably harder [Theorem 4.7].

B. N is a normal subgroup of the finite group G with $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$ if and only if,

$N \cap \mathfrak{f}S \subseteq \mathfrak{h}\mathfrak{f}S$ whenever the subgroup S of G is generated by $n+1$ elements one of which belongs to N, and

$T/\mathfrak{f}T$ is soluble for every subgroup T of the group of automorphisms, induced in N by $\mathfrak{f}G$.

The proof of this result depends very much on our result A; see Theorem 6.2. Note that the theorems stated at the outset are all fairly obvious special cases of the theorems A and B.

It will be noted that the finiteness of G is not really important for the above theorems; all that counts is the finiteness of $\mathfrak{f}G$ and N respectively. Slightly deeper are the generalisations that one obtains when requiring only that $\mathfrak{f}G$ and N be noetherian and soluble; see Theorem 4.1, Corollary 4.12, and Theorem 6.6.

Notations

$o(g)$ = order of group element g.
$o(G)$ = order of group G.
$x^y = y^{-1}xy$.
$x^{\sigma-1} = x^{\sigma}x^{-1}$ for x a group element and σ an endomorphism.
$x \circ y = x^{-1}y^{-1}xy$.
$X \circ Y = \{x \circ y$ for all x in X and y in $Y\}$.
$\mathfrak{z}X$ = center of group X.
$\mathfrak{h}X$ = hypercenter of group X = intersection of all normal subgroups N of X with $\mathfrak{z}(X/N) = 1$.

Commutator subgroup of $X = X' =$ derived group of $X = \mathfrak{d}X = X \circ X$.
Derived series of X, inductively defined by

$$X = X^{(0)} = \mathfrak{d}^0 X, \qquad \mathfrak{d}(\mathfrak{d}^i X) = (X^{(i)})' = \mathfrak{d}^{i+1}X = X^{(i+1)}.$$

Descending central chain of X, inductively defined by

$$X = \mathfrak{c}_0 X, \qquad X \circ \mathfrak{c}_i X = \mathfrak{c}_{i+1} X.$$

$\mathfrak{c}X =$ centralizer of subset X of group G (in G).
$\mathfrak{n}X =$ normalizer of subgroup X of group G (in G).
Factor of a group = epimorphic image of subgroup.
A group X is

noetherian, if all its subgroups are finitely generated;
artinian, if the minimum condition is satisfied by its subgroups;
soluble, if every epimorphic image, not 1, of X possesses an abelian normal subgroup, not 1;
nilpotent, if $\mathfrak{z}Y \neq 1$ whenever the epimorphic image Y of X is not 1;
of finite class, if $\mathfrak{c}_i X = 1$ for almost all i;
p-closed, if products of p-elements are p-elements.

1. The concept of functor

The domain of definition of the functors to be considered will be a non-vacuous class $\mathfrak{D}$ of groups, subject to the following requirements:

$\mathfrak{D}$ contains with any group all its subgroups and epimorphic images;
$\mathfrak{D}$ contains with any two groups their direct product.

This class $\mathfrak{D}$ may be the class of all groups or the class of all finite groups and so on. It will usually be kept fixed and will hardly be mentioned in the body of our investigation. Special choices of $\mathfrak{D}$ will be made only when discussing particular situations or when constructing examples.

A *functor* $\mathfrak{f}$ assigns to every group X in $\mathfrak{D}$ a well-determined subgroup $\mathfrak{f}X$ of X. All the functors appearing in the sequel will be subject to the following two requirements, and after some preliminary discussion of these requirements they will be used without any further reference to them:

(I) *If S is a subgroup of the group G in $\mathfrak{D}$, then $\mathfrak{f}S \subseteq \mathfrak{f}G$.*
(II) *If σ is an epimorphism of the group G in $\mathfrak{D}$, then $\mathfrak{f}(G^\sigma) = (\mathfrak{f}G)^\sigma$.*

Restatement of these conditions in the language of exact sequences.

(I) If $1 \to S \xrightarrow{\sigma} G$ is an exact sequence, then $1 \to \mathfrak{f}S \xrightarrow{\sigma} \mathfrak{f}G$ is an exact sequence.

(II) If $G \xrightarrow{\sigma} S \to 1$ is an exact sequence, then $\mathfrak{f}G \xrightarrow{\sigma} \mathfrak{f}S \to 1$ is an exact sequence.

Clearly these conditions are duals of each other, showing that our concept of functor is self-dual.

If η is a homomorphism of the group G in $\mathfrak{D}$ into the group H in $\mathfrak{D}$, then η is an epimorphism of G upon the subgroup G^{η} of H; and we deduce from (I) and (II) the validity of the following general monotonicity property:

(M) *If η is a homomorphism of the group G in $\mathfrak{D}$ into the group H in $\mathfrak{D}$, then* $\mathfrak{f}(G^{\eta}) \subseteq \mathfrak{f}H$.

From (M) it follows that $\mathfrak{f}G$ is always a fully invariant subgroup of G. Thus the center is not a functor in the sense in which we are going to use this term in the present investigation. As a matter of fact neither (I) nor (II) is satisfied by the operation assigning to every group G in $\mathfrak{D}$ its center $\mathfrak{z}G$. Typical examples of functors are the commutator subgroup $\mathfrak{d}X$, the members $\mathfrak{d}^{i}X$ of the derived series, and the members $\mathfrak{c}_i X$ of the descending central chain.

From (II) one deduces that every functor has the following property:

If σ is an epimorphism of the group G in $\mathfrak{D}$ upon the group H, then $\mathfrak{f}H = 1$ if, and only if, $\mathfrak{f}G$ is part of the kernel of σ.

Thus in particular $\mathfrak{f}(G/\mathfrak{f}G) = 1$. One notes furthermore that property (M) implies requirement (I) and the following weakened form of (II):

If σ is an isomorphism of G upon H, then $\mathfrak{f}H = (\mathfrak{f}G)^{\sigma}$.

But (II) is not a consequence of (M), as may be seen from the following

Example 1.1. Let $\mathfrak{D}$ be the universal class $\mathfrak{U}$ of all groups; and define $\mathfrak{f}$ by the equation:

$$\mathfrak{f}G = \mathfrak{d}G \cap G^{p}$$

where p is some fixed odd prime and G^{p} the subgroup generated by all the p^{th} powers of elements in G. It is fairly obvious that $\mathfrak{f}$ meets requirement (M). But $\mathfrak{f}$ does not meet requirement (II), as may be seen from the following example:

Let $A = \{a, b\}$ be the group generated by the elements a, b, subject to the following conditions:

$$a^{p} = b^{p} = (a \circ b)^{p} = 1, \qquad a(a \circ b) = (a \circ b)a,$$
$$b(a \circ b) = (a \circ b)b.$$

Thus $A^{p} = 1$ and $\mathfrak{d}A = \mathfrak{z}A = \{a \circ b\}$. Denote furthermore by B a cyclic group of order p^{2}; and let $G = A \otimes B$ be the direct product of A and B. It is clear then that

$$\mathfrak{d}G = \mathfrak{d}A, \qquad G^{p} = B^{p}, \qquad \mathfrak{f}G = \mathfrak{d}A \cap B^{p} = 1.$$

As $\mathfrak{z}G = \mathfrak{z}A \otimes B$, the subgroup $\{(a \circ b)t^{p}\}$ for t a generator of B is a normal subgroup of G; and $\mathfrak{f}[G/\{(a \circ b)t^{p}\}]$ is easily seen to be cyclic of order p, showing that (II) is not satisfied by $\mathfrak{f}$.

It is a consequence of this example that we use the term "functor" in a somewhat more restricted sense than may be usual.

Example 1.2. Let $\mathfrak{D}$ be the class $\mathfrak{F}$ of all finite groups; and denote by $\mathfrak{j}G$ for G a finite group the intersection of all maximal normal subgroups of G. Then $\mathfrak{j}$ is a functor, meeting the following two requirements:

$G/\mathfrak{j}G$ is a direct product of simple groups;
if the epimorphism σ maps G upon a direct product of simple groups, then $(\mathfrak{j}G)^{\sigma} = 1$.

For the simple verification of these two properties see e.g. Baer [6; Folgerung 4.5].

From the two facts just stated it follows that the functor $\mathfrak{j}$ meets the requirement (II). If on the other hand G is a finite simple group, then $\mathfrak{j}G = 1$. But there exists any number of finite simple groups, possessing subgroups which are not direct products of finite simple groups; and hence (I) is not satisfied by the functor $\mathfrak{j}$.

We pointed out before that $\mathfrak{f}G$ is, for every functor $\mathfrak{f}$, a fully invariant subgroup of G. This raises the question whether $\mathfrak{f}G$ might not always be a word subgroup of G. This is not the case as may be seen from the following example which appears to be of independent interest.

Example 1.3. Let $\mathfrak{D}$ be again the class $\mathfrak{F}$ of all finite groups, and denote by $\mathfrak{c}_\infty G$ the terminal member of the descending central series of G. Because of the finiteness of G we may characterize $\mathfrak{c}_\infty G$ likewise as the intersection of all normal subgroups X of G with nilpotent quotient group G/X. From this last remark it is easily seen that $\mathfrak{c}_\infty G$ is a functor. Consider on the other hand that $\mathfrak{c}_\infty G$ is nothing but the intersection of all the $\mathfrak{c}_i G$; and that the Theorem of Magnus states: the intersection of the members of the descending central chain of a free group is equal to 1; see Specht [1; p. 211, Satz 21]. Hence $\mathfrak{c}_\infty G$ cannot be a word subgroup of G. Our considerations show furthermore that $\mathfrak{c}_\infty G$ would cease to be a functor, if defined on the class $\mathfrak{U}$ of all groups.

LEMMA 1.4. *If $\mathfrak{f}$ is a functor, then $\mathfrak{f}(A \otimes B) = \mathfrak{f}A \otimes \mathfrak{f}B$ for all A, B in $\mathfrak{D}$.*

Proof. Denote by α the decomposition endomorphism of $G = A \otimes B$ which leaves invariant every element in A and maps B upon 1; and denote by β the complementary decomposition endomorphism of G which maps A onto 1 and leaves invariant every element in B. Then $\alpha + \beta = 1$ and we deduce from property (II) that

$$\mathfrak{f}A = (\mathfrak{f}G)^{\alpha}, \qquad \mathfrak{f}B = (\mathfrak{f}G)^{\beta},$$

$$\mathfrak{f}(A \otimes B) = \mathfrak{f}G = (\mathfrak{f}G)^{\alpha}(\mathfrak{f}G)^{\beta} = \mathfrak{f}A \otimes \mathfrak{f}B.$$

Construction of Functors. A. Suppose that the functors $\mathfrak{a}$ and $\mathfrak{b}$ are both defined on the same class $\mathfrak{D}$ of groups. Then their product $\mathfrak{a}\mathfrak{b}$ is well defined by the formula:

$$\mathfrak{a}\mathfrak{b}(G) = \mathfrak{a}(\mathfrak{b}G) \quad \text{for every } G \text{ in } \mathfrak{D}.$$

It is easily seen that with $\mathfrak{a}$ and $\mathfrak{b}$ their product $\mathfrak{ab}$ is also monotone.

B. Suppose that the functors $\mathfrak{a}$ and $\mathfrak{b}$ are both defined on the same class $\mathfrak{D}$ of groups. Then their intersection $\mathfrak{a} \cap \mathfrak{b}$ is defined by the formula:

$$(\mathfrak{a} \cap \mathfrak{b})G = \mathfrak{a}G \cap \mathfrak{b}G \quad \text{for every } G \text{ in } \mathfrak{D}.$$

Example 1.1 shows that $\mathfrak{a} \cap \mathfrak{b}$ need not be a functor.

C. Suppose that the functors $\mathfrak{a}$ and $\mathfrak{b}$ are both defined on the same class $\mathfrak{D}$ of groups. Then their join $\mathfrak{a} \vee \mathfrak{b}$ is defined by the formula:

$$(\mathfrak{a} \vee \mathfrak{b})G = (\mathfrak{a}G)(\mathfrak{b}G) \quad \text{for every } G \text{ in } \mathfrak{D}.$$

This is a well-determined functor. The join of infinitely many functors may be defined likewise; it is likewise a functor.

D. Suppose that the functors $\mathfrak{a}$ and $\mathfrak{b}$ are both defined on the same class $\mathfrak{D}$ of groups. Then their commutator $\mathfrak{a} \circ \mathfrak{b}$ is defined by the formula:

$$(\mathfrak{a} \circ \mathfrak{b})G = \mathfrak{a}G \circ \mathfrak{b}G \quad \text{for every } G \text{ in } \mathfrak{D}.$$

This is a well-determined functor.

Generation of Functors. Consider a functor $\mathfrak{f}$ on the class $\mathfrak{D}$ of groups. Consider a positive integer n. If G is a group in $\mathfrak{D}$, then *the subset* $\mathfrak{f}_n G$ *of* G is defined by the following rule:

The element c in G belongs to $\mathfrak{f}_n G$ if, and only if, there exists a subgroup U of G which is generated by n elements (or less) such that c belongs to $\mathfrak{f}U$.

From the monotonicity of $\mathfrak{f}$ it follows immediately that each of these subsets $\mathfrak{f}_n G$ of G is actually a subset of $\mathfrak{f}G$. One verifies furthermore:

(I.n) *If U is a subgroup of G, then $\mathfrak{f}_n U$ is a subset of $\mathfrak{f}_n G$.*

(II.n) *If σ is an epimorphism of G upon H, then $\mathfrak{f}_n H = (\mathfrak{f}_n G)^\sigma$.*

Note that (I.n) is a consequence of the monotonicity property (I) and that (II.n) is derived from (II) by remarking that subgroups of H are generated by n elements if, and only if, they are the images of subgroups, generated by n elements.

Next we note the obvious fact that the sets $\mathfrak{f}_n G$ form an ascending sequence of subsets of $\mathfrak{f}G$. We form *the join* $\mathfrak{f}_\infty G = \bigcup_{n=1}^{\infty} \mathfrak{f}_n G$ which is naturally a subset of $\mathfrak{f}G$. But if x and y are elements in $\mathfrak{f}_\infty G$, then there exist finitely generated subgroups X and Y of G such that x belongs to $\mathfrak{f}X$ and y to $\mathfrak{f}Y$. Naturally $U = \{X, Y\}$ is a finitely generated subgroup of G; and $\{\mathfrak{f}X, \mathfrak{f}Y\} \subseteq \mathfrak{f}U$ is a consequence of the monotonicity of $\mathfrak{f}$. It follows that x, y and xy^{-1} belong to $\mathfrak{f}U$. Hence xy^{-1} belongs to $\mathfrak{f}_\infty G$, showing that $\mathfrak{f}_\infty G$ is actually a characteristic subgroup of G.

We introduce now the following definitions which will prove fundamental in the sequel:

The functor $\mathfrak{f}$ *is an n-functor, if* $\mathfrak{f}G = \{f_n G\}$ *for every* G *in* $\mathfrak{D}$; *and the functor* $\mathfrak{f}$ *is finitely definable, if* $\mathfrak{f}G = \mathfrak{f}_\infty G$ *for every* G *in* $\mathfrak{D}$.

For every positive integer i the subgroup G^i generated by all the i^{th} powers of elements in G defines a 1-functor. Furthermore $\mathfrak{d}^k$ is a $2k$-functor and $\mathfrak{c}_k$ is a $(k + 1)$-functor. These three families of functors are defined on the universal class $\mathfrak{U}$ of all groups.

Next we discuss two functors defined on the class $\mathfrak{F}$ of all finite groups. We remember that a finite group G is nilpotent if, and only if, elements of relatively prime order in G commute. It follows that the terminal member $\mathfrak{c}_\infty G$ of the descending central chain of the finite group G—see Example 1.3—is generated by all the commutators $x \circ y$ for elements x, y in G with relatively prime orders. Hence $\mathfrak{c}_\infty$ is a 2-functor. Let furthermore $\mathfrak{d}^\infty G$ be the terminal member of the derived series $\mathfrak{d}^i G$ of the finite group G. This functor $\mathfrak{d}^\infty$ has the following characteristic property:

The epimorphism σ of G maps G upon a soluble group if, and only if, $(\mathfrak{d}^\infty G)^\sigma = 1$.

It has been shown by John Thompson that every finite simple group all of whose proper subgroups are soluble is one of the known simple groups and may therefore be generated by two elements. Consequently $\mathfrak{d}^\infty$ is a 2-functor.

Remark 1.5. The author has been unable to construct an example of a functor on the class $\mathfrak{F}$ of all finite groups which is not an n-functor for suitable n.

If G is a group in $\mathfrak{D}$, then *the subset* $\mathfrak{f}_\mathfrak{a} G$ of G is defined by the following rule:

The element c *in* G *belongs to* $\mathfrak{f}_\mathfrak{a} G$ *if, and only if, there exists a countable subgroup* U *of* G *such that* c *belongs to* $\mathfrak{f}U$.

One verifies without any trouble that $\mathfrak{f}_\mathfrak{a} G$ is a characteristic subgroup of G which is contained in $\mathfrak{f}G$, meeting the following requirements:

(I.$\mathfrak{a}$) If U is a subgroup of G, then $f_\mathfrak{a} U \subseteq \mathfrak{f}_\mathfrak{a} G$.

(II.$\mathfrak{a}$) If σ is an epimorphism of G upon H, then $\mathfrak{f}_\mathfrak{a} H = (\mathfrak{f}_\mathfrak{a} G)^\sigma$.

If it so happens that $\mathfrak{f}G = \mathfrak{f}_\mathfrak{a} G$ for every G in $\mathfrak{D}$, then we term $\mathfrak{f}$ *a countably definable functor.*

Example 1.6. Let $\mathfrak{D}$ be the class of all locally finite groups with minimum condition for all subgroups. If G is a group in $\mathfrak{D}$, then we define $\mathfrak{f}G$ as the intersection of all subgroups X of G with finite index $[G:X]$. It is well known and easily verified that $[G:\mathfrak{f}G]$ is finite, too; and thus one sees that $\mathfrak{f}$ is a functor on $\mathfrak{D}$.

If G belongs to $\mathfrak{D}$, and if U is a finitely generated subgroup of G, then U is finite so that $\mathfrak{f}U = 1$. This implies $\mathfrak{f}_\infty G = 1$. On the other hand $\mathfrak{f}G = \mathfrak{f}_\mathfrak{a} G$, showing that finite definability and countable definability are in general different properties of a functor.

A joint definition of n-functor, finitely definable functor, and countably definable functor may be obtained as follows. Denote by $\mathfrak{n}$ some finite or infinite cardinal number. If $\mathfrak{f}$ is a functor and G is a group in $\mathfrak{D}$, then denote by $\mathfrak{f}(\mathfrak{n}, G)$ the set of all the elements c in G with the following property:

There exists a subgroup C of G which is generated by less than $\mathfrak{n}$ elements such that c belongs to $\mathfrak{f}C$.

If $\mathfrak{n}$ happens to be a natural integer, then $\mathfrak{f}(\mathfrak{n}, G) = \mathfrak{f}_{\mathfrak{n}-1}G$; and $\mathfrak{f}(\aleph_0, G) = \mathfrak{f}_\infty G$ and $\mathfrak{f}(\aleph_1, G) = \mathfrak{f}_\mathfrak{a} G$. It is furthermore clear that always $\mathfrak{f}(\mathfrak{n}, G) \subseteq \mathfrak{f}G$, and $\mathfrak{f}(\mathfrak{n}, G)$ is for infinite $\mathfrak{n}$ a characteristic subgroup of G. We shall, however, not investigate these generalisations in the present paper.

2. Group-theoretical properties and functors

Of the many possible ways to relate group-theoretical properties and functors only one will be discussed here. This may be described in a general fashion as follows:

Suppose that $\mathfrak{D}$ is as usual a nonvacuous class of groups which contains with any group all its subgroups and epimorphic images, and with any pair of groups their direct product. Suppose furthermore that the functor $\mathfrak{f}$ is defined on $\mathfrak{D}$ and that the group-theoretical property $\mathfrak{P}$ is defined on $\mathfrak{D}$ (so that a group in $\mathfrak{D}$ may or may not be a $\mathfrak{P}$-group). Then we shall always say that

$\mathfrak{f}$ is a *co-$\mathfrak{P}$-functor*, if G is a $\mathfrak{P}$-group whenever $\mathfrak{f}G = 1$.

$\mathfrak{P}$ is a *co-$\mathfrak{f}$-property*, if $\mathfrak{f}G = 1$ whenever G is a $\mathfrak{P}$-group.

Thus the commutator subgroup functor $\mathfrak{d}$ is co-abelian and co-nilpotent and co-soluble whereas the property of being commutative is co-$\mathfrak{c}_i$ and co-$\mathfrak{d}^j$. Accordingly we shall term $\mathfrak{f}$ and $\mathfrak{P}$ *complementary*, provided

$$G \text{ is a } \mathfrak{P}\text{-group if, and only if, } \mathfrak{f}G = 1.$$

It is now almost obvious how to construct to a given property (functor) the complementary functor (property), and this we are going to do now in a systematic fashion. We set down the definitions:

The group G is a $\mathfrak{D}_\mathfrak{f}$-group if, and only if, $\mathfrak{f}G = 1$.

If G is a group in $\mathfrak{D}$, then $\mathfrak{f}_\mathfrak{P} G$ is the intersection of all the normal subgroups X of G such that G/X is a $\mathfrak{P}$-group.

It is clear that $\mathfrak{D}_\mathfrak{f}$ is always a well-determined group-theoretical property: there exist $\mathfrak{D}_\mathfrak{f}$-groups like 1, and isomorphic images of $\mathfrak{D}_\mathfrak{f}$-groups are $\mathfrak{D}_\mathfrak{f}$-groups. Likewise $f_\mathfrak{P} G$ is always a well-determined characteristic subgroup of G.

If $\mathfrak{a}$ and $\mathfrak{b}$ are functors on $\mathfrak{D}$ such that $\mathfrak{a}G \subseteq \mathfrak{b}G$ for every G in $\mathfrak{D}$, then every $\mathfrak{D}_\mathfrak{b}$-group is at the same time a $\mathfrak{D}_\mathfrak{a}$-group.

If $\mathfrak{A}$ and $\mathfrak{B}$ are group-theoretical properties on $\mathfrak{D}$ such that every $\mathfrak{A}$-group is at the same time a $\mathfrak{B}$-group, then $\mathfrak{f}_\mathfrak{B} G \subseteq \mathfrak{f}_\mathfrak{A} G$ for every G.

PROPOSITION 2.1. *If* $\mathfrak{f}$ *is a functor, then* $\mathfrak{f} = \mathfrak{f}_{\mathfrak{D}_\mathfrak{f}}$, *and subgroups, epimorphic images, and direct products of pairs of* $\mathfrak{D}_\mathfrak{f}$*-groups are* $\mathfrak{D}_\mathfrak{f}$*-groups.*

Proof. If σ is an epimorphism of G upon H, then by definition $\mathfrak{f}H = 1$ whenever H is a $\mathfrak{D}_\mathfrak{f}$-group. From the defining property (II) of functors and $\mathfrak{f}H = 1$ we deduce $(\mathfrak{f}G)^\sigma = \mathfrak{f}H = 1$; and this in turn implies that $\mathfrak{f}G$ is part of the kernel of σ. If $\mathfrak{f}G$ is part of the kernel of σ, then it follows likewise that $\mathfrak{f}H = (\mathfrak{f}G)^\sigma = 1$; and this implies by definition that H is a $\mathfrak{D}_\mathfrak{f}$-group. Hence H is a $\mathfrak{D}_\mathfrak{f}$-group if, and only if, $\mathfrak{f}G$ is part of the kernel of σ; and this is equivalent to saying

$\mathfrak{f}G$ is the intersection of all the normal subgroups X of G such that G/X is a $\mathfrak{D}_\mathfrak{f}$-group.

But this property is just an explicit statement of $\mathfrak{f} = \mathfrak{f}_{\mathfrak{D}_\mathfrak{f}}$. The other claims of Proposition 2.1 are fairly immediate consequences of the defining properties of a functor and of Lemma 1.4.

PROPOSITION 2.2. *If the minimum condition is satisfied by the normal subgroups of every group in* $\mathfrak{D}$, *then the following properties of the group-theoretical property* $\mathfrak{P}$ *(on* $\mathfrak{D}$*) are equivalent:*

(i) $\mathfrak{f}_\mathfrak{P}$ *is a functor and* $\mathfrak{P} = \mathfrak{D}_{\mathfrak{f}_\mathfrak{P}}$.

(ii) *Subgroups, epimorphic images, and direct products of pairs of* $\mathfrak{P}$*-groups are* $\mathfrak{P}$*-groups.*

(iii) *Subgroups of* $\mathfrak{P}$*-groups are* $\mathfrak{P}$*-groups, and the epimorphism* σ *maps* G *upon a* $\mathfrak{P}$*-group if, and only if,* $(\mathfrak{f}_\mathfrak{P} G)^\sigma = 1$.

Proof. Assume first the validity of (i). If G is a $\mathfrak{P}$-group, then by definition $\mathfrak{f}_\mathfrak{P} G = 1$. If S is a subgroup of G, then $\mathfrak{f}_\mathfrak{P} S = 1$, since $\mathfrak{f}_\mathfrak{P}$ is a functor, and hence S is a $\mathfrak{P}$-group, since $\mathfrak{P} = \mathfrak{D}_{\mathfrak{f}_\mathfrak{P}}$. Likewise $\mathfrak{f}_\mathfrak{P} H = 1$ for every epimorphic image H of G so that H too is a $\mathfrak{P}$-group. If finally A and B are $\mathfrak{P}$-groups, then $\mathfrak{f}_\mathfrak{P} A = 1$ and $\mathfrak{f}_\mathfrak{P} B = 1$ implying $\mathfrak{f}_\mathfrak{P}(A \otimes B) = 1$ by Lemma 1.4 so that $A \otimes B$ is a $\mathfrak{P}$-group too. Thus (ii) is a consequence of (i).

Assume next the validity of (ii). If G is any group in $\mathfrak{D}$, then there exists because of the minimum condition among the normal subgroups of G with $\mathfrak{P}$-quotient group a minimal one, say G^*. Then G/G^* is a $\mathfrak{P}$-group so that in particular $\mathfrak{f}_\mathfrak{P} G \subseteq G^*$ (by definition). If X is another normal subgroup of G with $\mathfrak{P}$-quotient group G/X, then $G/(G^* \cap X)$ is isomorphic to a subgroup of the direct product of the $\mathfrak{P}$-groups G/G^* and G/X. Apply (ii) to see that $G/(G^* \cap X)$ is a $\mathfrak{P}$-group. From the minimality of G^* we infer now that $G^* = G^* \cap X \subseteq X$; and this implies $G^* \subseteq \mathfrak{f}_\mathfrak{P} G$ so that (iii) is a consequence of (ii).

Assume finally the validity of (iii). If G is a $\mathfrak{P}$-group, then (by definition) $\mathfrak{f}_\mathfrak{P} G = 1$; and if conversely $\mathfrak{f}_\mathfrak{P} G = 1$, then (by the second part of (iii)) G is a $\mathfrak{P}$-group. Hence $\mathfrak{P} = \mathfrak{D}_{\mathfrak{f}_\mathfrak{P}}$. If X is any group, then every epimorphism σ of X upon Y induces an epimorphism of X upon $Y/(\mathfrak{f}_\mathfrak{P} X)^\sigma$ which latter group is a $\mathfrak{P}$-group by the second part of (iii). Hence $\mathfrak{f}_\mathfrak{P} Y \subseteq (\mathfrak{f}_\mathfrak{P} X)^\sigma$. If J is the inverse image of $\mathfrak{f}_\mathfrak{P} Y$ with respect to the epimorphism σ of X upon Y, then

X/J and $Y/\mathfrak{f}_{\mathfrak{P}}Y$ are isomorphic groups. But the latter group is again a $\mathfrak{P}$-group (by the second part of (iii)) so that $\mathfrak{f}_{\mathfrak{P}}X \subseteq J$ and

$$(\mathfrak{f}_{\mathfrak{P}}X)^{\sigma} \subseteq J^{\sigma} = \mathfrak{f}_{\mathfrak{P}}Y \subseteq (\mathfrak{f}_{\mathfrak{P}}X)^{\sigma}$$

implying $\mathfrak{f}_{\mathfrak{P}}Y = (\mathfrak{f}_{\mathfrak{P}}X)^{\sigma}$. If finally S is a subgroup of X, then

$$S/(S \cap \mathfrak{f}_{\mathfrak{P}}X) \simeq S\mathfrak{f}_{\mathfrak{P}}X/\mathfrak{f}_{\mathfrak{P}}X$$

is a subgroup of the $\mathfrak{P}$-group $X/\mathfrak{f}_{\mathfrak{P}}X$ and is consequently itself a $\mathfrak{P}$-group. But then we deduce from the second part of (iii) that $\mathfrak{f}_{\mathfrak{P}}S \subseteq S \cap \mathfrak{f}_{\mathfrak{P}}X \subseteq \mathfrak{f}_{\mathfrak{P}}X$, proving that $\mathfrak{f}_{\mathfrak{P}}$ is a functor and (i) a consequence of (iii).

Remark 2.3. We note that the minimum condition for the normal subgroups of groups in $\mathfrak{D}$ has been used only when deducing (iii) from (ii). But it is indispensable for the validity of Proposition 2.2, as may be seen from the following simple example:

Denote by $\mathfrak{D}$ the class of all groups and by $\mathfrak{P}$ the property of being of finite class—it would suffice to assume that $\mathfrak{P}$ is the property of being a finite nilpotent group or even a finite p-group for fixed prime p. Then $\mathfrak{P}$ certainly meets requirement (ii). If F is a (non-abelian) free group, then according to the Theorem of Magnus, 1 is the intersection of the members of the descending central chain of F; see Specht [1; p. 211, Satz 21]. It follows that $\mathfrak{f}_{\mathfrak{P}}F = 1$. If H is a finite, non-abelian, simple group on 2 generators, then $H = \mathfrak{f}_{\mathfrak{P}}H$ is an epimorphic image of F so that $\mathfrak{f}_{\mathfrak{P}}$ is not a functor, as it does not meet the defining requirement (II).

Remark 2.4. If $\mathfrak{D}$ is the class of all finite groups and $\mathfrak{P}$ the property of being a finite cyclic group, then $\mathfrak{f}_{\mathfrak{P}}G$ is for every finite group G just the commutator subgroup of G. Thus $\mathfrak{f}_{\mathfrak{P}}$ is a (co-abelian) functor, and $\mathfrak{D}_{\mathfrak{f}_{\mathfrak{P}}}$ is the property of being commutative. Hence $\mathfrak{P} \neq \mathfrak{D}_{\mathfrak{f}_{\mathfrak{P}}}$, proving that the second part of condition (i) is not a consequence of its first part. We mention in passing that the $\mathfrak{D}_{\mathfrak{f}_{\mathfrak{P}}}$-groups are just the "residually $\mathfrak{P}$-groups".

Because of the identity $\mathfrak{f} = \mathfrak{f}_{\mathfrak{D}_{\mathfrak{f}}}$ one may expect information concerning $\mathfrak{f}$ from properties of $\mathfrak{D}_{\mathfrak{f}}$. The following sample of such a result will prove useful when constructing examples.

LEMMA 2.5. *The following condition is necessary and sufficient for the functor* $\mathfrak{f}$ *to be an n-functor*:

(+) $\mathfrak{f}G = 1$ *if, and only if,* $\mathfrak{f}U = 1$ *for every subgroup* U *of* G *which is generated by n elements.*

Proof. $\mathfrak{f}G = 1$ implies $\mathfrak{f}U = 1$ for every subgroup U of G. Thus only the "if"-part of (+) is relevant.

Assume now that $\mathfrak{f}$ is an n-functor and that G is a group such that $\mathfrak{f}U = 1$ for every subgroup U of G which is generated by n elements. By definition $\mathfrak{f}_n G = 1$, and this implies $\mathfrak{f}G = \{\mathfrak{f}_n G\} = 1$, proving the necessity of (+).

Assume conversely the validity of (+). Consider a group G (in $\mathfrak{D}$), and let $G^{\wedge} = \{\mathfrak{f}_n G\}$. It is clear that $G^{\wedge}$ is a characteristic subgroup of G and that $G^{\wedge} \subseteq \mathfrak{f}G$. Denote by σ the canonical epimorphism of G upon $H = G/G^{\wedge}$. If the subgroup V of H is generated by n elements, then there exists a subgroup U of G which is generated by n elements such that $V = U^{\sigma}$. Since $\mathfrak{f}U$ is part of $\mathfrak{f}_n G$, we have

$$\mathfrak{f}V = (\mathfrak{f}U)^{\sigma} \subseteq G^{\wedge\sigma} = 1.$$

Application of $(+)$ shows $\mathfrak{f}H = 1$; and this implies

$$\mathfrak{f}G/G^{\wedge} = (\mathfrak{f}G)^{\sigma} = \mathfrak{f}H = 1.$$

Hence $\mathfrak{f}G = G^{\wedge} = \{\mathfrak{f}_n G\}$, proving that $\mathfrak{f}$ is an n-functor.

3. The hypercenter of $\mathfrak{f}G$

If A is any group, then its hypercenter $\mathfrak{h}A$ is the intersection of all the normal subgroups X of A with $\mathfrak{z}(A/X) = 1$. One may also define $\mathfrak{h}A$ as the terminal member of the transfinite ascending central chain. It is clear that the hypercenter is a characteristic subgroup. For the reader's convenience we collect a few properties of the hypercenter which we shall use in the future without explicit reference:

If σ is an epimorphism of A upon B, then $(\mathfrak{h}A)^{\sigma} \subseteq \mathfrak{h}B$.
If N is a normal subgroup of A, then $N \cap \mathfrak{h}A = 1$ and $N \cap \mathfrak{z}A = 1$ are equivalent properties of N.

For this and further properties of the hypercenter cp. Baer [1].

From now on we shall make use of the following properties of the class $\mathfrak{D}$ of groups and the functor $\mathfrak{f}$ on $\mathfrak{D}$ without restating these requirements:

$\mathfrak{D}$ contains with any group all its subgroups and epimorphic images, and with any two groups their direct product; and $\mathfrak{f}$ is a functor on $\mathfrak{D}$.

All groups considered will belong to $\mathfrak{D}$.

THEOREM 3.1. *The following properties of the normal subgroup N of G are equivalent:*

(1) $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.
(2) $N \circ \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.
(3) *If σ is an epimorphism of G upon H, and if M is a normal subgroup of H with $1 \subset M \subseteq N^{\sigma}$, then $M \cap \mathfrak{c}\mathfrak{f}H \neq 1$.*
(4) *If σ is an epimorphism of G upon H with $N^{\sigma} \neq 1$, then $N^{\sigma} \cap \mathfrak{c}\mathfrak{f}H \neq 1$.*
(5) $(\mathfrak{f}G)^{(i)} \circ (N \cap \mathfrak{f}G) \subseteq \mathfrak{h}\mathfrak{f}G$ *for at least one $i \geqq 0$.*

Notational Remark. If X and Y are normal subgroups of the group G, then $X^{(i)} \circ Y$ is defined inductively by the rules:

$$X^{(0)} \circ Y = Y, \qquad X^{(i+1)} \circ Y = X \circ [X^{(i)} \circ Y].$$

Note. Since $\mathfrak{c}\mathfrak{f}H$ is a characteristic subgroup of H, the subgroup $N^{\sigma} \cap \mathfrak{c}\mathfrak{f}H$ appearing in (4) is a normal subgroup of H which is centralized by $\mathfrak{f}H$. The group of automorphisms, induced in $N^{\sigma} \cap \mathfrak{h}\mathfrak{f}H$ by H, is therefore an epimorphic image of $H/\mathfrak{f}H$; and as such its $\mathfrak{f}$-subgroup equals 1. It is therefore what we termed in §2 a $\mathfrak{D}_{\mathfrak{f}}$-group.

Proof. Since N and $\mathfrak{f}G$ are normal subgroups of G, we have

$$N \circ \mathfrak{f}G \subseteq N \cap \mathfrak{f}G$$

so that (2) is a consequence of (1).

Assume next the validity of (2), and consider an epimorphism σ of G upon H and a normal subgroup M of H with $1 \subset M \subseteq N^{\sigma}$. If firstly $M \circ \mathfrak{f}H = 1$, then $M \subseteq \mathfrak{c}\mathfrak{f}H$ so that $1 \neq M = M \cap \mathfrak{c}\mathfrak{f}H$. If secondly $M \circ \mathfrak{f}H \neq 1$, then

$$1 \subset M \circ \mathfrak{f}H \subseteq N^{\sigma} \circ (\mathfrak{f}G)^{\sigma} = (N \circ \mathfrak{f}G)^{\sigma} \subseteq (\mathfrak{h}\mathfrak{f}G)^{\sigma} \subseteq \mathfrak{h}(\mathfrak{f}G)^{\sigma} = \mathfrak{h}\mathfrak{f}H;$$

and it is well known that this implies $J = (M \circ \mathfrak{f}H) \cap \mathfrak{z}\mathfrak{f}H \neq 1$. Hence

$$1 \subset J \subseteq M \cap \mathfrak{c}\mathfrak{f}H;$$

and thus we have shown in either case that (3) is a consequence of (2). It is clear that (4) is a consequence of (3); as a matter of fact (4) is prima facie a weaker statement than (3).

We assume next the validity of (4). Since $N \cap \mathfrak{h}\mathfrak{f}G$ is a normal subgroup of G, we may form $H = G/(N \cap \mathfrak{h}\mathfrak{f}G)$; and we denote by σ the canonical epimorphism of G upon H. Assume by way of contradiction that $N^{\sigma} \cap \mathfrak{f}H \neq 1$. There exist normal subgroups X of H with $X \subseteq N^{\sigma}$ and $X \cap \mathfrak{f}H = 1$; and among these normal subgroups X of H there exists a maximal one, say M (Maximum Principle of Set Theory). We let $H^* = H/M$, and we denote by τ the canonical epimorphism of G upon H^*. From $N^{\sigma} \cap \mathfrak{f}H \neq 1$ and $M \cap \mathfrak{f}H = 1$ we deduce $M \neq N^{\sigma}$; and from $M \subseteq N^{\sigma}$ and $N^{\tau} = N^{\sigma}/M$ we deduce now that $N^{\tau} \neq 1$. Application of (4) produces a normal subgroup J of H^* with $1 \subset J \subseteq N^{\tau}$ and $J \circ \mathfrak{f}H^* = 1$. Now $J = L/M$ where L is a normal subgroup of H satisfying $M \subset L \subseteq N^{\sigma}$; and from the maximality of M we deduce $K = L \cap \mathfrak{f}H \neq 1$. Clearly K is a normal subgroup of H with

$$1 \subset K \subseteq N^{\sigma} \cap \mathfrak{f}H;$$

and from $\mathfrak{f}H^* = M \cdot \mathfrak{f}H/M$ and $J \circ \mathfrak{f}H^* = 1$ we deduce

$$L \circ \mathfrak{f}H \subseteq M \cap \mathfrak{f}H = 1.$$

It follows that $K \subseteq \mathfrak{z}\mathfrak{f}H$. If $T = K^{\sigma^{-1}}$ is the inverse image of K under σ, then T is a normal subgroup of G which, because of

$$1 \subset K \subseteq N^{\sigma} \cap \mathfrak{z}\mathfrak{f}H \quad \text{and} \quad \mathfrak{f}H = \mathfrak{f}G/(N \cap \mathfrak{h}\mathfrak{f}G),$$

satisfies

$$N \cap \mathfrak{h}\mathfrak{f}G \subset T \subseteq N \cap \mathfrak{h}\mathfrak{f}G.$$

This is a contradiction proving

$$1 = N^{\sigma} \cap \mathfrak{f}H = (N \cap \mathfrak{f}G)/(N \cap \mathfrak{h}\mathfrak{f}G).$$

Hence $\mathfrak{f}G \cap N = N \cap \mathfrak{h}\mathfrak{f}G$, showing that (1) is a consequence of (4).

It is clear that (5) is a consequence of (1)—let $i = 0$. If conversely (5) is true, then there exists a minimal i with

$$(\mathfrak{f}G)^{(i)} \circ (N \cap \mathfrak{f}G) \subseteq \mathfrak{h}\mathfrak{f}G.$$

If i were positive, then we could apply the equivalence of (1) and (2) to deduce

$$(\mathfrak{f}G)^{(i-1)} \circ (N \cap \mathfrak{f}G) \subseteq \mathfrak{h}\mathfrak{f}G,$$

contradicting our choice of a minimal i. Hence $i = 0$, showing that (1) is a consequence of (5).

COROLLARY 3.2. *If N is a normal subgroup of G and $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$, then*
(a) *$N \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}U$ for every subgroup U of G, and*
(b) *$N^\sigma \cap \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$ for every epimorphism σ of G upon H.*

Proof. If U is a subgroup of G, then

$$N \cap \mathfrak{f}U = N \cap \mathfrak{f}G \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}G \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}U,$$

proving (a).

If σ is an epimorphism of G upon H, then we apply the equivalence of conditions (1) and (2) of Theorem 3.1 to show

$$N^\sigma \circ \mathfrak{f}H = N^\sigma \circ (\mathfrak{f}G)^\sigma = (N \circ \mathfrak{f}G)^\sigma \subseteq (\mathfrak{h}\mathfrak{f}G)^\sigma \subseteq \mathfrak{h}\mathfrak{f}H;$$

and a second application of Theorem 3.1 proves (b).

COROLLARY 3.3. *If N is a normal subgroup of G, if the kernel K of the epimorphism σ of G upon H is part of N, if U is a subgroup of G, then $N \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}U$ implies $N^\sigma \cap \mathfrak{f}(U^\sigma) \subseteq \mathfrak{h}\mathfrak{f}(U^\sigma)$.*

Proof. If d is an element in $N^\sigma \cap \mathfrak{f}(U^\sigma)$, then we deduce from $\mathfrak{f}(U^\sigma) = (\mathfrak{f}U)^\sigma$ the existence of an element s in N and an element t in $\mathfrak{f}U$ with $d = s^\sigma = t^\sigma$. The element ts^{-1} belongs to the kernel K of σ which by hypothesis is part of N. Hence $t = (ts^{-1})s$ is an element in $N \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}U$ so that $d = t^\sigma$ belongs to $(\mathfrak{h}\mathfrak{f}U)^\sigma \subseteq \mathfrak{h}\mathfrak{f}(U^\sigma)$, proving $N^\sigma \cap \mathfrak{f}(U^\sigma) \subseteq \mathfrak{h}\mathfrak{f}(U^\sigma)$.

Remark 3.4. The following simple construction shows that the condition $K \subseteq N$, imposed in Corollary 3.3, is indispensable. Let $\mathfrak{D}$ be the class of all groups (or the class of finite groups, etc.), and $\mathfrak{f}G = G$ for every G in $\mathfrak{D}$ (or $\mathfrak{f} = \mathfrak{d}$, etc.). Consider a pair N, K of isomorphic, finite, simple, non-abelian groups; and let $G = N \otimes K$ be their direct product. Then there exists a subgroup U of G such that

$$N \cap U = U \cap K = 1, \qquad NU = UK = G, \qquad U \simeq N \simeq K.$$

Clearly $N \cap \mathfrak{f}U = 1 \subseteq \mathfrak{h}\mathfrak{f}U$. Let $H = G/K$, and denote by σ the canonical epimorphism of G upon H. Then $N^\sigma = U^\sigma = H$ and hence

$$N^\sigma \cap \mathfrak{f}(U^\sigma) = H \not\subseteq 1 = \mathfrak{h}\mathfrak{f}(U^\sigma).$$

From the indications made one sees the possibility of many variations of this example.

We denote by $\mathfrak{C}\mathfrak{f}G$ *the product of all the normal subgroups* X *of* G *with* $X \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$. This is a well-determined characteristic subgroup of G.

PROPOSITION 3.5. $\mathfrak{C}\mathfrak{f}G \cap \mathfrak{f}G = \mathfrak{h}\mathfrak{f}G$ *and* $\mathfrak{C}\mathfrak{f}G/\mathfrak{h}\mathfrak{f}G = \mathfrak{C}(\mathfrak{f}G/\mathfrak{h}\mathfrak{f}G)$; *and* $\mathfrak{C}\mathfrak{f}G$ *is completely determined by these two properties.*

Proof. The set Θ of all the normal subgroups X of G with $X \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$ is because of Theorem 3.1 identical with the set of all normal subgroups X of G satisfying $X \circ \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$. It follows that $\mathfrak{C}\mathfrak{f}G \circ \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$; and Theorem 3.1 implies now $\mathfrak{C}\mathfrak{f}G \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$. From the obvious inclusion $\mathfrak{h}\mathfrak{f}G \subseteq \mathfrak{C}\mathfrak{f}G \cap \mathfrak{f}G$ we deduce now the validity of the first equation.

Denote by C the uniquely determined normal subgroup of G satisfying $\mathfrak{h}\mathfrak{f}G \subseteq C$ and $C/\mathfrak{h}\mathfrak{f}G = \mathfrak{C}(\mathfrak{f}G/\mathfrak{h}\mathfrak{f}G)$. A normal subgroup X of G is then part of C if, and only if, $X \circ \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$. A normal subgroup X of G is consequently part of C if, and only if, X belongs to Θ. Hence $C = \mathfrak{C}\mathfrak{f}G$, completing the proof.

The importance of the next result stems from Theorem 3.1,(2).

LEMMA 3.6. *If* N *and* K *are normal subgroups of* G *with* $N \circ K \subseteq \mathfrak{h}K$, *if the maximum condition is satisfied by the normal subgroups of* G *which are contained in* N, *then the group of automorphisms induced in* N *by* K *is of finite class.*

Proof. Among the normal subgroups of G which are contained in $N \circ K \subseteq N$ there exists one L which is maximal with respect to the following property:

There exist finitely many normal subgroups $L(i)$ of G such that

$$1 = L(0), \qquad K \circ L(i+1) \subseteq L(i) \subseteq L(i+1), \qquad L(t) = L.$$

Assume by way of contradiction that $L \neq N \circ K$. Then $L \subset N \circ K$. Let $H = G/L$, and denote by σ the canonical epimorphism of G upon H. Then we deduce from our hypotheses and constructions that

$$1 \subset (N \circ K)^\sigma = N^\sigma \circ K^\sigma \subseteq (\mathfrak{h}K)^\sigma \subseteq \mathfrak{h}(K^\sigma);$$

and this implies $1 \neq (N \circ K)^\sigma \cap \mathfrak{z}(K^\sigma) = W/L$ where the normal subgroup W of G satisfies

$$K \circ W \subseteq L \subset W \subseteq N \circ K,$$

contradicting the maximality of L. Hence $L = N \circ K$. If we let $N = L(t+1)$, then the $L(j)$ form a chain of normal subgroups of G connecting 1 and N with

$$K \circ L(j+1) \subseteq L(j) \subseteq L(j+1).$$

The elements in K induce in each $L(j+1)/L(j)$ the 1-automorphism only. Thus K induces in N a subgroup of what Specht [1; p. 349] terms the group of stability of the chain $L(j)$ of normal subgroups of N. Application of a

Theorem of Kaluschnin (see Specht [1; p. 366, Satz 44]) shows that the group of automorphisms induced in N by K is at most of class $t + 2$.

LEMMA 3.7. *If the normal subgroup N of G is a finite p-group, and if U is a subgroup of G such that*

$$N \cap \mathfrak{f}\{x, U\} \subseteq \mathfrak{h}\mathfrak{f}\{x, U\} \quad \textit{for every } x \textit{ in } N,$$

then a p-group of automorphisms is induced in N by $\mathfrak{f}U$.

Proof. It is clear that a finite group of automorphisms is induced in the finite normal subgroup N of G by the subgroup $\mathfrak{f}U$ of G; and this group of automorphisms is essentially the same as $\mathfrak{f}U/(\mathfrak{f}U \cap \mathfrak{c}N)$. This group possesses, as every finite group, a uniquely determined smallest normal subgroup with p-quotient group which we denote by $P/(\mathfrak{f}U \cap \mathfrak{c}N)$. Thus the normal subgroup K of $\mathfrak{f}U$ contains P if, and only if, $\mathfrak{f}U \cap \mathfrak{c}N \subseteq K$ and $\mathfrak{f}U/K$ is a p-group.

Let $X = \{x, U\}$ for x an element in N. Then by hypothesis $N \cap \mathfrak{f}X \subseteq \mathfrak{h}\mathfrak{f}X$. Since the finite normal p-subgroup $N \cap \mathfrak{f}X$ of $\mathfrak{f}X$ is part of the hypercenter of $\mathfrak{f}X$, every element in $\mathfrak{f}X$ induces a p-automorphism in $N \cap \mathfrak{f}X$; see Baer [1; p. 181, Lemma 3]. The finite group of automorphisms induced in $N \cap \mathfrak{f}X$ by $\mathfrak{f}X$ consists consequently of p-elements only and is as such a p-group. Because of $(N \cap X) \circ \mathfrak{f}X \subseteq N \cap \mathfrak{f}X$, the identity automorphishm is induced in $(N \cap X)/(N \cap \mathfrak{f}X)$ by $\mathfrak{f}X$. If we denote by Γ the group of automorphisms, induced in $N \cap X$ by $\mathfrak{f}X$, and by Γ_0 the subgroup of those automorphisms in Γ which induce the identity in $N \cap \mathfrak{f}X$, then Γ_0 is a normal subgroup of Γ with Γ/Γ_0 a p-group. Furthermore Γ_0 is part of the group of stability of the normal subgroup $N \cap \mathfrak{f}X$ of $N \cap X$; see Specht [1; p. 88]. If t is any element of $N \cap X$ and σ is any automorphism in Γ_0, then $t^{\sigma-1}$ belongs to $N \cap \mathfrak{f}X$ and is therefore a fixed element of σ. From $t^{\sigma} = t^{\sigma-1}t$ we deduce now by complete induction the validity of $t^{\sigma^i} = (t^{\sigma-1})^i t$ for every positive i. If p^e is the maximum order of the elements in the finite p-group $N \cap X$, then

$$t^{\sigma^{p^e}} = (t^{\sigma-1})^{p^e} t = t,$$

proving $\sigma^{p^e} = 1$. Thus every element in Γ_0 is a p-element so that the finite group Γ_0 is a p-group. Since Γ_0 and Γ/Γ_0 are p-groups, so is Γ. But the group of automorphisms, induced in $N \cap X$ by $\mathfrak{f}U \subseteq \mathfrak{f}X$, is a subgroup of Γ so that a p-group of automorphisms is induced in $N \cap X$ by $\mathfrak{f}U$. If $K = \mathfrak{f}U \cap \mathfrak{c}(N \cap X)$, then K is a normal subgroup of $\mathfrak{f}U$, and $\mathfrak{f}U/K$ is essentially the same as the group of automorphisms, induced in $N \cap X$ by $\mathfrak{f}U$. Thus $\mathfrak{f}U/K$ is a p-group. Clearly $\mathfrak{f}U \cap \mathfrak{c}N \subseteq K$. Recalling our characterization of P we find that $P \subseteq K$. Hence $X \circ P \subseteq X \circ K = 1$; and thus we have shown $N \circ P = 1$. Hence $P \subseteq \mathfrak{f}U \cap \mathfrak{c}N$ so that $\mathfrak{f}U/(\mathfrak{f}U \cap \mathfrak{c}N)$ is an epimorphic image of the finite p-group $\mathfrak{f}U/P$. But $\mathfrak{f}U/(\mathfrak{f}U \cap \mathfrak{c}N)$ is essentially the same as the group of automorphisms, induced in N by $\mathfrak{f}U$; and thus this group is a p-group too.

4. Groups with noetherian $\mathfrak{f}$-subgroup of finite class

The principal properties of engel elements will play a fundamental role in the proof of Theorem 4.1. We recall these.

If x and y are elements in some group G, then their iterated commutators $x^{(i)} \circ y$ are defined by

$$x^{(1)} \circ y = x \circ y = x^{-1}y^{-1}xy, \qquad x^{(i+1)} \circ y = x \circ [x^{(i)} \circ y].$$

The element e in the group G is termed

a left-engel element of G if, for every x in G, $e^{(i)} \circ x = 1$ for almost all i.

a right-engel element of G if, for every x in G, $x^{(i)} \circ e = 1$ for almost all i.

Heineken has shown that every right-engel element is also a left-engel element. Thus it is justified to say shortly engel element instead of left-engel element.

Before stating the main result concerning engel elements we recall that a group is termed *noetherian*, if all its subgroups are finitely generated, and that this is equivalent to requiring the validity of the maximum condition for subgroups.

(4.E) If e is an element in G such that $\{e^G\}$ is noetherian, then

e is a left-engel element of G if, and only if, $\{e^G\}$ is of finite class.

e is a right-engel element of G if, and only if, $\{e^G\} \subseteq \mathfrak{h}G$, so that e is contained in a finite term of the ascending central chain.

For proofs of these facts cp. Baer [5; p. 257, Satz L and Satz R].

THEOREM 4.1. *Suppose that $\mathfrak{f}$ is an n-functor and that $\mathfrak{f}G$ is noetherian.*

(a) *$\mathfrak{f}G$ is of finite class if, and only if, $\mathfrak{f}U$ is of finite class whenever the subgroup U of G is generated by $n + 1$ elements.*

(b) *$\mathfrak{f}G \subseteq \mathfrak{h}G$ if, and only if, $\mathfrak{f}U \subseteq \mathfrak{h}U$ whenever the subgroup U of G is generated by $n + 1$ elements.*

Note. In the course of this proof we shall use the functor-defining condition (I) only.

Proof. If firstly $\mathfrak{f}G$ is of finite class, and if U is a subgroup of G, then $\mathfrak{f}U \subseteq \mathfrak{f}G$ implies that $\mathfrak{f}U$ is of finite class too, proving the necessity of the condition given ad (a).

Assume conversely that $\mathfrak{f}U$ is of finite class whenever the subgroup U of G is generated by $n + 1$ elements. Consider an element c in $\mathfrak{f}_n G$ and an element x in G. We note first that c belongs to the noetherian characteristic subgroup $\mathfrak{f}G$ of G and that consequently $\{c^G\}$ is a noetherian normal subgroup of G. By definition of $\mathfrak{f}_n$ there exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. Then the subgroup $\{C, x\}$ of G is generated by $n + 1$ elements so that $\mathfrak{f}\{C, x\}$ is, by hypothesis, of finite class. Since c belongs to $\mathfrak{f}C \subseteq \mathfrak{f}\{C, x\}$, since $\mathfrak{f}\{C, x\}$ is a characteristic subgroup of $\{C, x\}$ and x is in $\{C, x\}$, it follows that $c^{(i)} \circ x = 1$ for almost all i. Hence c is an engel element of G; and application of (4.E) shows that $\{c^G\}$ is of finite class. From $\mathfrak{f}G = \{\mathfrak{f}_n G\}$ it follows now that the noetherian group $\mathfrak{f}G$ is the product of normal subgroups of finite class. Hence $\mathfrak{f}G$ is the product of finitely many normal subgroups of finite class; and this implies, as is well known, that $\mathfrak{f}G$ itself is of finite class; cp. e.g. Baer [3; p. 406, Lemma 4]. This proves (a).

If secondly $\mathfrak{f}G \subseteq \mathfrak{h}G$ and if U is a subgroup of G, then

$$\mathfrak{f}U \subseteq U \cap \mathfrak{f}G \subseteq U \cap \mathfrak{h}G \subseteq \mathfrak{h}U,$$

proving the necessity of the condition given ad (b).

Assume conversely that $\mathfrak{f}U \subseteq \mathfrak{h}U$ whenever the subgroup U of G is generated by $n + 1$ elements. Consider an element c in $\mathfrak{f}_n G$ and an element x in G. Then the normal subgroup $\{c^G\}$ of G is part of the noetherian characteristic subgroup $\mathfrak{f}G$ of G so that $\{c^G\}$ itself is noetherian. By definition of $\mathfrak{f}_n$ there exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. Then c belongs likewise to $\mathfrak{f}\{C, x\}$; and since $\{C, x\}$ is generated by $n + 1$ elements, we deduce $\mathfrak{f}\{C, x\} \subseteq \mathfrak{h}\{C, x\}$ from our hypothesis. Since x belongs to $\{C, x\}$ and c to $\mathfrak{h}\{C, x\}$, since $\{c^{\{C,x\}}\} \subseteq \{c^G\}$ is noetherian, application of (4.E) proves that $x^{(i)} \circ c = 1$ for almost all i. Hence c is a right-engel element of G; and a second application of (4.E) shows $\{c^G\} \subseteq \mathfrak{h}G$. Thus $\mathfrak{f}_n G$ has been shown to be part of $\mathfrak{h}G$. Since $\mathfrak{f}$ is supposed to be an n-functor, it follows that $\mathfrak{f}G = \{\mathfrak{f}_n G\} \subseteq \mathfrak{h}G$, as we claimed.

COROLLARY 4.2. *Assume that $\mathfrak{f}$ is an n-functor, that N is a normal subgroup of the group G, that Γ is the group of automorphisms, induced in N by G, and that $\mathfrak{f}G$ is noetherian. Then $\mathfrak{f}\Gamma$ is of finite class if, and only if,*

(∗) *the group of automorphisms, induced in N by $\mathfrak{f}U$, is of finite class whenever the subgroup U of G is generated by $n + 1$ elements.*

Proof. If U is a subgroup of G and if Θ is the group of automorphisms, induced in N by U, then $\mathfrak{f}\Theta$ is the group of automorphisms, induced in N by $\mathfrak{f}U$. It follows that $\mathfrak{f}\Theta \subseteq \mathfrak{f}\Gamma$, since $\mathfrak{f}U \subseteq \mathfrak{f}G$. Hence $\mathfrak{f}\Theta$ is noetherian; and $\mathfrak{f}\Theta$ is of finite class whenever $\mathfrak{f}\Gamma$ is of finite class. This shows in particular the necessity of condition (∗).

Assume conversely the validity of condition (∗). If the subgroup Θ of Γ is generated by $n + 1$ elements, then there exists a subgroup U of G which is generated by $n + 1$ elements such that Θ is the group of automorphisms, induced in N by U. Then $\mathfrak{f}\Theta$ is induced by $\mathfrak{f}U$; and it follows from (∗) that

$\mathfrak{f}\Theta$ is of finite class. Since $\mathfrak{f}\Gamma$ is noetherian, we may apply Theorem 4.1,(a) to show that $\mathfrak{f}\Gamma$ is of finite class.

We are now going to discuss the question—which will prove important in later applications—under which circumstances it is possible to substitute for the integer $n + 1$ appearing in Theorem 4.1,(a) the integer n. We want to point out immediately that a further improvement to $n - 1$ cannot be expected. For this discussion we shall need the following simple criterion for a group and one of its epimorphic images to have the same number of generators.

LEMMA 4.3. *If M is an abelian minimal normal subgroup of the finite group G, if there exist complements of M in G, and if all the complements of M in G are conjugate, if furthermore G/M is not cyclic, then G and G/M have the same rank.*

Terminological Notes. A *complement of M in G* is a subgroup C of G such that $G = MC$ and $1 = M \cap C$. These conditions naturally imply $G/M \simeq C$. The *rank* of the group X is the minimum number of elements, generating X.

Proof. Clearly it suffices to prove the following fact:

If G/M is generated by j elements, then so is G.

Our last hypothesis implies $1 < j$. We denote by J^* some set of j elements, generating G/M.

A subset of j elements in G is termed a set of representatives of J^*, if it contains exactly j elements and if every coset in J^* contains one and only one element in this set. If we let m denote the order of M, then m^j is the number of sets of representatives of J^*.

If C is a complement of M in G, then every coset of G modulo M contains one and only one element in C. We denote by $C \cap J^*$ the elements in C belonging to cosets in J^*. This is clearly a set of representatives of J^*, which we term singular. All the other sets of representatives of J^* we call regular.

The number s of singular sets of representatives of J^* is by construction equal to the number of complements of M in G. Since all complements are conjugate, this number equals $[G:\mathfrak{n}C]$ where C is any complement. Since $C \subseteq \mathfrak{n}C$ and $[G:C]$ is just the order m of M, we find that

$$s = [G:\mathfrak{n}C] \mid [G:C] = m$$

so that

$$s \leqq m < m^j,$$

since $1 < j$. Since s is the number of singular sets of representatives of J^* whereas m^j is the number of all the sets of representatives of J^*, there exists at least one regular set J of representatives of J^*. The subgroup $S = \{J\}$ is consequently generated by j elements, but is not a complement of M in G. Since J represents J^*, we find that $G/M = \{J^*\} = MS/M$; and this implies

$G = MS$. Since S is not a complement, it follows that $M \cap S \neq 1$. Thus there exists an element $a \neq 1$ in $M \cap S$. Since M is an abelian minimal normal subgroup of G, we infer

$$M = \{a^G\} = \{a^{MS}\} = \{a^S\} \subseteq S$$

so that $G = MS = S$ is generated by j elements, as we intended to show.

Remark 4.4. The indispensability of the hypothesis that G/M be non-cyclic is seen by considering the example of the essentially uniquely determined non-abelian group F of order pq for primes p and q with $p \equiv 1$ modulo q. Such a group is not generated by one element; it possesses a normal subgroup M of order p with G/M of order q so that G/M is cyclic, and clearly all the other hypotheses of Lemma 4.3 are satisfied by the pair M, G.

COROLLARY 4.5. *If M is an abelian minimal normal subgroup of the finite group G, and if G/M is not cyclic, then G and G/M have the same rank, provided at least one of the following conditions is satisfied by the pair M, G:*

(a) *M is a Hall subgroup of G.*

(b) *There exists a normal subgroup J of G such that M is a Hall subgroup of J and $M \not\subseteq \mathfrak{z}J$.*

Terminological Remark. A subgroup is a *Hall subgroup*, if order and index are relatively prime.

Proof. If M is an abelian normal Hall subgroup of the finite group X, then the Theorems of Schur-Zassenhaus assert the existence of complements of M in X and the conjugacy of all these complements; cp. Zassenhaus [1; pp. 125–126, Satz 25, Satz 27].

If M is a Hall subgroup of G, then the Theorems of Schur-Zassenhaus show the applicability of Lemma 4.3 so that G and G/M have equal rank.

Assume next the existence of a normal subgroup J of G such that M is a Hall subgroup of J, but $M \not\subseteq \mathfrak{z}J$. Since the characteristic subgroup $\mathfrak{z}J$ of the normal subgroup J of G is a normal subgroup of G, we deduce $M \cap \mathfrak{z}J = 1$ from the minimality of M. Application of the Theorems of Schur-Zassenhaus to the normal Hall subgroup M of J shows the existence of a complement C of M in J and the conjugacy of all the complements of M in J. Using this last fact we prove $G = J \cdot \mathfrak{n}C = MC\mathfrak{n}C = M \cdot \mathfrak{n}C$ by the so-called Frattini Argument; see Baer [8; p. 117, Lemma 2]. Since C and $M \cap \mathfrak{n}C$ are normal subgroups of $\mathfrak{n}C$, we have

$$C \circ (M \cap \mathfrak{n}C) \subseteq M \cap C = 1.$$

Thus $M \cap \mathfrak{n}C$ is centralized by C and by the abelian group M; and it is consequently part of the center of $MC = J$. Hence $M \cap \mathfrak{n}C \subseteq M \cap \mathfrak{z}J = 1$ so that $\mathfrak{n}C$ is a complement of M in G. If S is any complement of M in G, then $J \cap S$ is a complement of M in J. Clearly $S \subseteq \mathfrak{n}(J \cap S)$. Since S and $\mathfrak{n}(J \cap S)$ are complements of M in G, we conclude $S = \mathfrak{n}(J \cap S)$; and now it is clear that all the complements of M in G are conjugate, since all the complements of M in J are conjugate. Thus we have shown again the applicability of Lemma 4.3 so that the ranks of G and G/M are equal.

We turn now to the decisive

LEMMA 4.6. *Assume that* $\mathfrak{f}$ *is an n-functor with* $1 < n$ *and that* G *is a finite group. Then* $\mathfrak{f}G$ *is nilpotent, if firstly* $\mathfrak{f}S$ *is nilpotent for every subgroup* S *of* G *which is generated by n elements, and if secondly at least one of the following conditions* (A) *and* (B) *is satisfied*:

(A) $S/\mathfrak{f}S$ *is soluble for every proper subgroup* S *of* G *with* $S \subseteq \mathfrak{f}G$.

(B) *Every simple epimorphic image of* G *is generated by n elements, and there exist properties* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of finite groups, meeting the following requirements*:

(I) *The orders of* $\mathfrak{A}$*-groups and* $\mathfrak{B}$*-groups are relatively prime.*

(II) $S/\mathfrak{f}S$ *is an extension of an* $\mathfrak{A}$*-group by a* $\mathfrak{B}$*-group for every proper subgroup* S *of* G.

(III.$\mathfrak{A}$) *If there exist nonsoluble* $\mathfrak{A}$*-groups, then* $o(\mathfrak{f}S)$ *is prime to the order of every* $\mathfrak{A}$*-group for every proper subgroup* S *of* G *with nilpotent* $\mathfrak{f}S$.

(III.$\mathfrak{B}$) *If there exist nonsoluble* $\mathfrak{B}$*-groups, then* $o(\mathfrak{f}S)$ *is prime to the order of every* $\mathfrak{B}$*-group for every proper subgroup* S *of* G *with nilpotent* $\mathfrak{f}S$ *and* $S/\mathfrak{f}S$ *a* $\mathfrak{B}$*-group.*

(IV) *There does not exist a factor* H *of* G *with the following properties*:

(i) *there exists one and only one normal subgroup* M *of* H *with* $1 \subset M \subset H$;

(ii) M *is a simple non-abelian* $\mathfrak{B}$*-group*;

(iii) $M = \mathfrak{f}H$ *and* H/M *is a cyclic* $\mathfrak{A}$*-group of order a prime* $p \neq 2$;

(iv) *if t is of order p and x belongs to* M, *then* $\{t, x\}' \subseteq \mathfrak{f}\{t, x\}$ *and* $\{t, x\}'$ *is nilpotent*;

(v) *there does not exist a prime q such that* $\mathfrak{f}S$ *is a q-group whenever the subgroup* S *of* H *is generated by* 2 *elements*;

(vi) *there does not exist a prime q such that* M *is generated by n elements one of which is a q-element and such that* $o(\mathfrak{f}S)$ *is prime to q whenever the subgroup* S *of* H *is generated by* 2 *elements.*

Note on group-theoretical properties. If $\mathfrak{X}$ is a property of (finite) groups, then we term every group with property $\mathfrak{X}$ an $\mathfrak{X}$*-group*; and we require that isomorphic images of $\mathfrak{X}$-groups be again $\mathfrak{X}$-groups. If the prime x divides the order of some $\mathfrak{X}$-group, then x is said to be an $\mathfrak{X}$*-prime* or a prime *belonging* to $\mathfrak{X}$.

Discussion of condition (B). It is a well-known conjecture of Burnside that every finite simple group may be generated by 2 elements. If this conjecture should prove to be true, then the preamble of (B) may be omitted (because of $1 < n$).

Conditions (II) and (III) are certainly satisfied whenever the n-functor $\mathfrak{f}$ meets the following requirement:

(+) $\mathfrak{f}X = 1$ if, and only if, X is an extension of an $\mathfrak{A}$-group by a $\mathfrak{B}$-group.

Suitably selected properties $\mathfrak{A}$ and $\mathfrak{B}$ will naturally lead to a functor $\mathfrak{f}$ meeting our requirements (I)–(III) via the property (+). A typical example of such a pair of properties is obtained by considering two complementary sets $\mathfrak{A}$ and $\mathfrak{B}$ of primes and terming a group an $\mathfrak{X}$-group whenever its order is divisible by primes in $\mathfrak{X}$ only. It was our desire to find an example showing the indispensability of conditions (A), (B) that led us to the investigation of this class of functors. But our next remark will show that it is extremely unlikely that such an example can be found in this direction.

Condition (IV) is essentially an enumeration of properties of the "least criminal". If H is a group with properties (i)–(iii), then a consideration of (I) shows that M is a simple, non-abelian group possessing an automorphism of order a prime $p \neq 2$ which does not divide $o(M)$ and that H is not generated by n elements. According to presently available evidence it appears unlikely that such an M and such a p do exist.

In a later application of this lemma condition (v) will prove useful.

It should be noted finally that the prime 2 belongs to at most one of the properties $\mathfrak{A}$ and $\mathfrak{B}$. According to the Theorem of Walter Feit and John Thompson groups of odd order are soluble. It follows that at least one of the classes $\mathfrak{A}$ and $\mathfrak{B}$ consists of soluble groups only. Thus of the conditions (III.$\mathfrak{A}$) and (III.$\mathfrak{B}$) one is vacuous, though we do not know which one it is.

Proof. The proof will be effected in two essentially different steps: in the first of these no effective use will be made of conditions (A) and (B) which are going to come into play during the second part of the proof only.

If the lemma were false, then there would exist finite groups meeting all the requirements of the lemma, though their $\mathfrak{f}$-subgroups were not nilpotent; and among these groups there would exist one G of minimal order: the "least criminal". We are going to derive a number of properties of G.

As G is a "criminal", we have

(1) $\mathfrak{f}G$ is not nilpotent.

Next we note that all the properties imposed on G, with the exception of (1), are inherited by all the factors of G. As G is a "least criminal", we have

(2) $\mathfrak{f}F$ is nilpotent for every proper factor F of G.

If G were generated by n elements, then we could deduce the nilpotency of $\mathfrak{f}G$ from our first hypothesis in contradiction of (1). Hence

(3) G is not generated by n elements.

Assume next by way of contradiction that the minimal normal subgroup K of G is not part of $\mathfrak{f}G$. Then $K \cap \mathfrak{f}G = 1$. Noting that epimorphisms map $\mathfrak{f}$-subgroups upon $\mathfrak{f}$-subgroups, we deduce from (2) the nilpotency of

$$\mathfrak{f}(G/K) = K\mathfrak{f}G/K \simeq \mathfrak{f}G/(K \cap \mathfrak{f}G) = \mathfrak{f}G$$

contradicting (1). Consequently

(4) every minimal normal subgroup of G is part of $\mathfrak{f}G$.

Assume by way of contradiction the existence of two different minimal normal subgroups A and B of G. If X is one of them, then we deduce $X \subseteq \mathfrak{f}G$ from (4), and the nilpotency of $\mathfrak{f}(G/X) = \mathfrak{f}G/X$ from (2). But $A \cap B = 1$ so that $\mathfrak{f}G$ is isomorphic to a subgroup of the direct product of the two nilpotent groups $\mathfrak{f}G/A$ and $\mathfrak{f}G/B$. Consequently $\mathfrak{f}G$ itself would be nilpotent contradicting (1). Hence

(5) there exists one and only one minimal normal subgroup M of G.

Clearly M is a characteristic subgroup of G such that $M \subseteq \mathfrak{f}G$ by (4), and $\mathfrak{f}G/M$ is nilpotent by (2). Next we prove the following fundamental fact.

(6) M is not soluble.

If this were false, then M would be an elementary abelian p-group, since the minimal normal subgroup M is free of proper characteristic subgroups. We noted before that $\mathfrak{f}G/M$ is nilpotent; and as such $\mathfrak{f}G/M$ is the direct product of a p-group P/M and a group Q/M of order prime to p. These direct factors of $\mathfrak{f}G/M$ are characteristic subgroups of $\mathfrak{f}G/M$; and since $\mathfrak{f}G$ and M are characteristic subgroups of G, so are P and Q.

Since the p-group M is clearly a Hall subgroup of Q, and since the abelian group M is part of its centralizer $\mathfrak{c}M$, we find that M is a Hall subgroup of $D = Q \cap \mathfrak{c}M$. Application of a Theorem of Schur (see Zassenhaus [1; p. 125, Satz 25]) shows the existence of a complement Q^* of M in D; and since M is centralized by D, we have $D = M \otimes Q^*$. As the orders of M and Q^* are relatively prime, Q^* is a characteristic subgroup of D. By noting that M and consequently $\mathfrak{c}M$ are characteristic subgroups of G, and recalling that Q is a characteristic subgroup of G, it follows that D and consequently Q^* is a characteristic subgroup of G. Thus $Q^* \neq 1$ and (5) would imply $1 \subset M \subseteq Q^*$ which is impossible, since the orders of M and Q^* are relatively prime. Hence $Q^* = 1$ so that $D = M$.

If $M = Q$, then $\mathfrak{f}G = P$ would be a p-group and hence nilpotent, contradicting (1). Thus $Q \cap \mathfrak{c}M = M \subset Q$ so that the Hall subgroup M of the characteristic subgroup Q of G is not part of $\mathfrak{z}Q$. Hence condition (b) of Corollary 4.5 is satisfied, showing that G and G/M are generated by the same number of elements. Apply (3) to show that G/M is not generated by n elements.

Consider an element c in $\mathfrak{f}_n G$. Then there exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. Since C is generated by n elements and G/M is not, we have $MC \neq G$. Application of (2) shows the nilpotency of $\mathfrak{f}(MC)$ so that $\mathfrak{f}(MC) = A \otimes B$ is the direct product of a p-group A and a group B of order prime to p. Since B is a characteristic subgroup of the characteristic subgroup $\mathfrak{f}(MC)$ of MC, it is a characteristic subgroup of MC. Thus M and B are characteristic subgroups of relatively prime order of MC; and this implies $M \circ B \subseteq M \cap B = 1$ so that $B \subseteq \mathfrak{c}M$.

But $B \subseteq \mathfrak{f}(MC) \subseteq \mathfrak{f}G$ implies $B \subseteq Q$, since the order of B is prime to p. Hence $B \subseteq Q \cap \mathfrak{c}M = M$ so that $B = 1$, as the orders of M and B are relatively prime. This shows that $\mathfrak{f}(MC) = A$ is a p-subgroup of $\mathfrak{f}G$; and as such it is part of P. Thus the element c belongs to $\mathfrak{f}C \subseteq \mathfrak{f}(MC) \subseteq P$ so that $\mathfrak{f}_n G \subseteq P$. Since $\mathfrak{f}$ is an n-functor, it follows that

$$\mathfrak{f}G = \{\mathfrak{f}_n G\} \subseteq P \text{ is a } p\text{-group};$$

and this contradicts (1), proving (6).

(7) $$\mathfrak{c}M = 1.$$

If this were false, then $M \subseteq \mathfrak{c}M$ would be a consequence of (5). But then M would be abelian, contradicting (6).

(8) $\mathfrak{f}S = 1$, if S is a subgroup of G with $M \subseteq S \subset G$.

From (2) and $S \neq G$ we conclude the nilpotency of $\mathfrak{f}S$. Since M and $\mathfrak{f}S$ are normal subgroups of S, their intersection $M \cap \mathfrak{f}S$ is a nilpotent normal subgroup of M. But the minimal normal subgroup M of G is not soluble by (6); and thus 1 is the only nilpotent normal subgroup of M. Hence $M \cap \mathfrak{f}S = 1$. But M and $\mathfrak{f}S$ normalize each other so that $M \circ \mathfrak{f}S \subseteq M \cap \mathfrak{f}S = 1$. Consequently $\mathfrak{f}S \subseteq \mathfrak{c}M = 1$ by (7), proving (8).

(9) G/M is generated by n elements.

Assume the falsity of (9). Since $\mathfrak{f}$ is an n-functor, we deduce $\{\mathfrak{f}_n G\} = \mathfrak{f}G \neq 1$ from (1) so that there exists an element $c \neq 1$ in $\mathfrak{f}_n G$. There exists furthermore a subgroup C of G, generated by n elements, such that c belongs to $\mathfrak{f}C$. Since G/M is supposedly not generated by n elements, we have $MC \neq G$; and application of (8) shows

$$1 \subset \mathfrak{f}C \subseteq \mathfrak{f}(MC) = 1,$$

a contradiction proving (9).

(10) $o(M)$ and $o(\mathfrak{f}G)$ have the same prime divisors.

By (4), M is part of $\mathfrak{f}G$ so that every prime divisor of $o(M)$ is likewise a prime divisor of $o(\mathfrak{f}G)$. Assume by way of contradiction the existence of a prime divisor p of $o(\mathfrak{f}G)$ which does not divide $o(M)$. Then p is necessarily a divisor of $o(\mathfrak{f}G/M)$. We recall furthermore that $\mathfrak{f}G/M$ is nilpotent by (2) and that therefore the totality P/M of p-elements in $\mathfrak{f}G/M$ is a characteristic subgroup of $\mathfrak{f}G/M$, implying that P itself is a characteristic subgroup of G. If S is a p-Sylow subgroup of P, then S is a complement of M in P, since P/M is a p-group whereas $o(M)$ is prime to p.

Consider now any prime divisor q of $o(M)$. If Q is a q-Sylow subgroup of M, then we deduce from (6) that Q is not a normal subgroup of G so that its normalizer $\mathfrak{n}Q \neq G$, implying the nilpotency of $\mathfrak{f}\mathfrak{n}Q$ by (2). By Frattini's Argument (see Baer [8; p. 117, Lemma 1]) we conclude that $G = M\mathfrak{n}Q$;

and this implies by $M \subseteq P$ and Dedekind's modular law that $P = M(P \cap \mathfrak{n}Q)$. Since the order of M is prime to p, it follows that a p-Sylow subgroup T of $P \cap \mathfrak{n}Q$ is a p-Sylow subgroup of P. Hence S and T are conjugate in P so that there exists an element x in P with $T^x = S$. Then $Q^x = Q^*$ is a q-Sylow subgroup of M, and $\mathfrak{f}\mathfrak{n}Q^* = (\mathfrak{f}\mathfrak{n}Q)^x$ is nilpotent. Furthermore $S = T^x$ is a p-Sylow subgroup of $(P \cap \mathfrak{n}Q)^x = P \cap \mathfrak{n}Q^*$. It follows that S is a complement of $M \cap \mathfrak{n}Q^*$ in $P \cap \mathfrak{n}Q^*$.

The natural epimorphism of $G = M \cdot \mathfrak{n}Q^*$ upon G/M maps $\mathfrak{n}Q^*$ upon G/M and consequently $\mathfrak{f}\mathfrak{n}Q^*$ upon $\mathfrak{f}(G/M) = \mathfrak{f}G/M$ so that $\mathfrak{f}G = M \cdot \mathfrak{f}\mathfrak{n}Q^*$. Since

$$S \subseteq P \cap \mathfrak{n}Q^* \subseteq \mathfrak{f}G \cap \mathfrak{n}Q^* = M \cdot \mathfrak{f}\mathfrak{n}Q^* \cap \mathfrak{n}Q^* = \mathfrak{f}\mathfrak{n}Q^*(M \cap \mathfrak{n}Q^*),$$

and since S is a p-group whereas the order of $\mathfrak{f}\mathfrak{n}Q^*(M \cap \mathfrak{n}Q^*)/\mathfrak{f}\mathfrak{n}Q^*$ is a divisor of $o(M)$ and hence prime to p, we conclude that $S \subseteq \mathfrak{f}\mathfrak{n}Q^*$. Since S is a p-Sylow subgroup of $P \cap \mathfrak{n}Q^*$, it is likewise a p-Sylow subgroup of $P \cap \mathfrak{f}\mathfrak{n}Q^*$. Noting that P is a characteristic subgroup of G and that

$$\mathfrak{f}\mathfrak{n}Q^*/(P \cap \mathfrak{f}\mathfrak{n}Q^*) \simeq P\mathfrak{f}\mathfrak{n}Q^*/P \subseteq \mathfrak{f}G/P,$$

a group of order prime to p, we see that S is a p-Sylow subgroup of the nilpotent group $\mathfrak{f}\mathfrak{n}Q^*$. Hence S is a characteristic subgroup of $\mathfrak{f}\mathfrak{n}Q^*$ and consequently of $\mathfrak{n}Q^*$. Thus S and Q^* are normal subgroups of $\mathfrak{n}Q^*$. Since their orders are relatively prime, it follows that $S \circ Q^* \subseteq S \cap Q^* = 1$. Hence S is centralized by Q^*. Thus we have shown that $\mathfrak{c}S$ contains a q-Sylow subgroup of M for every prime divisor q of $o(M)$, proving that S is centralized by M itself. Hence

$$1 \subset S \subseteq \mathfrak{c}M = 1$$

by (7). This contradiction shows that every prime divisor of $o(\mathfrak{f}G)$ is a divisor of $o(M)$, proving (10).

(11) There exists a proper subgroup S of G with $S \subseteq \mathfrak{f}G$ such that $S/\mathfrak{f}S$ is not soluble.

Assume the falsity of (11). It is a consequence of (4) and (5) that $M \subseteq \mathfrak{f}G$. Assume $M \neq G$. Then it follows from (8) that $\mathfrak{f}M = 1$ so that by hypothesis $M = M/\mathfrak{f}M$ is soluble contradicting (6). Hence $M = G$ implying $M = \mathfrak{f}G$. But then it follows from our hypothesis that $S/\mathfrak{f}S$ is soluble for every proper subgroup of G; and this implies by (2) that every proper subgroup of G is soluble. Because of the minimality of the normal subgroup M of G and because of $M = G$, we find that G is a simple group all of whose proper subgroups are soluble. It has been shown recently by John Thompson that such a group is generated by 2 elements. But $1 < n$ so that G is generated by n elements, contradicting (3) and proving (11).

Since (11) amounts to saying that condition (A) is not satisfied by G, $\mathfrak{f}$, we may restate (11) as follows:

(11*) Condition (B) is satisfied by G and $\mathfrak{f}$.

Accordingly we shall assume in the sequel the validity of (B).

If $G = M$, then G would be a simple group; and G would be, as a consequence of (B), generated by n elements, contradicting (3). Hence

(12) $$M \neq G.$$

By (12) and (8), $\mathfrak{f}M = 1$. By (12) and (II), M is an extension of an $\mathfrak{A}$-group by a $\mathfrak{B}$-group. By (I), the orders of $\mathfrak{A}$-groups and of $\mathfrak{B}$-groups are relatively prime. Since M is free of proper characteristic subgroups, it follows now that M is either an $\mathfrak{A}$-group or a $\mathfrak{B}$-group. Assume by way of contradiction that M is an $\mathfrak{A}$-group. By (6), M is not soluble. Thus not all $\mathfrak{A}$-groups are soluble. By (1), $1 \neq \mathfrak{f}G = \{\mathfrak{f}_n G\}$, as $\mathfrak{f}$ is an n-functor. Hence there exists an element $c \neq 1$ in $\mathfrak{f}_n G$. Consequently there exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. By (3), $C \neq G$, and, by (2), $\mathfrak{f}C$ is nilpotent. By (III.$\mathfrak{A}$), the order of $\mathfrak{f}C$ is prime to every prime belonging to $\mathfrak{A}$. Since $c \neq 1$ belongs to $\mathfrak{f}C$, there exists a prime divisor b of $o(\mathfrak{f}C)$ which does not belong to $\mathfrak{A}$. From $\mathfrak{f}C \subseteq \mathfrak{f}G$ we deduce that b is a divisor of $o(\mathfrak{f}G)$; and this implies by (10) that b is a divisor of $o(M)$. Since M is an $\mathfrak{A}$-group, we have shown that the prime b both belongs and does not belong to $\mathfrak{A}$. This contradiction proves that

(13) M is a $\mathfrak{B}$-group.

Assume by way of contradiction that every prime divisor of $o(G/M)$ belongs to $\mathfrak{B}$. This is, by (13), equivalent to saying that every prime divisor of $o(G)$ belongs to $\mathfrak{B}$. By (6) and (13), M is a nonsoluble $\mathfrak{B}$-group. If $S \subset G$, then $\mathfrak{f}S$ is nilpotent by (2); and $S/\mathfrak{f}S$ is by (II) an extension of an $\mathfrak{A}$-group A by a $\mathfrak{B}$-group. Applying (I) and thefact that every prime divisor of $o(G)$ belongs to $\mathfrak{B}$ we conclude that $A = 1$ so that $S/\mathfrak{f}S$ is a $\mathfrak{B}$-group. Apply (III.$\mathfrak{B}$) to see that $o(\mathfrak{f}S)$ is not divisible by primes belonging to $\mathfrak{B}$. As every prime divisor of $o(G)$ belongs to $\mathfrak{B}$, it follows that $o(\mathfrak{f}S) = 1$ and hence that $\mathfrak{f}S = 1$. If c is an element in $\mathfrak{f}_n G$, then there exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. From (3) we conclude $C \neq G$ so that $\mathfrak{f}C = 1$ and hence $c = 1$. This implies $\mathfrak{f}_n G = 1$, and since $\mathfrak{f}$ is an n-functor, it follows that $\mathfrak{f}G = \{\mathfrak{f}_n G\} = 1$, contradicting (1). Hence there exists a prime divisor p of $o(G/M)$ which does not belong to $\mathfrak{B}$.

By Cauchy's Theorem there exists an element of order p in G/M, and consequently there exists a subgroup T of G which contains M such that T/M is cyclic of order p. Assume by way of contradiction that $T \neq G$. Then we deduce $\mathfrak{f}T = 1$ from (8) so that T is, by (II), an extension of a normal $\mathfrak{A}$-subgroup L by the $\mathfrak{B}$-group T/L. Now $o(T) = o(M)p$, and M is a $\mathfrak{B}$-group by (13), whereas the prime p does not belong to $\mathfrak{B}$. The orders $o(L)$ of the $\mathfrak{A}$-group L and $o(M)$ of the $\mathfrak{B}$-group M are by (I) relatively prime. Consequently $o(L) = p$ implying $T = M \otimes L$. Now we deduce $1 \subset L \subseteq \mathfrak{c}M = 1$ from (7), a contradiction proving $G = T$. Hence $[G:M] = p$ is a prime not belonging to $\mathfrak{B}$.

Since p is a divisor of $o(G)$, there exists by Cauchy's Theorem an element s of order p in G. Since $\mathfrak{c}M = 1$ by (7), there exists an element t in M with $st \neq ts$. From $1 < n$ and our general hypothesis we deduce the nilpotency of $\mathfrak{f}(\{s, t\})$. Since s is not in $M \cap \{s, t\}$, as p is not a divisor of $o(M)$, and since $[G:M] = p$, we have $[\{s, t\}:M \cap \{s, t\}] = p$ so that $\{s, t\} = (M \cap \{s, t\})\{s\}$ and $1 = (M \cap \{s, t\}) \cap \{s\}$. Since t belongs to $M \cap \{s, t\}$, and since $st \neq ts$, it is impossible that $\{s\}$ is a normal subgroup of $\{s, t\}$, as this would imply $\{s, t\} = (M \cap \{s, t\}) \otimes \{s\}$. Next we note that p is the highest power of p dividing $o(G)$ so that $\{s\}$ is a p-Sylow subgroup of G. If s were contained in $\mathfrak{f}(\{s, t\})$, then $\{s\}$ would be a p-Sylow subgroup of the nilpotent group $\mathfrak{f}(\{s, t\})$ and as such $\{s\}$ would be a characteristic subgroup of $\mathfrak{f}(\{s, t\})$ and hence of $\{s, t\}$ which we proved to be impossible. Thus s does not belong to $\mathfrak{f}(\{s, t\})$ so that p is a divisor of $[\{s, t\}:\mathfrak{f}(\{s, t\})]$. Since $\{s, t\}$ is, by (3) and $1 < n$ a proper subgroup of G, we deduce from (II) that $\{s, t\}/\mathfrak{f}(\{s, t\})$ is an extension of an $\mathfrak{A}$-group $A/\mathfrak{f}(\{s, t\})$ by the $\mathfrak{B}$-group $\{s, t\}/A$. Since p is a divisor of $[\{s, t\}:\mathfrak{f}(\{s, t\})]$ without belonging to $\mathfrak{B}$, and since p is the highest order of p dividing $o(G)$, we conclude that the $\mathfrak{A}$-group $A/\mathfrak{f}(\{s, t\})$ is a cyclic group of order p whose $\mathfrak{f}$-subgroup equals 1. Thus p belongs to $\mathfrak{A}$, and cyclic groups of order p are $\mathfrak{A}$-groups with $\mathfrak{f}$-subgroup $= 1$.

This last remark implies $\mathfrak{f}G \subseteq M$; and we deduce $M = \mathfrak{f}G$ from (4) and (5). Since there exist by (6) and (13) nonsoluble $\mathfrak{B}$-groups (like M), it follows from the Theorem of Feit-Thompson that the prime 2 belongs to $\mathfrak{B}$ so that $p \neq 2$. Thus we have verified the following facts:

(14) $\mathfrak{f}G = M$, and $G/\mathfrak{f}G$ is a cyclic $\mathfrak{A}$-group of order a prime $p \neq 2$ belonging to $\mathfrak{A}$.

We note next that during the proof of (14) we have verified the following facts: if s is an element of order p in G and t is an element in M with $st \neq ts$, and if $R = \{s, t\}$, then $R/\mathfrak{f}R$ is an extension of the cyclic $\mathfrak{A}$-group $A/\mathfrak{f}R$ of order p by the $\mathfrak{B}$-group R/A. Since $[R:A]$ is prime to p, the element s belongs to A. Naturally $M \cap R$ is a normal subgroup of R which contains t so that $R = A(M \cap R)$. From (14) we deduce $\mathfrak{f}R \subseteq \mathfrak{f}G \cap R = M \cap R$ so that $\mathfrak{f}R \subseteq A \cap M \subseteq A$. But $[A:\mathfrak{f}R] = p$ so that either $A \cap M = \mathfrak{f}R$ or $A \cap M = A$. In the latter case $A \subseteq M$, and hence $R = A$, an impossibility, so that $A \cap M = \mathfrak{f}R$. Hence $s \circ t$ is an element in

$$A \circ (M \cap R) \subseteq A \cap M \cap R = \mathfrak{f}R;$$

and now we deduce from $R = \{s, t\}$ that $R' \subseteq \mathfrak{f}R$. From $1 < n$ and our general hypothesis we conclude furthermore the nilpotency of $\mathfrak{f}R$ and hence that of R'. If $st = ts$, then $R' = 1$; and we have verified the following fact:

(15) If s an element of order p in G and t is an element in M, then $\{s, t\}'$ is a nilpotent subgroup of $\mathfrak{f}(\{s, t\})$.

Since the minimal normal subgroup M of G is not soluble by (6), it is the direct product of all its simple normal subgroups; and these are all isomorphic and non-abelian; cp. Specht [1; p. 274, Satz 28/29] and Zassenhaus [1; p. 86, Satz 12]. Assume now by way of contradiction that M is not simple. We recall that M is by (13) a $\mathfrak{B}$-group, and G/M is by (14) a cyclic $\mathfrak{A}$-group of order a prime $p \neq 2$ which is prime to $o(M)$. Hence there exists an element t of order p in G, and we have $G = M\{t\}$ and $1 = M \cap \{t\}$.

Let E be a simple and non-abelian direct factor of M. Then $E \neq M$ so that E is not normal in G. Since E is normalized by M, it is not normalized by t. Since $o(t) = p \neq 2$, the conjugate subgroups E^{t^i} with $0 \leqq i < p$ are pairwise different and in particular E, E^t, E^{t^2} are pairwise different. Since the product of these normal subgroups E^{t^i} of M is normalized both by M and by t, it is normalized by G; and we deduce from the minimality of M that

$$M = E \otimes E^t \otimes E^{t^2} \otimes \cdots \otimes E^{t^{p-1}}.$$

Consider some prime divisor q of $o(E)$. Since E is a non-abelian simple group, E is not q-closed so that there exist two q-elements a, b in E whose product ab is not a q-element. The elements a, a^{-t}, and b^{t^2} belong to different components of the above direct decomposition so that they commute pairwise; and likewise a^{-1} and b^t commute. It follows that $x = ab^{t^2}$ is a q-element; and that

$$x^{-1} \circ t = xx^{-t} = aa^{-t}b^{t^2}b^{-t^3}, \qquad t^{-1} \circ x^{-1} = x^{t^{-1}}x^{-1} = a^{t^{-1}}a^{-1}b^tb^{-t^2}.$$

We apply (15) to see the nilpotency of $\{x, t\}'$. But this subgroup contains $x^{-1} \circ t$ and $t^{-1} \circ x^{-1}$, and the E^t-components of these elements are a^{-t} and b^t respectively; here we use $2 \neq p$ again. The projection of the nilpotent group $\{x, t\}'$ into E^t contains the q-elements a^t and b^t whose product $(ab)^t$ is not a q-element. This projection is thus at the same time nilpotent and not q-closed, a contradiction proving that

(16) M is simple.

Assume by way of contradiction the existence of a prime q with the property:

(+) If the subgroup T of G is generated by 2 elements, then $\mathfrak{f}T$ is a q-group.

Because of (13), (14) there exists a p-Sylow subgroup P of G which is cyclic of order $p \neq 2$ such that $G = MP$ and $1 = M \cap P$. If r is a prime divisor of $o(M)$ and R is an r-Sylow subgroup of M, then we deduce $G = M \cdot \mathfrak{n}R$ from Frattini's Argument; see Baer [8; p. 117, Lemma 1]. Since p is, by (13), no divisor of $o(M)$, it is a divisor of $o(\mathfrak{n}R)$ so that $\mathfrak{n}R$ contains a p-Sylow subgroup of G which naturally has the form P^g for g in G. Let $S(r) = R^{g^{-1}}$. Then

$$P \subseteq (\mathfrak{n}R)^{g^{-1}} = \mathfrak{n}S(r).$$

Consider next a prime divisor $r \neq q$ of $o(M)$ and an element x in $S(r)$. Application of (15) shows that

$$\{x, P\}' \subseteq \mathfrak{f}(\{x, P\}) \quad \text{a } q\text{-group by } (+).$$

From $\{x, P\} = \{x^P\}P$ we conclude that

$$\{x, P\}' \subseteq \{x^P\} \subseteq \{S(r)^P\} = S(r) \quad \text{an } r\text{-group}$$

so that $\{x, P\}' = 1$ as a consequence of $r \neq q$. Hence $x \circ P = 1$ showing that x, and hence $S(r)$, is part of $\mathfrak{c}P$. Since this is true for every $r \neq q$, we conclude that $M = (M \cap \mathfrak{c}P)S(q)$. Since $\{P^G\}$ is a normal subgroup, not 1, of G, we deduce from (5) that

$$M \subseteq \{P^G\} = \{P^{S(q)}\} \subseteq PS(q).$$

But the order of M is prime to p, and $S(q)$ is normalized by P. Hence $M \subseteq S(q)$ contradicting (6), and thus we have shown that

(17) There does not exist a prime q with the property: $\mathfrak{f}T$ is a q-group whenever the subgroup T of G is generated by 2 elements.

Assume next by way of contradiction the existence of a prime q with the following two properties:

(a) M is generated by n elements one of which is a q-element;

(b) $o(\mathfrak{f}T)$ is prime to q whenever the subgroup T of G is generated by 2 elements.

According to (a) there exists an n-element set E of generators of M which contains a q-element e. There exists a q-Sylow subgroup Q of M which contains e; and $G = M \cdot \mathfrak{n}Q$ by Frattini's Argument; see Baer [8; p. 117, Lemma 1]. By (13), (14), and (I) we conclude that a p-Sylow subgroup P of $\mathfrak{n}Q$ is a p-Sylow subgroup of G with $o(P) = p$ and $G = MP$, $1 = M \cap P$. If we let $F = \{e, P\}$, then we deduce $F' \subseteq \mathfrak{f}F$ from (15) so that $o(F')$ is prime to q by (b). On the other hand we have $F = \{e^P\}P$ so that

$$F' \subseteq \{e^P\} \subseteq \{Q^P\} = Q \quad \text{a } q\text{-group.}$$

Consequently $F' = 1$, and this implies $e \circ P = 1$.

Consider now an element $t \neq 1$ in P. Then $P = \{t\}$ and $et = te$. As the orders of e and t are relatively prime, the elements e and t are both contained in $\{et\}$. The n-element set consisting of et and the elements, not e, in E consequently generates G, contradicting (3), so that we have shown:

(18) There does not exist a prime q with the following two properties:

(a) M is generated by n elements one of which is a q-element;

(b) $o(\mathfrak{f}T)$ is prime to q whenever the subgroup T of G is generated by 2 elements.

Combining (5), (6), (13), (14), (15), (16), (17), and (18) we obtain a contradiction to (IV) proving our lemma.

THEOREM 4.7. *If $\mathfrak{f}$ is an n-functor with $1 < n$, and if G is a group with finite $\mathfrak{f}G$, then the following two conditions are necessary and sufficient for nil-*

potency of $\mathfrak{f}G$:

(a) *if the subgroup* S *of* G *is generated by* n *elements, then* $\mathfrak{f}S$ *is nilpotent; and*

(b) *if* $S \neq G$ *is a subgroup of* $\mathfrak{f}G$, *then* $S/\mathfrak{f}S$ *is soluble.*

Remark. If we require instead of the finiteness of $\mathfrak{f}G$ only the finiteness of $\mathfrak{f}G/\mathfrak{h}\mathfrak{f}G$, then Theorem 4.7 shows that the conditions (a) and (b) are necessary and sufficient for the nilpotency of the finite group $\mathfrak{f}G/\mathfrak{h}\mathfrak{f}G$. But nilpotency of $\mathfrak{f}G/\mathfrak{h}\mathfrak{f}G$ is equivalent to $\mathfrak{f}G = \mathfrak{h}\mathfrak{f}G$ which in turn is equivalent to the nilpotency of $\mathfrak{f}G$.

Proof. The conditions (a) and (b) are just strongly weakened forms of the nilpotency of $\mathfrak{f}G$, since $\mathfrak{f}S \subseteq \mathfrak{f}G$ for every subgroup S of G; and hence they are necessary conditions.

Assume conversely the validity of (a) and (b). Since $\mathfrak{c}\mathfrak{f}G$ is a characteristic subgroup of G, and since $G/\mathfrak{c}\mathfrak{f}G = H$ is essentially the same as the group of automorphisms, induced in $\mathfrak{f}G$ by G, we deduce the finiteness of H from the finiteness of $\mathfrak{f}G$. Next we note that

$$(+) \qquad \mathfrak{f}H = \mathfrak{c}\mathfrak{f}G \cdot \mathfrak{f}G/\mathfrak{c}\mathfrak{f}G \simeq \mathfrak{f}G/\mathfrak{z}\mathfrak{f}G.$$

If a subgroup of H is generated by n elements, then it is an epimorphic image of a subgroup of G which is generated by n elements so that (a) is satisfied by H. If a proper subgroup of H is part of $\mathfrak{f}H$, then it is an epimorphic image of a proper subgroup of G which is part of $\mathfrak{f}G$ so that (b) is satisfied by H. Application of Lemma 4.6 upon the finite group H shows the nilpotency of $\mathfrak{f}H$; and this implies because of (+) the nilpotency of $\mathfrak{f}G$.

Remark 4.8 *on co-soluble functors.* A functor $\mathfrak{f}$ may be termed co-soluble, whenever $X/\mathfrak{f}X$ is soluble for every finite group X. Clearly condition (b) of Theorem 4.7 may be omitted whenever the functor under discussion is co-soluble. The functors mentioned in the introduction are all co-soluble; they are even "co-nilpotent". Whether condition (b) is indispensable, we have not been able to decide.

Remark 4.9. The condition (a) cannot, in general, be weakened to

(a′) If the subgroup S of G is generated by $n - 1$ elements, then $\mathfrak{f}S$ is nilpotent,

as may be seen by the following discussion of co-soluble 2-functors $\mathfrak{f}$ like commutator subgroup or terminal member of descending central series, etc. In this case (a′) asserts that $\mathfrak{f}S$ is nilpotent whenever S is cyclic, an assertion that is always true. But if G is a finite, non-abelian, simple group, then $G = \mathfrak{f}G$ is certainly not nilpotent. See §8.C and Theorem A.1.

Remark 4.10. The hypothesis $1 < n$—which we did not have to make in Theorem 4.1—is indispensable, as may be seen from the following simple consideration: let p be any prime, and denote by $\mathfrak{f}G$ for every finite group G

the intersection of all the normal subgroups N of G with p-quotient group G/N. Then $G/\mathfrak{f}G$ is likewise a p-group; and it is fairly obvious that $\mathfrak{f}$ is a co-soluble 1-functor. Condition (a) is trivially satisfied, since it asserts only that $\mathfrak{f}S$ is nilpotent in case S is cyclic. But if G is a finite, non-abelian, simple group, then $G = \mathfrak{f}G$ is certainly not nilpotent.

Intermezzo 4.11 *on a second application of Lemma* 4.6. Consider a pair of properties $\mathfrak{A}, \mathfrak{B}$, of finite groups and an n-functor $\mathfrak{f}$ with $1 < n$, subject to the following requirements:

(I) The orders of $\mathfrak{A}$-groups and $\mathfrak{B}$-groups are relatively prime.

(II) If X is a finite group with $\mathfrak{f}X = 1$, then X is an extension of an $\mathfrak{A}$-group by a $\mathfrak{B}$-group.

(III.$\mathfrak{A}$) If there exist nonsoluble $\mathfrak{A}$-groups, if X is a finite group with nilpotent $\mathfrak{f}X$, then $o(\mathfrak{f}X)$ is prime to every prime, belonging to $\mathfrak{A}$.

(III.$\mathfrak{B}$) If there exist nonsoluble $\mathfrak{B}$-groups, if X is a finite group with nilpotent $\mathfrak{f}X$ and $X/\mathfrak{f}X$ a $\mathfrak{B}$-group, then $o(\mathfrak{f}X)$ is prime to every prime, belonging to $\mathfrak{B}$.

(IV) There does not exist a finite group G, possessing one and only one proper normal subgroup M with the following properties:

(i) M is a non-abelian, simple $\mathfrak{B}$-group;
(ii) $M = \mathfrak{f}G$;
(iii) G/M is a cyclic $\mathfrak{A}$-group of order a prime $p \neq 2$;
(iv) G is not generated by n elements;
(v) if $o(x) = p$ and y belongs to M, then $\{x, y\}'$ is nilpotent.

The following proposition is then an immediate consequence of Lemma 4.6

$\mathfrak{f}G$ for G a finite group is nilpotent if, and only if, $\mathfrak{f}S$ is nilpotent whenever the subgroup S of G is generated by n elements.

A simple construction of 2-functors of this type may be given: Denote by $\mathfrak{p}$ some set of primes and by $\mathfrak{p}'$ its complement. Let $\mathfrak{A}$ be the class of all finite $\mathfrak{p}$-groups ($=$ groups whose orders are divisible by primes in $\mathfrak{p}$ only); and let $\mathfrak{B}$ be the class of all finite $\mathfrak{p}'$-groups. Characterize the functor $\mathfrak{f}$ by the property:

(+) $\mathfrak{f}G = 1$ for G a finite group if, and only if, G is $\mathfrak{p}$-closed.

Then $\mathfrak{f}$ is clearly a 2-functor; and requirement (I) is trivially satisfied. (II) and (III) hold in the stronger forms:

(II*) X is a finite group with $\mathfrak{f}X = 1$ if, and only if, X is an extension of an $\mathfrak{A}$-group by a $\mathfrak{B}$-group.

(III*) If X is a finite group with nilpotent $\mathfrak{f}X$, then $\mathfrak{f}X$ is a $\mathfrak{B}$-group; and $\mathfrak{f}X = 1$ in case $X/\mathfrak{f}X$ is a $\mathfrak{B}$-group.

It is a consequence of the Theorem of Walter Feit and John Thompson that (IV) will be true whenever 2 belongs to $\mathfrak{p}$ since then $\mathfrak{B}$-groups are soluble as groups of odd order. Whether or not (IV) is satisfied in case 2 does not belong to $\mathfrak{p}$, appears still to be an open question. However, there seems to be good reason to believe that a group G may be generated by two elements, if it possesses a non-abelian, simple, normal subgroup M whose index $[G:M]$ is an odd prime, not dividing $o(M)$. If this should happen to be true, then we would be assured of the validity of (IV) in either case.

COROLLARY 4.12. *If $\mathfrak{f}$ is an n-functor with $1 < n$, then the following properties of the group G with noetherian $\mathfrak{f}G$ are equivalent:*

(i) *$\mathfrak{f}G$ is of finite class.*

(ii) (a) *$\mathfrak{f}S$ is of finite class whenever the subgroup S of G is generated by n elements.*
(b) *$\mathfrak{f}G$ is soluble.*

(iii) (a) *$\mathfrak{f}S$ is of finite class whenever the subgroup S of G is generated by n elements.*
(b) *$\mathfrak{f}S$ is soluble whenever the subgroup S of G is generated by $n + 1$ elements.*

Terminological Reminder. The group X is called *soluble*, if every epimorphic image, not 1, of X possesses an abelian normal subgroup different from 1. For noetherian X this requirement is equivalent to $X^{(i)} = 1$ for almost all i.

Proof. That (ii) is a consequence of (i), is immediately deduced from $\mathfrak{f}S \subseteq \mathfrak{f}G$; and for the same reason (iii) is a consequence of (ii).

Assume the validity of (iii) and consider a subgroup H of G which is generated by $n + 1$ elements. Since $\mathfrak{f}G$ is noetherian, so is $\mathfrak{f}H \subseteq \mathfrak{f}G$; and it is a consequence of (iii.b) that $\mathfrak{f}H$ is soluble. Suppose that K is a normal subgroup of $\mathfrak{f}H$ with finite $\mathfrak{f}H/K$. Since the noetherian group $\mathfrak{f}H$ is finitely generated, there exists a characteristic subgroup N of $\mathfrak{f}H$ with $N \subseteq K$ and finite $\mathfrak{f}H/N$; see Baer [7; p. 331, Folgerung 3]. Then N is likewise a characteristic subgroup of H and we may form the epimorphic image $J = H/N$. From the solubility of $\mathfrak{f}H$ we deduce that $\mathfrak{f}J = \mathfrak{f}H/N$ is a finite soluble group. If the subgroup S of J is generated by n elements, then S is an epimorphic image of a subgroup of H and G which is generated by n elements; and application of (iii.a) shows that $\mathfrak{f}S$ is a finite nilpotent group. Hence we may apply Theorem 4.7 showing the nilpotency of the finite soluble group $\mathfrak{f}J$. From $N \subseteq K$ we deduce that $\mathfrak{f}H/K$ is an epimorphic image of $\mathfrak{f}H/N = \mathfrak{f}J$. Consequently $\mathfrak{f}H/K$ is nilpotent; and we have shown

(+) Every finite epimorphic image of $\mathfrak{f}H$ is nilpotent.

As a noetherian soluble group with the property (+) the group $\mathfrak{f}H$ is of finite class; see Baer [1; p. 205, Theorem and p. 170, Lemma 4]. Since $\mathfrak{f}G$ is noetherian, we may apply Theorem 4.1,(a) to show that $\mathfrak{f}G$ is of finite class. Hence (i) is a consequence of (iii).

Remark 4.13. Note that the full strength of the requirement that $\mathfrak{f}G$ be noetherian and that condition (iii.b) be satisfied came into play only after the residual property (+) had been verified.

PROPOSITION 4.14. *If $\mathfrak{f}$ is a co-soluble n-functor with $1 < n$, if $\mathfrak{f}S$ is finite for every finitely generated subgroup S of G, and if $\mathfrak{f}T$ is nilpotent whenever the subgroup T of G is generated by n elements, then every finitely generated subgroup of $\mathfrak{f}G$ is finite and nilpotent.*

Proof. Suppose that c is an element in $\mathfrak{f}_n G$ and that x is any element in G. Then there exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. The subgroup $X = \{C, x\}$ of G is finitely generated so that by hypothesis $\mathfrak{f}X$ is finite. Our third hypothesis shows that $\mathfrak{f}Y$ is a finite nilpotent group whenever the subgroup Y of X is generated by n elements. Since $\mathfrak{f}$ is a co-soluble n-functor with $1 < n$, application of Theorem 4.7 shows that $\mathfrak{f}X$ is a finite nilpotent characteristic subgroup of X. Since c belongs to $\mathfrak{f}C \subseteq \mathfrak{f}X$, it follows from (4.E) that c is a left-engel element of X. As x belongs to X, it follows that $c^{(i)} \circ x = 1$ for almost all i; and thus we have shown that every element in $\mathfrak{f}_n G$ is a left-engel element of G.

Consider next a finite subset F of $\mathfrak{f}_n G$. If s is an element in F, then there exists a subgroup s^* of G which is generated by n elements such that s belongs to s^*. The subgroup F^* of G which is generated by all the subgroups s^* with s in F is finitely generated so that $\mathfrak{f}F^*$ is finite by hypothesis. If s is an element in F, then s belongs to $\mathfrak{f}s^* \subseteq \mathfrak{f}F^*$ so that F is a subset of $\mathfrak{f}F^*$. Every element in F is a left-engel element of G and hence of the finite group $\mathfrak{f}F^*$. Application of (4.E) shows that every element in F belongs to a finite nilpotent normal subgroup of $\mathfrak{f}F^*$; and the set F is consequently likewise contained in a finite nilpotent normal subgroup of $\mathfrak{f}F^*$. It follows in particular that every finite subset F of $\mathfrak{f}_n G$ generates a finite nilpotent subgroup of G.

Consider finally a finite subset A of $\mathfrak{f}G$. Since $\mathfrak{f}$ is an n-functor, $\mathfrak{f}G$ is generated by $\mathfrak{f}_n G$; and there exists consequently a finite subset B of $\mathfrak{f}_n G$ such that A is contained in $\{B\}$. Since $\{B\}$ has been shown to be finite and nilpotent, its subgroup $\{A\}$ is likewise finite and nilpotent, as was to be shown.

COROLLARY 4.15. *If $\mathfrak{f}$ is a co-soluble n-functor with $1 < n$, if G is a locally finite group with artinian $\mathfrak{f}G$, then $\mathfrak{f}G$ is nilpotent if, and only if, $\mathfrak{f}S$ is nilpotent whenever the subgroup S of G is generated by n elements.*

Terminological Reminder. The group A is *artinian*, if the minimum condition is satisfied by its subgroups. The group L is *nilpotent*, if every epimorphic image, not 1, of L has a center different from 1.

Proof. The necessity of our condition is an immediate consequence of $\mathfrak{f}S \subseteq \mathfrak{f}G$. If conversely our condition is satisfied, then all the hypotheses of Proposition 4.14 are satisfied too. Hence every finitely generated subgroup of $\mathfrak{f}G$ is finite and nilpotent. Application of Baer [9; p. 21, Satz 4.1] shows the nilpotency of $\mathfrak{f}G$.

The following result will prove useful in an important application.

LEMMA 4.16. *If* $\mathfrak{f}$ *is an* n*-functor with* $1 < n$, *if* p *is a prime, and if* G *is a group with finite* $\mathfrak{f}G$, *then the following properties are equivalent*:

(i) $\mathfrak{f}G$ *is a* p*-group.*

(ii) (a) $\mathfrak{f}S$ *is a* p*-group whenever the subgroup* S *of* G *is generated by* n *elements*;
(b) $S/\mathfrak{f}S$ *is soluble whenever* $S \neq G$ *is a subgroup of* $\mathfrak{f}G$.

(iii) (a) $\mathfrak{f}S$ *is a* p*-group whenever the subgroup* S *of* G *is generated by* n *elements*;
(b) $S/\mathfrak{f}S$ *is* p*-closed whenever* $S \neq G$ *is a subgroup of* $\mathfrak{f}G$.

(iv) (a) $\mathfrak{f}S$ *is a* p*-group whenever the subgroup* S *of* G *is generated by* n *elements*;
(b) $\mathfrak{f}G$ *is* p*-closed.*

Terminological Reminder. A group is p-*closed* if products of p-elements are again p-elements.

Proof. That (ii) is a consequence of (i), follows from $\mathfrak{f}S \subseteq \mathfrak{f}G$ (and the finiteness of $\mathfrak{f}G$). If (ii) is true, then we deduce from Theorem 4.7 the nilpotency of $\mathfrak{f}G$; and this shows that (iii) is a consequence of (ii).

Assume next by way of contradiction that (iii) is true and (iv) is false. Then there exists a pair of p-elements a, b in $\mathfrak{f}G$ whose product ab is not a p-element; and $S = \{a, b\}$ is a subgroup of $\mathfrak{f}G$, generated by $2 \leqq n$ elements, which is not p-closed. Apply (iii.a) to see that $\mathfrak{f}S$ is a p-group. Hence $S/\mathfrak{f}S$ is not p-closed; and it follows from (iii.b) that $S = G$ implying that $\mathfrak{f}G = \mathfrak{f}S$ is a p-group. This is a contradiction, proving that (iv) is a consequence of (iii).

Assume finally that (iv) is true. Consider an element c in $\mathfrak{f}_n G$. Then there exists a subgroup C of G, generated by n elements, such that c belongs to $\mathfrak{f}C$. By (iv.a), $\mathfrak{f}C$ is a p-group so that c is a p-element. Since $\mathfrak{f}$ is an n-functor, $\mathfrak{f}G = \{\mathfrak{f}_n G\}$ is by (iv.b) a p-closed group, generated by p-elements; and as such $\mathfrak{f}G$ is a p-group, completing the proof of the equivalence of (i)–(iv).

Remark 4.17. The second criterion, contained in Lemma 4.16, admits of an interesting application to the situation discussed in Corollary 4.15; cp. in particular Lemma 4.6,(IV.v).

5. Finitely and countably definable functors

For these functors our problems admit of a comparatively simple solution. We begin by proving a useful general property of countably definable functors.

LEMMA 5.1. *If* $\mathfrak{f}$ *is a countably definable functor and* U *a countable subgroup of* G, *then there exists a countable subgroup* V *of* G *with* $U \subseteq V$ *and* $\mathfrak{f}V = V \cap \mathfrak{f}G$.

Proof. We recall first that $\mathfrak{f}G = \mathfrak{f}_a G$ because of the countable definability of $\mathfrak{f}$; and this implies because of the definition of $\mathfrak{f}_a$:

(1) To every element x in $\mathfrak{f}G$ there exists a countable subgroup $\hat{x}$ of G such that x belongs to $\mathfrak{f}\hat{x}$.

If X is a countable subgroup of G, then $X \cap \mathfrak{f}G$ is likewise countable. It follows that

$$X^0 = \{X, \hat{x} \text{ for } x \text{ in } X \cap \mathfrak{f}G\}$$

is likewise countable. If x is an element in $X \cap \mathfrak{f}G$, then x belongs to $\mathfrak{f}\hat{x} \subseteq \mathfrak{f}X^0$. Hence

$$X \cap \mathfrak{f}G \subseteq X \cap \mathfrak{f}X^0 \subseteq X \cap \mathfrak{f}G;$$

and thus we have shown

(2) Every countable subgroup X of G is contained in a countable subgroup X^0 of G with $X \cap \mathfrak{f}G = X \cap \mathfrak{f}X^0$.

If U is a countable subgroup of G, then one may derive from (2) by complete induction the existence of countable subgroups $U(i)$ of G with

$$U = U(0), \qquad U(i) \subseteq U(i+1), \qquad U(i) \cap \mathfrak{f}G = U(i) \cap \mathfrak{f}U(i+1).$$

The join $V = \bigcup_{i=0}^{\infty} U(i)$ of this ascending chain of countable subgroups of G is likewise a countable subgroup of G. Clearly $U \subseteq V$ and $\mathfrak{f}V \subseteq V \cap \mathfrak{f}G$. If x belongs to $V \cap \mathfrak{f}G$, then x belongs to some $U(i)$ and hence to

$$U(i) \cap \mathfrak{f}G \subseteq \mathfrak{f}U(i+1) \subseteq \mathfrak{f}V.$$

Thus $\mathfrak{f}V = V \cap \mathfrak{f}G$, as we wanted to show.

THEOREM 5.2. *If $\mathfrak{f}$ is a countably definable functor, and if N is a normal subgroup of G, then*

$$N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$$

if, and only if,

$$N \cap \mathfrak{f}A \subseteq \mathfrak{h}\mathfrak{f}A \quad \textit{for every countable subgroup } A \textit{ of } G.$$

Proof. If $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$, then we deduce $N \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}U$ for every subgroup U of G from Corollary 3.2; and this proves the necessity of our condition.

We assume conversely the validity of our condition. Let $H = G/(N \cap \mathfrak{h}\mathfrak{f}G)$ and denote by σ the canonical epimorphism of G upon H. Then

$$\mathfrak{f}H = (\mathfrak{f}G)^{\sigma} = \mathfrak{f}G/(N \cap \mathfrak{h}\mathfrak{f}G),$$

and consequently

$$\begin{aligned} N^{\sigma} \cap \mathfrak{h}\mathfrak{f}H &= [N/(N \cap \mathfrak{h}\mathfrak{f}G)] \cap \mathfrak{h}[\mathfrak{f}G/(N \cap \mathfrak{h}\mathfrak{f}G)] \\ &= [N/(N \cap \mathfrak{h}\mathfrak{f}G)] \cap [\mathfrak{h}\mathfrak{f}G/(N \cap \mathfrak{h}\mathfrak{f}G)] = 1; \end{aligned}$$

and thus we have shown

(1) $$1 = N^\sigma \cap \mathfrak{h}\mathfrak{f}H = N^\sigma \cap \mathfrak{z}\mathfrak{f}H.$$

Suppose next that A is a countable subgroup of H. Then there exists a countable subgroup B of G with $B^\sigma = A$. It is a consequence of our hypothesis that $N \cap \mathfrak{f}B \subseteq \mathfrak{h}\mathfrak{f}B$; and application of Corollary 3.3 shows

(2) $N^\sigma \cap \mathfrak{f}A \subseteq \mathfrak{h}\mathfrak{f}A$ for every countable subgroup A of H.

If U is a countable subgroup of $\mathfrak{f}H$, then there exists because of (1) to every element $u \neq 1$ in $N^\sigma \cap U$ an element u' in $\mathfrak{f}H$ with $u \circ u' \neq 1$. Because of the countability of U and $N^\sigma \cap U$ we obtain a countable subgroup V of H by adjoining to U all these elements u' for u in $N^\sigma \cap U$; and it follows from our construction that $U \subseteq V \subseteq \mathfrak{f}H$ and $N^\sigma \cap U \cap \mathfrak{z}V = 1$. We state this result for future reference:

(3) To every countable subgroup U of $\mathfrak{f}H$ there exists a countable subgroup V with $U \subseteq V \subseteq \mathfrak{f}H$ and $N^\sigma \cap U \cap \mathfrak{z}V = 1$.

Assume now by way of contradiction that $N^\sigma \cap \mathfrak{f}H \neq 1$. Then there exists a countable subgroup W with $1 \subset W \subseteq N^\sigma \cap \mathfrak{f}H$. Because of (3) there exists a countable subgroup $U(1)$ with $W \subseteq U(1) \subseteq \mathfrak{f}H$ and $N^\sigma \cap W \cap \mathfrak{z}U(1) = 1$; and application of Lemma 5.1 shows the existence of a countable subgroup $V(1)$ with $U(1) \subseteq V(1)$ and $\mathfrak{f}V(1) = V(1) \cap \mathfrak{f}H$. Clearly $\mathfrak{f}U(1) \subseteq \mathfrak{f}V(1)$.

Assume now that we have constructed a countable subgroup $V(i)$ for some positive i. From $\mathfrak{f}V(i) \subseteq \mathfrak{f}H$ and (3) we deduce the existence of a countable subgroup $U(i + 1)$ with

$$\mathfrak{f}V(i) \subseteq U(i+1) \subseteq \mathfrak{f}H \quad \text{and} \quad N^\sigma \cap \mathfrak{f}V(i) \cap \mathfrak{z}U(i+1) = 1.$$

Then $\{V(i), U(i + 1)\}$ is likewise a countable subgroup of H; and application of Lemma 5.1 shows the existence of a countable subgroup $V(i + 1)$ of H with $\{V(i), U(i + 1)\} \subseteq V(i + 1)$ and $\mathfrak{f}V(i + 1) = V(i + 1) \cap \mathfrak{f}H$. By construction $U(i + 1) \subseteq \mathfrak{f}V(i + 1)$. We note the principal features of this construction:

(a) $$\begin{gathered} \mathfrak{f}V(i) \subseteq U(i+1) \subseteq \mathfrak{f}V(i+1), \qquad V(i) \subseteq V(i+1), \\ \mathfrak{f}V(i) = V(i) \cap \mathfrak{f}H, \qquad N^\sigma \cap \mathfrak{f}V(i) \cap \mathfrak{z}U(i+1) = 1. \end{gathered}$$

Since the $U(i)$ as well as the $V(i)$ form ascending chains of countable subgroups, their joins

$$U = \bigcup_{i=1}^\infty U(i) \quad \text{and} \quad V = \bigcup_{i=1}^\infty V(i)$$

are again countable subgroups of H. We derive further properties of U and V. Firstly we have

$$\mathfrak{f}V \subseteq V \cap \mathfrak{f}H = \mathfrak{f}H \cap \bigcup_{i=1}^\infty V(i) = \bigcup_{i=1}^\infty [\mathfrak{f}H \cap V(i)] = \bigcup_{i=1}^\infty \mathfrak{f}V(i) \subseteq \mathfrak{f}V,$$

proving

$$\text{(b)} \qquad \mathfrak{f}V = V \cap \mathfrak{f}H = \bigcup_{i=1}^{\infty} \mathfrak{f}V(i).$$

A second application of (a) shows

$$U = \bigcup_{i=1}^{\infty} U(i) \subseteq \bigcup_{i=1}^{\infty} \mathfrak{f}V(i) \subseteq \bigcup_{i=1}^{\infty} U(i+1) = U,$$

proving because of (b)

$$\text{(c)} \qquad U = \mathfrak{f}V.$$

Consider next an element t in $N^{\sigma} \cap \mathfrak{z}\mathfrak{f}V$. Because of (b) and (c) the element t belongs to almost all $U(i)$ and $\mathfrak{f}V(i)$. If t belongs to $\mathfrak{f}V(i) \subseteq \mathfrak{f}V = U$, then it belongs to $U(i+1)$ and hence to $\mathfrak{z}U(i+1)$ so that t belongs to $N^{\sigma} \cap \mathfrak{f}V(i) \cap \mathfrak{z}U(i+1) = 1$. Hence $t = 1$; and we have shown

$$\text{(d)} \qquad N^{\sigma} \cap \mathfrak{z}\mathfrak{f}V = N^{\sigma} \cap \mathfrak{z}U = 1.$$

Because of the countability of V and (2) we have $N^{\sigma} \cap \mathfrak{f}V \subseteq \mathfrak{h}\mathfrak{f}V$; and thus it follows from $W \subseteq U(1)$ and (d) that

$$1 \subset W \subseteq N^{\sigma} \cap U(1) \subseteq N^{\sigma} \cap U = N^{\sigma} \cap \mathfrak{f}V = N^{\sigma} \cap \mathfrak{h}\mathfrak{f}V = 1,$$

a contradiction. Hence $N^{\sigma} \cap \mathfrak{f}H = 1$; and this is equivalent to

$$N \cap \mathfrak{f}G = N \cap \mathfrak{h}\mathfrak{f}G,$$

as we wanted to prove.

THEOREM 5.3. *If $\mathfrak{f}$ is a finitely definable functor, and if N is a noetherian normal subgroup of G, then*

$$N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$$

if, and only if,

$$N \cap \mathfrak{f}S \subseteq \mathfrak{h}\mathfrak{f}S \quad \textit{for every finitely generated subgroup } S \textit{ of } G.$$

Proof. The necessity of our condition is an immediate consequence of Corollary 3.2. If conversely our condition is satisfied by N, then there exists to every element t in $N \cap \mathfrak{f}G$ a finitely generated subgroup T of G such that t belongs to $\mathfrak{f}T$, since $\mathfrak{f}G = \mathfrak{f}_{\infty} G$. If x is an element in $\mathfrak{f}G = \mathfrak{f}_{\infty} G$, then there exists a finitely generated subgroup X of G such that x belongs to $\mathfrak{f}X$. Since $\{T, X\}$ is likewise finitely generated, we deduce

$$N \cap \mathfrak{f}\{T, X\} \subseteq \mathfrak{h}\mathfrak{f}\{T, X\}$$

from our condition. Because of the monotonicity of $\mathfrak{f}$ we have

$$\{t, x\} \subseteq \{\mathfrak{f}T, \mathfrak{f}X\} \subseteq \mathfrak{f}\{T, X\}.$$

Since t belongs to N, it belongs to $\mathfrak{h}\mathfrak{f}\{T, X\}$, and x belongs to $\mathfrak{f}\{T, X\}$.

Since t belongs to a noetherian normal subgroup of $\mathfrak{f}\{T, X\}$ and to $\mathfrak{h}\mathfrak{f}\{T, X\}$, it is by (4.E) a right-engel element of $\mathfrak{f}\{T, X\}$. Since x is in $\mathfrak{f}\{T, X\}$, we have

$$x^{(i)} \circ t = 1 \quad \text{for almost all } i \text{ and every } x \text{ in } \mathfrak{f}G.$$

Thus we have shown that t is a right-engel element of $\mathfrak{f}G$. Since t belongs to a noetherian normal subgroup of $\mathfrak{f}G$, a second application of (4.E) shows that t belongs to $\mathfrak{h}\mathfrak{f}G$; and hence we have shown $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.

6. The normal subgroups hypercentralized by the $\mathfrak{f}$-subgroup

It is the principal aim of this section to relate the following properties of a normal subgroup N of a group G:

(I) $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.

(II) $N \cap \mathfrak{f}\{g_0, \cdots, g_n\} \subseteq \mathfrak{h}\mathfrak{f}\{g_0, \cdots, g_n\}$ *whenever at least one of the* $n + 1$ *elements* g_i *in* G *belongs to* N.

It is an immediate consequence of Corollary 3.2,(a) that (I) implies (II); and we shall prove under suitable additional assumptions that (I) is a consequence of (II). It is therefore convenient to introduce short names for these two properties. Accordingly we shall say that the normal subgroup N of G is $\mathfrak{f}$*-hypercentralized by* G, *or that* N *is an* $\mathfrak{f}$*-hypercentralized normal subgroup of* G, whenever (I) is satisfied by N, G; and in case (II) is satisfied by N, G, we shall speak of an $\mathfrak{f}$*-*n*-hypercentralized normal subgroup* N *of* G.

We begin by collecting a number of frequently used simple properties of $\mathfrak{f}$-n-hypercentralized normal subgroups.

LEMMA 6.1. *Assume that* N *is an* $\mathfrak{f}$-n*-hypercentralized normal subgroup of* G.

(A) *If* U *is a subgroup of* G, *then* $N \cap U$ *is an* $\mathfrak{f}$-n*-hypercentralized normal subgroup of* U.

(B) *If* σ *is an epimorphism of* G *upon* H, *and if the kernel of* σ *is contained in* N, *then* N^{σ} *is* $\mathfrak{f}$-n*-hypercentralized by* H.

(C) $\{c^N\}$ *is of finite class for every* c *in* $\mathfrak{f}_n G$, *provided* N *is noetherian.*

(D) $N \circ \{\mathfrak{f}_n G\}$ *is of finite class, if* N *is noetherian.*

Proof. The validity of (A) is quite obvious. To prove (B) consider $n + 1$ elements h_i in H at least one of which belongs to N^{σ}. Then there exist $n + 1$ elements g_i in G at least one of which belongs to N and such that $g_i^{\sigma} = h_i$. Application of (II) shows

$$N \cap \mathfrak{f}\{g_0, \cdots, g_n\} \subseteq \mathfrak{h}\mathfrak{f}\{g_0, \cdots, g_n\};$$

and application of Corollary 3.3 implies

$$N^{\sigma} \cap \mathfrak{f}\{h_0, \cdots, h_n\} \subseteq \mathfrak{h}\mathfrak{f}\{h_0, \cdots, h_n\},$$

proving (B).

If c is an element in $\mathfrak{f}_n G$, then $C = \{N, c\} = N\{c\}$ is noetherian as an extension of the noetherian group N by the cyclic group C/N. To every element y in C there exist an element x in N and an integer j with $y = c^j x$. From the definition of $\mathfrak{f}_n G$ we deduce the existence of a subgroup U of G, generated by n elements, such that c belongs to $\mathfrak{f}U$. Let $V = \{U, x\}$. Then $\mathfrak{f}U \subseteq \mathfrak{f}V$ so that c belongs to $\mathfrak{f}V$, too. Since V is generated by $n + 1$ elements at least one of which is in N, we may apply (II) proving

$$N \cap \mathfrak{f}V \subseteq \mathfrak{h}\mathfrak{f}V.$$

Since x belongs to the normal subgroup $N \cap V$ of V, and since c belongs to the characteristic subgroup $\mathfrak{f}V$ of V, the commutator $c \circ x$ belongs to $(N \cap V) \cap \mathfrak{f}V = N \cap \mathfrak{f}V \subseteq \mathfrak{h}\mathfrak{f}V$. Hence $c \circ y = c \circ (c^j x) = c \circ x$ belongs to $\mathfrak{h}\mathfrak{f}V$. Since c is in $\mathfrak{f}V$, it follows from (4.E) that

$$1 = c^{(i)} \circ (c \circ y) = c^{(i+1)} \circ y$$

for almost all i. Hence c is an engel element of the noetherian group C. Application of (4.E) shows that $\{c^C\} = \{c^N\}$ is of finite class, proving (C).

Let $P = \prod_{c \in \mathfrak{f}_n G} [N \cap \{c^N\}]$. This is by (C) a product of normal subgroups of finite class of N; and since N is noetherian, P is a product of finitely many of its normal subgroups of finite class. Applying a well known result (see e.g. Baer [3; p. 406, Lemma 4]) we conclude that P is of finite class. From the normality of N we deduce $N \circ c \subseteq N \cap \{c^N\}$ so that $N \circ \{\mathfrak{f}_n G\} \subseteq P$, proving (D).

THEOREM 6.2. *If $\mathfrak{f}$ is an n-functor with $1 < n$, then the following properties of the finite normal subgroup N of G are equivalent:*

(1) $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.

(2) *N is an $\mathfrak{f}$-$(n+1)$-hypercentralized normal subgroup of G.*

(3) (a) *N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G.*
(b) *$\mathfrak{f}G$ induces in N a nilpotent group of automorphisms.*

(4) (a) *N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G.*
(b) *If X is a subgroup of the group of automorphisms, induced in N by $\mathfrak{f}G$, then $X/\mathfrak{f}X$ is soluble.*

(5) (a) *N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G.*
(b) *If σ is an epimorphism of G upon H, if the minimal normal subgroup M of H is an elementary abelian p-group, contained in N^σ, if Γ is the group of automorphisms, induced in M by H, if $\mathfrak{f}\Lambda$ is a p-group whenever the subgroup Λ of Γ is generated by n elements, then $\Delta/\mathfrak{f}\Delta$ is p-closed for every subgroup $\Delta \neq \Gamma$ of $\mathfrak{f}\Gamma$.*

Proof. It is an immediate consequence of Corollary 3.2,(a) that (1) implies (2). Assume next the validity of (2). Then clearly (3.a) is satisfied too. The group Γ of automorphisms, induced in N by G, is finite, since N is finite; and $\mathfrak{f}\Gamma$ is induced by $\mathfrak{f}G$ in N. Suppose now that the subgroup Λ of Γ is generated by $n+1$ elements. Then there exists a subgroup U of G which is generated by $n+1$ elements and which induces Λ in N. The group of automorphisms induced in N by $\mathfrak{f}U$, is essentially the same as $\mathfrak{f}U/(\mathfrak{c}N \cap \mathfrak{f}U)$; and it is finite because of the finiteness of N. Denote by T the uniquely determined normal subgroup of $\mathfrak{f}U$ which contains $\mathfrak{c}N \cap \mathfrak{f}U$ and such that $T/(\mathfrak{c}N \cap \mathfrak{f}U)$ is the terminal member of the descending central chain of $\mathfrak{f}U/(\mathfrak{c}N \cap \mathfrak{f}U)$. Then a normal subgroup K of $\mathfrak{f}U$ contains T if, and only if, $\mathfrak{c}N \cap \mathfrak{f}U \subseteq K$ and $\mathfrak{f}U/K$ is nilpotent. If x is any element in N, then $X = \{x, U\}$ is generated by $n + 2$ elements at least one of which belongs to N. Thus we may apply (2) to show

$$(N \cap X) \circ \mathfrak{f}X \subseteq N \cap \mathfrak{f}X \subseteq \mathfrak{h}\mathfrak{f}X;$$

note that $N \cap X$ and $\mathfrak{f}X$ are normal subgroups of X. Since $N \cap X$ is finite we may apply Lemma 3.6, proving that a nilpotent group of automorphisms is induced in $N \cap X$ by $\mathfrak{f}X$; and from $\mathfrak{f}U \subseteq \mathfrak{f}X$ we deduce that a nilpotent group of automorphisms is induced in $N \cap X$ by $\mathfrak{f}U$. If X^* is the normal subgroup of $\mathfrak{f}U$ which induces the identity automorphism in $N \cap X$, then clearly $\mathfrak{c}N \cap \mathfrak{f}U \subseteq X^*$ and $\mathfrak{f}U/X^*$ is nilpotent. We deduce $T \subseteq X^*$ from our characterization of T, so that in particular

$$x \circ T \subseteq (N \cap X) \circ X^* = 1.$$

This implies $N \circ T = 1$; and the group of automorphisms, induced in N by $\mathfrak{f}U$, is consequently an epimorphic image of the nilpotent group $\mathfrak{f}U/T$. But this group of automorphisms is just $\mathfrak{f}\Lambda$; and we have therefore shown that $\mathfrak{f}\Lambda$ is nilpotent whenever the subgroup Λ of Γ is generated by $n + 1$ elements. Application of Theorem 4.1,(a) shows the nilpotency of $\mathfrak{f}\Gamma$, proving the validity of (3.b).

Condition (4) is nothing but a considerably weakened form of condition (3). Assume next the validity of (4). Then we note first the identity of conditions (4.a) and (5.a). Consider an epimorphism σ of G upon H and a minimal normal subgroup M of H with the following properties:

M is an elementary abelian p-group; $M \subseteq N^\sigma$.

If Γ is the group of automorphisms, induced in M by H, and if the subgroup Λ of Γ is generated by n elements, then $\mathfrak{f}\Lambda$ is a p-group.

Consider next a subgroup $\Theta \neq \Gamma$ of $\mathfrak{f}\Gamma$. We note that Γ is an epimorphic image of the group of automorphisms, induced in N by G, and that therefore $\mathfrak{f}\Gamma$ is an epimorphic image of the group of automorphisms induced in N by $\mathfrak{f}G$. Apply condition (4.b) to show the solubility of $\Theta/\mathfrak{f}\Theta$. Thus Γ is a finite group, satisfying condition (ii) of Lemma 4.16. It follows that $\Delta/\mathfrak{f}\Delta$ is p-closed whenever the subgroup $\Delta \neq \Gamma$ of Γ is a subgroup of $\mathfrak{f}\Gamma$; and this shows that (5) is a consequence of (4).

Assume finally the validity of condition (5), and assume by way of contradiction that $N \cap \mathfrak{f}G \not\subseteq \mathfrak{h}\mathfrak{f}G$. Then $N \cap \mathfrak{h}\mathfrak{f}G \subset N \cap \mathfrak{f}G$. Denote by σ the canonical epimorphism of G upon $H = G/(N \cap \mathfrak{h}\mathfrak{f}G)$. Then

$$N^\sigma = N/(N \cap \mathfrak{h}\mathfrak{f}G), \qquad \mathfrak{f}H = \mathfrak{f}G/(N \cap \mathfrak{h}\mathfrak{f}G),$$

$$N^\sigma \cap \mathfrak{f}H = (N \cap \mathfrak{f}G)/(N \cap \mathfrak{h}\mathfrak{f}G) \neq 1.$$

Since N is finite, there exists a minimal normal subgroup M of H with

(i) $$M \subseteq N^{\sigma} \cap \mathfrak{f}H.$$

If the subgroup U of H is generated by $n + 1$ elements one of which is contained in N^{σ}, then there exists a subgroup V of G, generated by $n + 1$ elements one of which belongs to N, with $V^{\sigma} = U$. By (5.a) we have

$$N \cap \mathfrak{f}V \subseteq \mathfrak{h}\mathfrak{f}V.$$

Since the kernel of σ is part of N, we may apply Corollary 3.3 so that

$$M \cap \mathfrak{f}U \subseteq N^{\sigma} \cap \mathfrak{f}(V^{\sigma}) \subseteq \mathfrak{h}\mathfrak{f}(V^{\sigma}) = \mathfrak{h}\mathfrak{f}U;$$

in other words:

(ii) M is an $\mathfrak{f}$-n-hypercentralized normal subgroup of H.

From the construction of σ and the fundamental properties of the hypercenter we conclude that $N^{\sigma} \cap \mathfrak{h}\mathfrak{f}H = 1$; and this implies by (i) that

(iii) $$1 = \mathfrak{h}\mathfrak{f}H \cap M = M \cap \mathfrak{z}\mathfrak{f}H.$$

Recalling that $\mathfrak{f}$ is an n-functor and that M is finite, we deduce from (ii) and Lemma 6.1,(D) that

$$M \circ \mathfrak{f}H = M \circ \{\mathfrak{f}_n H\}$$

is a nilpotent normal subgroup of H which is part of the minimal normal subgroup M of H. If $M \circ \mathfrak{f}H = 1$, then we would deduce from (i) that $M \subseteq \mathfrak{z}\mathfrak{f}H$, contradicting (iii); and thus it follows that $M = M \circ \mathfrak{f}H$ is nilpotent. Since M as a minimal normal subgroup is free of proper characteristic subgroups, it follows that

(iv) M is an elementary abelian p-group.

Denote by Γ the group of automorphisms, induced in M by H. If the subgroup Λ of Γ is generated by n elements, then there exists a subgroup U of H which is generated by n elements such that Λ is induced in M by U. If x is an element in M, then

$$M \cap \mathfrak{f}\{x, U\} \subseteq \mathfrak{h}\mathfrak{f}\{x, U\}$$

by (ii). Application of (iv) and Lemma 3.7 shows that a p-group of automorphisms is induced in M by $\mathfrak{f}U$; and thus we have shown:

(v) If the subgroup Λ of Γ is generated by n elements, then $\mathfrak{f}\Lambda$ is a p-group.

Combining (iv), (v), and condition (5.b) we obtain

(vi) If $\Lambda \neq \Gamma$ is a subgroup of $\mathfrak{f}\Gamma$, then $\Lambda/\mathfrak{f}\Lambda$ is p-closed.

Since $\mathfrak{f}$ is an n-functor with $1 < n$ and Γ is finite, properties (v) and (vi) together with Lemma 4.16 show that

(vii) $\mathfrak{f}\Gamma$ is a p-group.

Now a p-group of automorphisms of a finite p-group, not 1, always possesses fixed elements, not 1. By (vii) therefore $\mathfrak{f}\Gamma$ possesses fixed elements, not 1. Since $\mathfrak{f}\Gamma$ is induced by $\mathfrak{f}H$ in M, this is equivalent to saying that $M \cap \mathfrak{c}\mathfrak{f}H \neq 1$. Since M is a minimal normal subgroup of H which is by (i) part of $\mathfrak{f}H$, we deduce from (iii) that

$$1 \subset M = M \cap \mathfrak{c}\mathfrak{f}H \subseteq M \cap \mathfrak{z}\mathfrak{f}H = 1,$$

a contradiction arising from the assumption that $N \cap \mathfrak{f}G \nsubseteq \mathfrak{h}\mathfrak{f}G$. Thus (1) is a consequence of (5), completing the proof.

Remark 6.3. Condition (4.b) is automatically satisfied—and may then be omitted—whenever the functor $\mathfrak{f}$ is co-soluble. We have pointed out that there exists a great variety of such functors. The perfect functors, introduced and discussed in §7, provide by Theorem 7.6 a second class of functors for which condition (4.b) may be omitted; and it is a consequence of Corollary 7.8 that the class of co-soluble functors and the class of perfect functors do hardly overlap.

Remark 6.4. It is as yet undecided whether the conditions (3.b), (4.b), (5.b) are indispensable or not. Condition (5.b) shows that the class of functors, discussed in the Intermezzo 4.11, will not provide examples showing the indispensability of condition (5.b), that on the contrary in the presence of such functors (5.b) may—as a consequence of Lemma 4.6—be omitted. The impossibility of substituting $n - 1$ for n in conditions (3.a), (4.a) may be deduced from the examples constructed in §8.C and from Theorem A.1 below.

Remark 6.5. The hypothesis $1 < n$ has been used in part of our proof only, mainly when making use of Corollary 4.14. But it is an indispensable hypothesis, as will be shown by an example, constructed in §8.A below.

THEOREM 6.6. *If $\mathfrak{f}$ is an n-functor with $1 < n$, then the following properties of the noetherian normal subgroup N of G are equivalent*:

(1) $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.

(2) *N is an $\mathfrak{f}$-$(n+1)$-hypercentralized normal subgroup of G.*

(3) (a) *N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G.*
(b) *The group of automorphisms, induced in N by $\mathfrak{f}G$, is of finite class.*

(4) (a) *N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G.*
(b) *If Λ is a subgroup of the group of automorphisms, induced in N by $\mathfrak{f}G$, then $\Lambda/\mathfrak{f}\Lambda$ is soluble.*

(5) (a) *If σ is an epimorphism of G upon H whose kernel is part of N, and if $N^\sigma \circ \mathfrak{f}H$ is finite, then $N^\sigma \cap \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$.*
(b) *$N \circ \mathfrak{f}G$ is soluble.*

Proof. It is a consequence of Corollary 3.2,(a) that (1) implies (2) and (3.a); and (3.b) is a consequence of $N \circ \mathfrak{f}G \subseteq N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$ and Lemma 3.6. Since groups of finite class are soluble, (4) is a consequence of (3).

Assume next that at least one of the conditions (2) and (4) is satisfied by N. Consider an epimorphism σ of G upon H whose kernel is part of N such that $N^\sigma \circ \mathfrak{f}H$ is finite. Then we deduce from Lemma 6.1,(B) and (2) that

(2*) $N^\sigma \circ \mathfrak{f}H$ is an $\mathfrak{f}$-$(n+1)$-hypercentralized normal subgroup of H;

and we deduce from Lemma 6.1,(B) and (4.a) that

(4*.a) $N^\sigma \circ \mathfrak{f}H$ is an $\mathfrak{f}$-n-hypercentralized normal subgroup of H.

If Λ is a subgroup of the group of automorphisms, induced in $N^\sigma \circ \mathfrak{f}H$ by $\mathfrak{f}H$, then Λ is an epimorphic image of a subgroup Δ of the group of automorphisms, induced in N by $\mathfrak{f}G$. Consequently $\Delta/\mathfrak{f}\Delta$ is soluble by (4.b), and its epimorphic image $\Lambda/\mathfrak{f}\Lambda$ is likewise soluble. Hence the following property is a consequence of (4.b):

(4*.b) If Λ is a subgroup of the group of automorphisms, induced in $N^\sigma \circ \mathfrak{f}H$ by $\mathfrak{f}H$, then $\Lambda/\mathfrak{f}\Lambda$ is soluble.

Thus we have shown that $N^\sigma \circ \mathfrak{f}H$ is a finite normal subgroup of H which satisfies (2*) or (4*). An immediate application of Theorem 6.2 shows the validity of $N^\sigma \circ \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$; and now we deduce from Theorem 3.1 that $N^\sigma \cap \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$. Condition (5.a) is therefore a consequence of (2) as well as of (4).

If (2) or (4) is satisfied, then N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G. Since $\mathfrak{f}$ is an n-functor and N is noetherian, it follows from Lemma 6.1,(D) that $N \circ \mathfrak{f}G = N \circ \{\mathfrak{f}_n G\}$ is of finite class and a fortiori soluble: condition (5.b). Thus we have shown that (5) is both a consequence of (2) and a consequence of (4).

Assume now the validity of (5). Suppose that L is a normal subgroup of $N \circ \mathfrak{f}G$ with finite $(N \circ \mathfrak{f}G)/L$. Since N is noetherian, $N \circ \mathfrak{f}G$ is finitely generated, and consequently there exists a characteristic subgroup K of $N \circ \mathfrak{f}G$ such that $(N \circ \mathfrak{f}G)/K$ is finite and $K \subseteq L$; see Baer [7; p. 331, Folgerung 3]. Then K is a normal subgroup of G. The canonical epimorphism σ of G upon G/K has the following properties:

The kernel of σ is part of N;
$N^\sigma \circ \mathfrak{f}H = (N \circ \mathfrak{f}G)^\sigma = (N \circ \mathfrak{f}G)/K$ is finite.

Application of (5.a) shows that $N^\sigma \circ \mathfrak{f}H \subseteq N^\sigma \cap \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$. Thus $N^\sigma \circ \mathfrak{f}H = (N \circ \mathfrak{f}G)/K$ and its epimorphic image $(N \circ \mathfrak{f}G)/L$ are finite nilpotent groups. Using (5.b) we see that $N \circ \mathfrak{f}G$ is a noetherian soluble group all of whose finite epimorphic images are nilpotent. Applying Baer [1; p. 205, Theorem and p. 170, Lemma 4] we see that

(i) $N \circ \mathfrak{f}G$ is of finite class.

Since $N \circ \mathfrak{f}G$ is a noetherian group of finite class, we may apply Baer [7; p. 301, Hauptsatz 1] to obtain

(ii) There exists a positive integer k such that 1 is the intersection of all the characteristic subgroups X of $N \circ \mathfrak{f}G$ whose index $[(N \circ \mathfrak{f}G):X]$ is a prime power p^x with $x \leqq k$ (and variable prime p).

If p is some prime, then denote by $K(p)$ the intersection of all the characteristic subgroups X of $N \circ \mathfrak{f}G$ with $[(N \circ \mathfrak{f}G):X]$ a divisor of p^k. Since $N \circ \mathfrak{f}G$ is finitely generated, there exists only a finite number of such normal subgroups X; see Baer [7; p. 331, Lemma 4]. It follows that

(iii) $K(p)$ is a characteristic subgroup of $N \circ \mathfrak{f}G$, and $(N \circ \mathfrak{f}G)/K(p)$ is a finite p-group.

The following definition will be needed in the sequel: If A is a normal subgroup of B, then

$$A = B^{(0)} \circ A, \qquad B \circ (B^{(j)} \circ A) = B^{(j+1)} \circ A.$$

It is clear that $B^{(j)} \circ A$ is a normal subgroup of B, that the $B^{(j)} \circ A$ form a descending chain of normal subgroups, and that $B^{(j)} \circ A$ is part of the $(j + 1)^{\text{st}}$ term of the descending central chain of B.

Denote by σ the canonical epimorphism of G upon $H = G/K(p)$. Then the kernel of σ is part of N, and

$$N^\sigma \circ \mathfrak{f}H = (N \circ \mathfrak{f}G)^\sigma = (N \circ \mathfrak{f}G)/K(p) \quad \text{is finite.}$$

Application of condition (5) shows that $N^\sigma \circ \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$. Since $N^\sigma \circ \mathfrak{f}H$ is finite, we deduce from Theorem 3.1 that

$$(+) \qquad (\mathfrak{f}H)^{(j)} \circ (N^\sigma \circ \mathfrak{f}H) = 1 \quad \text{for almost all } j.$$

Consider next a normal subgroup X of $\mathfrak{f}H$ with $X \subseteq N^\sigma \circ \mathfrak{f}H$ and $[(N^\sigma \circ \mathfrak{f}H):X]$ a divisor of p^k. Since at most $k + 1$ terms of a composition chain of $\mathfrak{f}H/X$ are contained in $(N \circ \mathfrak{f}H)/X$, it follows from (+) that

$$(\mathfrak{f}H)^{(k)} \circ (N^\sigma \circ \mathfrak{f}H) \subseteq X.$$

But by our construction of $K(p)$ and σ the intersection of all these normal subgroups X of $N^\sigma \circ \mathfrak{f}H$ is 1; and this proves

$$(\mathfrak{f}H)^{(k)} \circ (N^\sigma \circ \mathfrak{f}H) = 1.$$

Recalling the definition of H and σ it follows that

$$\text{(iv)} \qquad (\mathfrak{f}G)^{(k)} \circ (N \circ \mathfrak{f}G) \subseteq K(p) \quad \text{for every } p.$$

Combining (ii) with the definition of $K(p)$ we see that $1 = \bigcap_p K(p)$; and hence it follows from (iv) that

$$(\mathfrak{f}G)^{(k)} \circ (N \circ \mathfrak{f}G) = 1.$$

But this fact implies clearly $N \circ \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$; and application of Theorem 3.1 shows the validity of $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$ so that (1) is a consequence of (5).

Remark 6.7. Condition (5.b) has been used only in the derivation of property (i); and an inspection of this proof shows that it would have sufficed to substitute for (5.b) the following slightly weaker condition:

(+) $N \circ \mathfrak{f}G$ possesses a soluble subgroup of finite index.

Note that $N \circ \mathfrak{f}G$ is noetherian and that we may use Baer [7; p. 331, Folgerung 3]. Groups with property (+) may be termed almost-soluble; cp. Baer [4; p. 276, Satz 3].

Remark 6.8. The impossibility of proving a theorem of the type of Theorem 6.6 without imposing any hypothesis like the maximum condition on N will be put into evidence by a simple example in §8.B.

PROPOSITION 6.9. *If $\mathfrak{f}$ is a co-soluble n-functor with $1 < n$, if N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G, and if $N \cap \mathfrak{f}S$ is finite for every finitely generated subgroup S of G, then every finitely generated subgroup X of $N \cap \mathfrak{f}G$ is finite and nilpotent; and if the element g in $\mathfrak{f}G \cap \mathfrak{n}X$ induces in X an automorphism of order m, then g commutes with the elements of order prime to m in X.*

Proof. Consider a finitely generated subgroup S of G. Then, by hypothesis, $N \cap \mathfrak{f}S$ is a finite, $\mathfrak{f}$-n-hypercentralized normal subgroup of S so that because of the co-solubility of $\mathfrak{f}$, condition (4) of Theorem 6.2 is satisfied by $N \cap \mathfrak{f}S$. Application of Theorem 6.2 shows then the validity of

(a) $N \cap \mathfrak{f}S \subseteq \mathfrak{h}\mathfrak{f}S$ for every finitely generated subgroup S of G.

Consider next a finite subset X of $N \cap \mathfrak{f}G$. Since $\mathfrak{f}$ is an n-functor, there exists to every element x in X a subgroup x^* of G which is generated by n elements such that x belongs to $\mathfrak{f}x^*$. If X^* is the subgroup of G which is generated by all the subgroups x^* for x in X, then

(b′) X^* is finitely generated,

since X is a finite set and every subgroup x^* is finitely generated;

(b″) X is a subset of $\mathfrak{f}X^*$,

since every element x in X belongs to $\mathfrak{f}x^* \subseteq \mathfrak{f}X^*$. From

$$\{X\} \subseteq N \cap \mathfrak{f}X^* \subseteq \mathfrak{h}\mathfrak{f}X^*,$$

by (a) and from our third hypothesis we conclude that

(c) $\{X\}$ is a finite nilpotent group.

If finally g is any element in $\mathfrak{f}G \cap \mathfrak{n}\{X\}$, then there exists a subgroup g^* of G which is generated by n elements such that g belongs to $\mathfrak{f}g^*$. Let $Y = \{X^*, g^*\}$. Then Y is finitely generated so that $N \cap \mathfrak{f}Y$ is finite by hypothesis. Application of (a) shows that

(d) $$\{X\} \subseteq N \cap \mathfrak{f}X^* \subseteq N \cap \mathfrak{f}Y \subseteq \mathfrak{h}\mathfrak{f}Y;$$

and g belongs to $\mathfrak{f}g^* \subseteq \mathfrak{f}Y$. Since g belongs to $\mathfrak{n}\{X\}$, and since $\{X\}$ is finite by (c), an automorphism is induced in $\{X\}$ by g whose order is a positive integer m. From (d) we deduce that the normal subgroup $\{X\}$ of $\{X, g\}$ is contained in the hypercenter $\mathfrak{h}\{X, g\}$. It is an immediate consequence of Baer [1; p. 179, Lemma 2] that g commutes with every element in $\{X\}$ whose order is prime to m.

COROLLARY 6.10. *If $\mathfrak{f}$ is a co-soluble n-functor with $1 < n$, and if N is a normal subgroup of G such that*

(a) $N \cap \mathfrak{f}S$ is finite for every finitely generated subgroup S of G and

(b) $N \cap \mathfrak{f}G$ is artinian,

then $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$ if, and only if, N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G.

Proof. It is an immediate consequence of Corollary 3.2,(a) that N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G, if $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$. We assume conversely that N is an $\mathfrak{f}$-n-hypercentralized normal subgroup of G. Because of (a) and Proposition 6.9

(1) Finitely generated subgroups of $N \cap \mathfrak{f}G$ are finite and nilpotent.

Because of (b) and (1) we may apply Baer [9; p. 21, Satz 4.1] to show that

(2) $N \cap \mathfrak{f}G$ is nilpotent and almost abelian.

Assume now by way of contradiction that $N \cap \mathfrak{f}G \not\subseteq \mathfrak{h}\mathfrak{f}G$. Denote by σ the canonical epimorphism of G upon $H = G/(N \cap \mathfrak{h}\mathfrak{f}G)$. Then

(3) $$1 \subset (N \cap \mathfrak{f}G)/(N \cap \mathfrak{h}\mathfrak{f}G) = (N \cap \mathfrak{f}G)^\sigma = N^\sigma \cap \mathfrak{f}H.$$

Since the kernel of σ is part of N, application of Corollary 3.3 shows that

(4) N^σ is an $\mathfrak{f}$-n-hypercentralized normal subgroup of H;

and we deduce from (b), (2), and (3) that

(5) $N^\sigma \cap \mathfrak{f}H$ is nilpotent, almost abelian, and artinian.

Since $N^\sigma \cap \mathfrak{f}H$ is an artinian normal subgroup of H which is different from 1, there exists a minimal normal subgroup M of H which is part of $N^\sigma \cap \mathfrak{f}H$. By (5), M is nilpotent, almost abelian, and artinian; and because of its minimality M is free of proper characteristic subgroups. Hence

(6) M is a finite, elementary abelian p-group.

With M its centralizer $\mathfrak{c}M$ is a normal subgroup of H; and $H/\mathfrak{c}M$ is essentially the same as the group of automorphisms, induced in M by H. It follows in particular that $H/\mathfrak{c}M$ is finite. If h is any element in $\mathfrak{f}H$, then h induces in

M an automorphism of positive order k. Since M is by (6) a finite p-group which is part of $N^\sigma \cap \mathfrak{f}H$, and since we may apply Proposition 6.9 by (a) and (4), it follows that k is a power of p. Thus we have shown that every element in the finite group $(\mathfrak{c}M\mathfrak{f}H)/\mathfrak{c}M$ is a p-element; and this implies

(7) $(\mathfrak{c}M\mathfrak{f}H)/\mathfrak{c}M$ is a p-group.

A p-group of automorphisms of a finite p-group, not 1, possesses fixed elements, not 1; and this is equivalent to saying that $M \cap \mathfrak{c}\mathfrak{f}H \neq 1$. Since $\mathfrak{c}\mathfrak{f}H$ is a characteristic subgroup of H and M is a minimal normal subgroup of H, we conclude that

$$M \subseteq \mathfrak{c}\mathfrak{f}H \cap \mathfrak{f}H \cap N = \mathfrak{z}\mathfrak{f}H \cap N = 1$$

by (3); and this is the desired contradiction which proves our corollary.

7. Perfect functors

Suppose that N is a normal subgroup of the group G, and denote by Γ the group of automorphisms, induced in N by G. Then Γ is essentially the same as $G/\mathfrak{c}N$, and $\mathfrak{f}G$ is mapped onto $\mathfrak{f}\Gamma$ by the natural epimorphism of G upon Γ. Consequently $\mathfrak{f}\Gamma = 1$ is equivalent to the fact that N is centralized by $\mathfrak{f}G$, a fact that may be expressed shortly by $N \circ \mathfrak{f}G = 1$. Of this equivalence we shall make considerable use in the sequel.

Proposition 7.1. *If $\mathfrak{f}$ is an n-functor and N a normal subgroup of the group G, then $N \circ \mathfrak{f}G = 1$ if, and only if, $N \circ \mathfrak{f}U = 1$ for every subgroup U of G which is generated by n elements.*

Proof. The necessity of our condition is an immediate consequence of $\mathfrak{f}U \subseteq \mathfrak{f}G$. If conversely our condition is satisfied, then we consider an element c in $\mathfrak{f}_n G$. There exists a subgroup C of G which is generated by n elements such that c belongs to $\mathfrak{f}C$. By hypothesis $N \circ \mathfrak{f}C = 1$; and this implies $N \circ c = 1$. Consequently $N \circ \mathfrak{f}_n G = 1$. Since $\mathfrak{f}$ is an n-functor, this implies

$$N \circ \mathfrak{f}G = N \circ \{\mathfrak{f}_n G\} = 1,$$

as we wanted to show.

Definition 7.2. *The functor $\mathfrak{f}$ is perfect, if $\mathfrak{f}G = (\mathfrak{f}G)'$ for every group G.*

For perfect functors it will be possible to improve considerably the results of §6; and these improvements will be strongly related to considerations of the type of Proposition 7.1.

Lemma 7.3. *The following properties of the functor $\mathfrak{f}$ are equivalent:*
(i) *$\mathfrak{f}$ is perfect.*
(ii) *$\mathfrak{f}X = 1$ whenever $\mathfrak{d}^i(\mathfrak{f}X) = 1$ for almost all i.*
(iii) *$\mathfrak{f}X = 1$ whenever $\mathfrak{f}X$ is of finite class.*
(iv) *$\mathfrak{f}X = 1$ whenever $\mathfrak{f}X$ is abelian.*

Proof. Assume first that $\mathfrak{f}$ is perfect, and consider a group X such that $\mathfrak{d}^k(\mathfrak{f}X) = 1$. Because of the perfectness of $\mathfrak{f}$ we have $\mathfrak{f}X = \mathfrak{d}^i(\mathfrak{f}X)$ for every positive i; and this shows that (ii) is a consequence of (i). It is almost obvious that (iii) is a consequence of (ii) and that (iv) is a consequence of (iii). Assume finally the validity of (iv). If X is any group, then $(\mathfrak{f}X)'$ is a characteristic subgroup of X, and thus we may form the epimorphic image $Y = X/(\mathfrak{f}X)'$ of X. Then $\mathfrak{f}Y = \mathfrak{f}X/(\mathfrak{f}X)'$ is abelian so that $\mathfrak{f}Y = 1$ by (iv). Hence $\mathfrak{f}X = (\mathfrak{f}X)'$, proving the perfectness of $\mathfrak{f}$.

PROPOSITION 7.4. *If $\mathfrak{f}$ is a perfect functor, then the following properties of the noetherian normal subgroup N of G are equivalent:*

(i) $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.
(ii) *The group of automorphisms, induced in N by $\mathfrak{f}G$, is of finite class.*
(iii) $N \circ \mathfrak{f}G = 1$.
(iv) $N \cap \mathfrak{f}G \subseteq \mathfrak{z}\mathfrak{f}G$.

Proof. Since N and $\mathfrak{f}G$ are normal subgroups of G, we have $N \circ \mathfrak{f}G \subseteq N \cap \mathfrak{f}G$. It is now an immediate consequence of Lemma 3.6 that (i) implies (ii).

Assume next the validity of (ii), and denote by Γ the group of automorphisms, induced in N by G. Then $\mathfrak{f}\Gamma$ is the group of automorphisms, induced in N by $\mathfrak{f}G$. Hence $\mathfrak{f}\Gamma$ is by (ii) of finite class; and we conclude $\mathfrak{f}\Gamma = 1$ from the perfectness of $\mathfrak{f}$ and Lemma 7.3. Hence $\mathfrak{f}G$ induces the 1-automorphism in N; and this is equivalent to $N \circ \mathfrak{f}G = 1$. Thus (iii) is a consequence of (ii).

It is fairly obvious that (iii) implies (iv) and that (i) is a consequence of (iv), showing the equivalence of (i)–(iv).

Remark 7.5. The proof shows that the hypothesis that N be noetherian is too strong. It suffices to assume that the maximum condition be satisfied by the normal subgroups of N.

THEOREM 7.6. *If $\mathfrak{f}$ is a perfect n-functor with $1 < n$, then the noetherian normal subgroup N of G is $\mathfrak{f}$-hypercentralized if, and only if, it is $\mathfrak{f}$-n-hypercentralized.*

Proof. If N is $\mathfrak{f}$-hypercentralized by G, then it is a consequence of Theorem 6.6 that N is $\mathfrak{f}$-n-hypercentralized by G. Assume conversely that N is $\mathfrak{f}$-n-hypercentralized by G. Consider an element x in N and an element c in $\mathfrak{f}_n G$. Then there exists a subgroup V of G which is generated by n elements such that c belongs to $\mathfrak{f}V$. Since $U = \{V, x\}$ is generated by $n + 1$ elements at least one of which belongs to N, we deduce

$$(N \cap U) \cap \mathfrak{f}U = N \cap \mathfrak{f}U \subseteq \mathfrak{h}\mathfrak{f}U$$

from our hypothesis. Since $\mathfrak{f}$ is perfect and $N \cap U$ a normal subgroup of U, we deduce

$$(N \cap U) \circ \mathfrak{f}U = 1$$

from Proposition 7.4. But x belongs to $N \cap U$, and c to $\mathfrak{f}U$. Hence $x \circ c = 1$; and thus we have shown that $N \circ \mathfrak{f}_n G = 1$. Since $\mathfrak{f}$ is an n-functor, this implies $1 = N \circ \{\mathfrak{f}_n G\} = N \circ \mathfrak{f}G$ so that N is $\mathfrak{f}$-hypercentralized by G (see Proposition 7.4).

Remark 7.7. The functor $\mathfrak{d}^\infty$, defined on the class of all finite groups, is perfect by Lemma 7.3, and it is a 2-functor, by a Theorem of John Thompson, as we have pointed out before. Thus it follows from Theorem 7.6 that $\mathfrak{d}^\infty$-hypercentralization and $\mathfrak{d}^\infty$-2-hypercentralization are equivalent properties of a normal subgroup N of a finite group G.

Corollary 7.8. *The functor* $\mathfrak{f}$, *defined on the class* $\mathfrak{F}$ *of all finite groups, is the functor* $\mathfrak{d}^\infty$ *if, and only if,* $\mathfrak{f}$ *is a perfect functor and* $G/\mathfrak{f}G$ *is soluble for every finite group* G.

Proof. It is almost obvious that the functor $\mathfrak{d}^\infty$ has the two properties in question. Assume conversely that $\mathfrak{f}$ is perfect and $G/\mathfrak{f}G$ soluble for every finite group G. If S is any finite soluble group, then we deduce from the perfectness of $\mathfrak{f}$ and Lemma 7.3 that $\mathfrak{f}S = 1$. If conversely T is a finite group with $\mathfrak{f}T = 1$, then $T = T/\mathfrak{f}T$ is soluble because of the second property of $\mathfrak{f}$. The finite group G is consequently soluble if, and only if, $\mathfrak{f}G = 1$. If N is a normal subgroup of the finite group G, then G/N is soluble if, and only if, $\mathfrak{f}G \subseteq N$; and this implies $\mathfrak{f}G = \mathfrak{d}^\infty G$ and hence $\mathfrak{f} = \mathfrak{d}^\infty$.

8. Counterexamples

A. If k is a positive integer, then a 1-functor $\mathfrak{f} = \mathfrak{f}_k$ defined by the rule

$$\mathfrak{f}G = G^k$$

is for every G in $\mathfrak{D}$ the subgroup, generated by all the k^{th} powers of elements in G.

We select now $\mathfrak{D}$ as the class of all finite groups, and k as a prime power with $2 < k$. Furthermore let p be a prime divisor of $k - 1$. If K is a cyclic group of order k, then K possesses an automorphism σ of order p. Denote by J the group, obtained by adjoining to K an element s, subject to the relations:

$$s^p = 1, \qquad s^{-1}xs = x^\sigma \quad \text{for every } x \text{ in } K.$$

Since 1 is the only fixed element of the automorphism σ, it follows that $\mathfrak{n}\{s\} = \mathfrak{c}\{s\} = \{s\}$; and this implies that every element in J, but not in K, has order p. Consequently J is generated by its elements of order p.

It is easy to construct now a group G, possessing a normal subgroup N with the following properties:

(a) N is an elementary abelian p-group;

(b) $N = \mathfrak{c}N$;

(c) $G/N \simeq J$.

Since J is generated by its p-elements, and since N is a p-group, one sees readily that

(d) G is generated by its p-elements.

Since the integers p and k are relatively prime, as the prime p is a divisor of $k - 1$, it follows that every p-element is a k^{th} power (of some power of itself). Apply (d) to show that

(e) $G = G^k$.

If x is an element in N and y an element in G, then we let $Y = \{x, y\}$. Clearly $N \cap Y$ is an elementary abelian normal p-subgroup of Y; and since x belongs to $N \cap Y$, we have $Y = (N \cap Y)\{y\}$ so that $Y/(N \cap Y)$ is a cyclic group of order a divisor of pk. Thus

$$[Y/(N \cap Y)]^k = (N \cap Y)Y^k/(N \cap Y)$$

is cyclic of order a divisor of p. Hence

(f) $\{x, y\}^k$ is a p-group whenever x is in N and y is in G.

It is clear that J is soluble; but the element s of order p, prime to k, does not commute with the elements of order, not 1, in K. Hence J is not nilpotent. Consequently

(g) G is soluble, but not nilpotent.

The elements of order k in G induce because of (b) in N automorphisms of order k. As k and p are relatively prime and N is a p-group, it follows from Baer [2; p. 42, Theorem 3] that N is not part of the hypercenter of G. Combining this with (e) we obtain

(h) $$N \cap G^k = N \not\subseteq \mathfrak{h}G = \mathfrak{h}(G^k).$$

By (f), N is an $\mathfrak{f}$-1-hypercentralized normal subgroup of G; and by (h), N is not an $\mathfrak{f}$-hypercentralized normal subgroup of G. Since G is soluble, condition (4) of Theorem 6.2 is satisfied by the normal subgroup N of G and the functor $\mathfrak{f}$. But neither condition (1) nor (3) is satisfied, showing the indispensability of the hypothesis $1 < n$. Cp. also in this context Remark 4.10.

B. Consider an odd prime p and countably infinite, elementary abelian p-groups U, V, W. Then we may use the elements in W as indices for a basis $b(w)$ of V; and there exists one and essentially only one group R containing V as normal subgroup and W as subgroup, subject to the following conditions:

$$R = VW = WV, \qquad 1 = V \cap W,$$

$$w^{-1}b(x)w = b(xw) \quad \text{for all } x, w \text{ in } W.$$

Then $\mathfrak{z}R = 1$, and finitely many elements in R generate a finite p-subgroup of R.

Since R is likewise a countably infinite group, we may use the elements in R as indices for a basis $u(r)$ of U. Then there exists one and essentially only one group G containing U as a normal subgroup and R as a subgroup, subject to the following conditions:

$$G = UR = RU, \qquad 1 = R \cap U,$$

$$r^{-1}u(y)r = u(yr) \quad \text{for all } r, y \text{ in } R.$$

Then $\mathfrak{z}G = 1$, and finitely many elements in G generate a finite p-subgroup of G.

One sees easily that the commutator subgroup $\mathfrak{d}G$ of G is contained in UV and is of a type very similar to that of UV. In particular $\mathfrak{z}\mathfrak{d}G = \mathfrak{h}\mathfrak{d}G = 1$.

If the subgroup S of G is generated by finitely many elements, then it is a finite p-group so that $\mathfrak{d}S = \mathfrak{h}\mathfrak{d}S$. Hence

$$U \cap \mathfrak{d}S \subseteq \mathfrak{d}S = \mathfrak{h}\mathfrak{d}S.$$

But $U \cap \mathfrak{d}G$ is infinite whereas $\mathfrak{z}\mathfrak{d}G = \mathfrak{h}\mathfrak{d}G = 1$. Recall finally that $\mathfrak{d}$ is a 2-functor; and we see that Theorems 5.3 and 6.6 cease to be valid without the hypothesis that N be noetherian.

C. The author is indebted to Professor Paul Fong (Berkeley) for pointing out to him the following class of examples. Denote by F a finite group, meeting the following requirements:

(a) F is of class three: $\mathfrak{c}_3 F = 1$.

(b) F is not of class two: $\mathfrak{c}_2 F \neq 1$.

(c) Subgroups S of F, generated by two elements, are of class two: $\mathfrak{c}_2 S = 1$.

There exist many such groups like the finite groups B, generated by more than two elements, subject to the identical relation $x^3 = 1$ only; see Levi-van der Waerden.

Denote by p any prime, not dividing $o(F)$. Then there exist elementary abelian p-groups N with a group Γ of automorphisms, isomorphic to F. We form the product $G = N\Gamma$ within the holomorph of N.

Consider a subgroup S of G with $N \subset S$. Then $[S:N]$ is different from 1 and prime to p. Consequently there exists in S an element s of order a prime $q \neq p$. The element s induces in N an automorphism of order q so that S is not nilpotent. We may state this result as follows:

(1) N is a maximal nilpotent subgroup of G.

Since $G/N \simeq F$ is of class three, we deduce from (1) that

(2) $$N = \mathfrak{c}_3 F;$$

and combining (1), (2) with $G/N \simeq F$ and (b) we find that

(3) $N \subset \mathfrak{c}_2 G$ and $\mathfrak{c}_2 G$ is not nilpotent.

Consider next a subgroup S of G which is generated by two elements, and form $T = NS$. Then T/N is generated by two elements; and we deduce from $G/N \simeq F$ and (c) that T/N is of class two. This is equivalent to $\mathfrak{c}_2 T \subseteq N$. We restate this as follows:

(4) If the subgroup S of G is generated by two elements, then $\mathfrak{c}_2 S$ and $\mathfrak{c}_2(NS)$ are elementary abelian p-groups, contained in N.

From the properties of the group G which we have derived so far we are going to deduce a number of properties of the functor $\mathfrak{c}_2$. We first state the following rather obvious fact.

(I) $\mathfrak{c}_2$ is a co-nilpotent 3-functor.

(II) $\mathfrak{c}_2$ is not a 2-functor.

Proof. If this were false, then we would deduce from (4), the co-solubility of $\mathfrak{c}_2$ and from Theorem 4.7 the nilpotency of $\mathfrak{c}_2 G$, contradicting (3).

(III) It is impossible to substitute $n - 1$ for n in condition (a) of Theorem 4.7.

Proof. Let in Theorem 4.7 the functor $\mathfrak{f}$ be $\mathfrak{c}_2$. Then $n = 3$, and (III) is a consequence of (3), (4).

(IV) It is impossible to substitute $n - 1$ for n in conditions (3.a) and (4.a) of Theorem 6.2.

Proof. Let in Theorem 6.2 the functor $\mathfrak{f}$ be $\mathfrak{c}_2$. Then $n = 3$. If we consider now the normal subgroup N of the group G under consideration, then it follows from (1) and (3) that

$$N \cap \mathfrak{c}_2 G = N \not\subseteq \mathfrak{h}\mathfrak{c}_2 G$$

as N is a normal Hall subgroup of $\mathfrak{h}\mathfrak{c}_2 G$. But G induces in N the nilpotent group Γ of automorphisms so that condition (3.b) of Theorem 6.2 is satisfied; and condition (4.b) of Theorem 6.2 holds true, since $\mathfrak{c}_2$ is co-soluble. Now (IV) is a consequence of (4).

Appendix. On the characterization of the invariant n of an n-functor.

We want to show in some special, though interesting situations that the properties contained in Theorem 4.7 and in Theorem 6.2 are characteristic for the invariant n of an n-functor.

THEOREM A.1. *If $\mathfrak{f}$ is a co-soluble functor on the class $\mathfrak{F}$ of all finite groups, then the following properties of the integer $n > 1$ are equivalent:*

(i) *$\mathfrak{f}$ is an n-functor.*

(ii) *If* $\mathfrak{f}S$ *is nilpotent whenever the subgroup* S *of the finite group* G *is generated by* n *elements, then* $\mathfrak{f}G$ *is nilpotent.*

(iii) *If* N *is an* $\mathfrak{f}$-n-*hypercentralized normal subgroup of the finite group* G, *then* $N \cap \mathfrak{f}G \subseteq \mathfrak{h}\mathfrak{f}G$.

Proof. That (i) implies (ii) is a consequence of Theorem 4.7; and that (i) implies (iii) may be deduced from Theorem 6.2.

We assume now that at least one of the conditions (ii) and (iii) is satisfied. Consider a finite group G with the following property:

(+) $\mathfrak{f}S = 1$ whenever the subgroup S of G is generated by n elements.

Select any prime p which does not divide the order $o(G)$ of G. Then there exist an elementary abelian p-group N and a group Γ of automorphisms of N which is isomorphic to G, and Γ may be selected in such a way that Γ induces a group of permutations, isomorphic to G, on a suitable basis of N. Next we form the product $H = N\Gamma$ within the holomorph of N. It is clear then that N and H have the following properties:

(a) N is a normal Hall subgroup of H; the group Γ of automorphisms is induced in N by H; and $H/N \simeq G$, $N = \mathfrak{c}N$.

Consider a subgroup S of H with $N \subset S$. Then $o(N)$ and $[S:N]$ are relatively prime and $[S:N] \neq 1$. By Cauchy's Theorem there exists an element s in S whose order is a prime $q \neq p$. This element does not belong to N and induces consequently by (a) in N an automorphism of order q. It follows that the p-group N (with $p \neq q$) is not part of the hypercenter $\mathfrak{h}S$ of S; see Baer [2; p. 42, Theorem 3]. Thus we have established the following facts:

(b) If S is a subgroup of H with $N \subset S$, then $N \not\subseteq \mathfrak{h}S$ so that S is not nilpotent.

Assume next that the subgroup S of H is generated by n elements; and let $T = NS$. Then T/N is generated by n elements; and T/N is isomorphic to a subgroup of G. Application of (+) shows therefore that $\mathfrak{f}(T/N) = 1$; and this is equivalent to $\mathfrak{f}T \subseteq N$. This implies in particular that $\mathfrak{f}S \subseteq \mathfrak{f}T \subseteq N$ so that $\mathfrak{f}S$ is an elementary abelian p-group. If furthermore x is an element in N and $X = \{x, S\}$, then $X \subseteq T$ so that $\mathfrak{f}X \subseteq \mathfrak{f}T \subseteq N$, implying again that $\mathfrak{f}X$ is an elementary abelian p-group. Thus we have shown

(c′) If the subgroup S of H is generated by n elements, then $\mathfrak{f}S$ is an elementary abelian p-group.

(c″) If the subgroup X of H is generated by $n + 1$ elements one of which belongs to N, then $\mathfrak{f}X$ is an elementary abelian p-group.

If condition (ii) holds, then we deduce the nilpotency of $\mathfrak{f}H$ from (c′).

Thus

(d′) If condition (ii) is satisfied by n, then $\mathfrak{f}H$ is nilpotent.

Similarly we deduce from (c″) the validity of the following statement:

(d″) If condition (iii) is satisfied by n, then $N \cap \mathfrak{f}H \subseteq \mathfrak{h}\mathfrak{f}H$.

Since N and $\mathfrak{h}\mathfrak{f}H$ are nilpotent normal subgroups of H, their product $N \cdot \mathfrak{h}\mathfrak{f}H$ is a nilpotent normal subgroup of the finite group H. Application of (b) shows that $\mathfrak{h}\mathfrak{f}H \subseteq N\mathfrak{h}\mathfrak{f}H = N$. If condition (ii) is true, then we deduce from (c′) that $\mathfrak{f}H$ is nilpotent, implying $\mathfrak{f}H = \mathfrak{h}\mathfrak{f}H \subseteq N$. If condition (iii) is true, then we deduce from (d″) that $\mathfrak{h}\mathfrak{f}H = N \cap \mathfrak{h}\mathfrak{f}H = N \cap \mathfrak{f}H$. Naturally automorphisms of order prime to p are induced by the elements in $\mathfrak{f}H$ in N and a fortiori in $\mathfrak{h}\mathfrak{f}H \subseteq N$. Application of Baer [2; p. 42, Theorem 3] shows that $\mathfrak{h}\mathfrak{f}H$ is centralized by $\mathfrak{f}H$. If x is an element in N and t is an element in $\mathfrak{f}H$, then $x \circ t$ belongs to $N \circ \mathfrak{f}H \subseteq N \cap \mathfrak{f}H = \mathfrak{h}\mathfrak{f}H$ so that $x \circ t$ is centralized by t. From $x^t = x(x \circ t)$ we deduce by complete induction that $x^{t^i} = x(x \circ t)^i$; and this implies in particular that $x^{t^p} = x(x \circ t)^p = x$, since $N^p = 1$. Thus t^p commutes with every x in N; and we have shown that N is centralized by t^p for every t in $\mathfrak{f}H$. In other words: $(\mathfrak{f}H)^p \subseteq \mathfrak{c}N = N$ by (a). This implies that $\mathfrak{f}H$ is a p-group, since N is a p-group. But N is a normal Hall subgroup; and thus it follows that N is the totality of all p-elements in H, implying $\mathfrak{f}H \subseteq N$. Accordingly we have shown in both cases that

(e) $$\mathfrak{f}H \subseteq N.$$

But (e) is clearly equivalent to $\mathfrak{f}G = 1$ so that we have derived from (ii) as well as from (iii) the validity of

(iv) If $\mathfrak{f}S = 1$ whenever the subgroup S of the finite group G is generated by n elements, then $\mathfrak{f}G = 1$.

In the presence of (iv) it is easy to prove that $\mathfrak{f}$ is an n-functor. For let G be some finite group. Then $C = \{\mathfrak{f}_n G\}$ is a characteristic subgroup of G. Thus we may form $H = G/C$. If the subgroup S of H is generated by n elements, then there exists a subgroup T of G which is generated by n elements such that $S = CT/C$. By definition $\mathfrak{f}T \subseteq \mathfrak{f}_n G \subseteq C$ so that $\mathfrak{f}S = 1$. Application of (iv) shows now that $1 = \mathfrak{f}H = \mathfrak{f}G/C$. Hence $\mathfrak{f}G = C = \{\mathfrak{f}_n G\}$ so that $\mathfrak{f}$ is an n-functor; and (i) has been shown to be a consequence of (iv), completing the proof.

Remark A.2. Inspection of the proof shows that the hypothesis of cosolubility of $\mathfrak{f}$ has been used only when deriving (ii) and (iii) from (i), but not when deducing (iv) from (ii) and from (iii) nor in the deduction of (i) from (iv).

Remark A.3. It is apparent from our proof that the domain $\mathfrak{D}$ of definition of $\mathfrak{f}$ did not have to be the class $\mathfrak{F}$ of all finite groups. It would have sufficed to impose upon $\mathfrak{D}$ the following two requirements:

$\mathfrak{D}$-groups are finite.

G is a $\mathfrak{D}$-group, whenever there exists a normal elementary abelian p-Hall subgroup P of G such that G/P is a $\mathfrak{D}$-group.

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UNIVERSITÄT
FRANKFURT AM MAIN, GERMANY