COMMUTATORS AND CUP PRODUCTS

BY
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1. Introduction

There has been evidence that cup products in the cohomology groups of a space Y are related to commutators for the suspension ΣY of Y. There is the result of Berstein and Ganea [4], $\smile$-long $Y \leqq$ conil ΣY, which asserts that if the basic n-fold co-commutator map ψ_n in $\pi(\Sigma Y, {}^n\Sigma Y)$ is trivial, then all cup products in Y of length n vanish. Furthermore, in [3] it is proved that ψ_n has finite order if and only if all rational cup products of length n vanish. In this paper we make explicit the relationship between cup products in Y of length n and the basic n-fold co-commutator map ψ_n of ΣY. Thus we obtain a new characterization of the cup product of a space in terms of a commutator in the suspension of the space.

We proceed as follows. For any X which is an H'-space (i.e., a space of normalized Lusternik-Schnirelmann category $\leqq 1$) and any spaces A_1 and A_2, we define in §3 a product $\pi(X, A_1) \times \pi(X, A_2) \rightarrow \pi(X, A_1 \flat A_2)$, where $A_1 \flat A_2$ is a space obtained from A_1 and A_2. This product, called the *flat product*, is defined by means of an H'-space commutator. In §4 we take A_i to be an Eilenberg-MacLane space $K(G_i, m_i + 1)$ and we choose a certain map

$$\theta : A_1 \flat A_2 \rightarrow K(G_1 \otimes G_2, m_1 + m_2 + 1).$$

By composing the flat product with θ we obtain a *cohomology flat product* which assigns to $\alpha_1 \in H^{m_1+1}(X; G_1)$ and $\alpha_2 \in H^{m_2+1}(X; G_2)$ an element

$$\langle \alpha_1, \alpha_2 \rangle \in H^{m_1+m_2+1}(X; G_1 \otimes G_2).$$

Our main result (Theorem 4.4) is that if the H'-space X is the suspension ΣY of Y, then the cohomology flat product for X is, up to natural isomorphism, the cup product in the cohomology of Y. This proves a conjecture which appeared in [1, p. 22].

A few applications of Theorem 4.4 are given in §5. We easily obtain another proof of the inequality $\smile$-long $Y \leqq$ conil ΣY of [4]. We also improve on a result from [3] by showing that $N \cdot \psi_n = 0$ implies that N times any n-fold cup product is zero. Our final application concerns a modified distributive law in which the deviation from distributivity is given in terms of a dual Hopf invariant and a flat product.

In the Eckmann-Hilton theory H'-spaces or spaces of normalized category $\leqq 1$ are dual to H-spaces and the functor $\flat$ is dual to the smashed product $\#$. Thus the flat product is the dual of the generalized Samelson product for an

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H-space (see [2]) and the cohomology flat product is the dual of the Samelson product in the homotopy groups of an H-space. It is well known that Whitehead products in the homotopy groups of a space are essentially Samelson products in the homotopy groups of the loops on the space. This fact and Theorem 4.4 exhibit the duality that exists between Whitehead products and cup products.[1] It should be noted that another approach to this duality, entirely different from ours, has been given by Hilton [7, §16].

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2. Preliminaries

By a space we shall always mean a path-connected space with a base point which has the based homotopy type of a CW-complex. We further assume that all maps and homotopies keep base points fixed. The same symbol is used for a map and its homotopy class. We let $\pi(X, Y)$ denote the collection of homotopy classes of maps from the space X into the space Y. Maps $f : X' \to X$ and $g : Y \to Y'$ induce $f^* : \pi(X, Y) \to \pi(X', Y)$ and $g_* : \pi(X, Y) \to \pi(X, Y')$ in the obvious way. We consistently adopt the following notation: CX for the reduced cone over X, ΣX for the reduced suspension of X and ΩY for the loop space of Y. The sets $\pi(\Sigma X, Y)$ and $\pi(X, \Omega Y)$ each have a group structure and there is a natural isomorphism, called the *adjoint isomorphism*,

$$\tau : \pi(\Sigma X, Y) \xrightarrow{\approx} \pi(X, \Omega Y).$$

The wedge $X \vee Y$ is considered a subset of the cartesian product $X \times Y$. The fibre of the inclusion map $X \vee Y \to X \times Y$ is denoted $X \flat Y$. Clearly $X \flat Y$ can be regarded as the space of paths in $X \times Y$ that begin at the base point and end in $X \vee Y$. The map $i : X \flat Y \to X \vee Y$ which projects a path onto its end point is essentially the inclusion of the fibre into the total space. Furthermore, we denote by $X \# Y$ the identification space $X \times Y/X \vee Y$. Thus $X \# Y$ is the cofibre of the (self-dual) map $X \vee Y \to X \times Y$ and $X \flat Y$ is the fibre.

We call X an *H'-space* if there is a $\phi : X \to X \vee X$ such that $j\phi = \Delta$ in $\pi(X, X \times X)$, where $j : X \vee X \to X \times X$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. H'-spaces are exactly the spaces of normalized Lusternik-Schnirelmann category $\leqq 1$ [7, p. 245]. If X is an H'-space, then there is a binary operation or multiplication in $\pi(X, Y)$ for which the constant map e in $\pi(X, Y)$ is a unit (i.e., $\alpha \cdot e = \alpha = e \cdot \alpha$ for all $\alpha \in \pi(X, Y)$). We

[1] Although the smashed product of two spheres is a sphere, the flat product of two Eilenberg-MacLane spaces is not an Eilenberg-MacLane space. Thus the main (technical) difficulty in the definition of the cohomology flat product was to choose a suitable map θ so that Theorem 4.4 holds. Remark 4.10 asserts that there is essentially one way of doing this.

call an H'-space X *inversive* if there exist $\lambda, \rho : X \to X$ such that $\lambda \cdot \iota = e$ and $\iota \cdot \rho = e$ in $\pi(X, X)$, where ι and e respectively denote the identity map and the constant map in $\pi(X, X)$. If X is inversive, then any α in $\pi(X, Y)$ has a left inverse $L(\alpha) = \alpha \circ \lambda$ and a right inverse $R(\alpha) = \alpha \circ \rho$. We call an H'-space X *associative* if $(\iota \vee \phi)\phi = (\phi \vee \iota)\phi$ in $\pi(X, X \vee X \vee X)$. If X is an associative, inversive H'-space, then $\lambda = \rho$ and $\pi(X, Y)$ is a group. An example of such a space is a reduced suspension. If X and X' are H'-spaces, then $f : X' \to X$ is called a *homomorphism* if $\phi f = (f \vee f)\phi'$ in $\pi(X', X \vee X)$. Such a map induces a homomorphism $f^* : \pi(X, Y) \to \pi(X', Y)$ of multiplicative structures. If X and X' are inversive, we require a homomorphism $f : X' \to X$ to satisfy the additional conditions $f\lambda' = \lambda f$ and $f\rho' = \rho f$. Then the induced map $f^* : \pi(X, Y) \to \pi(X', Y)$ preserves inverses.

In the sequel we only consider H'-spaces which are inversive. The following proposition shows that this is not a serious restriction.

PROPOSITION 2.1. *Every 1-connected H'-space is inversive. In addition, every homomorphism of 1-connected H'-spaces is a homomorphism of inversive H'-spaces.*

Since we shall not need this result, we omit its proof. (See [7, p. 230] for a proof of the dual proposition.)

3. The flat product

Throughout this section A_1 and A_2 are arbitrary spaces and X is an inversive H'-space. We adopt the following notation:

$l_i : A_i \to A_1 \vee A_2$	for the injections,	
$p_i : A_1 \vee A_2 \to A_i$	for the projections,	
$j : A_1 \vee A_2 \to A_1 \times A_2$	for the inclusion,	$i = 1, 2.$

We now define the flat product.

Elements $\alpha_1 \in \pi(X, A_1)$ and $\alpha_2 \in \pi(X, A_2)$ determine $\alpha_1' = l_{1*}(\alpha_1)$ and $\alpha_2' = l_{2*}(\alpha_2)$ in $\pi(X, A_1 \vee A_2)$. We form the commutator

$$(\alpha_1', \alpha_2') = L(\alpha_2' \cdot \alpha_1') \cdot (\alpha_1' \cdot \alpha_2'),$$

where L denotes the left inverse in $\pi(X, A_1 \vee A_2)$. Now

$$p_{1*}(\alpha_1', \alpha_2') = L(p_{1*}\, \alpha_2' \cdot p_{1*}\, \alpha_1') \cdot (p_{1*}\, \alpha_1' \cdot p_{1*}\, \alpha_2') = L(e \cdot \alpha_1) \cdot (\alpha_1 \cdot e) = e,$$

and similarly $p_{2*}(\alpha_1', \alpha_2') = e$. Therefore $j_*(\alpha_1', \alpha_2') = e$. Since

$$A_1 \flat A_2 \xrightarrow{i} A_1 \vee A_2 \xrightarrow{j} A_1 \times A_2$$

is essentially a fibre sequence, there is an exact sequence

$$\pi(X, A_1 \flat A_2) \xrightarrow{i_*} \pi(X, A_1 \vee A_2) \xrightarrow{j_*} \pi(X, A_1 \times A_2). \tag{3.1}$$

We then conclude that (α_1', α_2') is in the image of i_*. We prove a lemma which guarantees that the pre-image of (α_1', α_2') is unique.

LEMMA 3.2. *If X is an H'-space then $i_* : \pi(X, A_1 \flat A_2) \to \pi(X, A_1 \vee A_2)$ is one-one for any A_1 and A_2.*

Proof. Suppose X is a suspension, $X = \Sigma Y$. Then the commutative diagram

$$\begin{array}{ccc} \pi(\Sigma Y, A_1 \flat A_2) & \xrightarrow{i_*} & \pi(\Sigma Y, A_1 \vee A_2) \\ \approx \big\downarrow \tau & & \approx \big\downarrow \tau \\ \pi(Y, \Omega(A_1 \flat A_2)) & \xrightarrow{\Omega i_*} & \pi(Y, \Omega(A_1 \vee A_2)) \end{array}$$

and the existence of a left homotopy inverse for Ωi [7, p. 112] show that i_* is one-one. However it is known that every H'-space is dominated by a suspension (see [5, pp. 624–629]). This fact now establishes Lemma 3.2.

We return to the definition of the flat product. By exactness of the sequence (3.1) there is an element in $\pi(X, A_1 \flat A_2)$, written $\{\alpha_1, \alpha_2\}$, such that $i_*\{\alpha_1, \alpha_2\}$ is the commutator (α_1', α_2'). By Lemma 3.2, $\{\alpha_1, \alpha_2\}$ is determined uniquely.

DEFINITION 3.3. The *flat product* of $\alpha_1 \in \pi(X, A_1)$ and $\alpha_2 \in \pi(X, A_2)$ is the unique element $\{\alpha_1, \alpha_2\} \in \pi(X, A_1 \flat A_2)$ which is defined by the equation

$$i_*\{\alpha_1, \alpha_2\} = (\alpha_1', \alpha_2'),$$

where $\alpha_i' = l_{i*}(\alpha_i)$. Here it is assumed that X is an inversive H'-space.

Remark 3.4. If we set $X = \Sigma Y$, we obtain by means of the adjoint isomorphism τ a product $[\beta_1, \beta_2] \in \pi(Y, \Omega(A_1 \flat A_2))$ of $\beta_1 \in \pi(Y, \Omega A_1)$ and $\beta_2 \in \pi(Y, \Omega A_2)$, for any spaces A_1, A_2 and Y. This product was briefly considered in [1] under the name of the *dual product.*

The following proposition is easily verified.

PROPOSITION 3.5. (a) *If $f : X' \to X$ is a homomorphism of inversive H'-spaces then $f^*\{\alpha_1, \alpha_2\} = \{f^*\alpha_1, f^*\alpha_2\}$ for all $\alpha_i \in \pi(X, A_i)$, $i = 1, 2$.*

(b) *If $g_i : A_i \to A_i'$ are any maps then $(g_1 \flat g_2)_*\{\alpha_1, \alpha_2\} = \{g_{1*}\alpha_1, g_{2*}\alpha_2\}$ for all $\alpha_i \in \pi(X, A_i)$.*

4. The cohomology flat product

In this section we show how the flat product gives rise to a binary cohomology operation for H'-spaces. Our main result (Theorem 4.4) then asserts the equivalence, for a suspension, of this product with the cup product. We begin by borrowing some notation and a lemma from [6].

Let A_1 and A_2 be any spaces, $q : \Omega A_1 \times \Omega A_2 \to \Omega A_1 \# \Omega A_2$ the projection and C the reduced cone functor. Then there is a homotopy equivalence

$$k : C\Omega A_1 \times \Omega A_2 \cup \Omega A_1 \times C\Omega A_2 \to \Sigma(\Omega A_1 \# \Omega A_2)$$

defined as the composition of the natural map

$$s : C\Omega A_1 \times \Omega A_2 \cup \Omega A_1 \times C\Omega A_2 \to \Sigma(\Omega A_1 \times \Omega A_2)$$

with

$$\Sigma q : \Sigma(\Omega A_1 \times \Omega A_2) \to \Sigma(\Omega A_1 \# \Omega A_2).$$

We also observe that it is not difficult to define a map

$$h : C\Omega A_1 \times \Omega A_2 \cup \Omega A_1 \times C\Omega A_2 \to A_1 \flat A_2 \qquad [6, \text{p. } 135].$$

If A_1 and A_2 are 1-connected, then h is a homotopy equivalence [6, p. 135]. Next we define $\bar{c} : \Sigma(\Omega A_1 \times \Omega A_2) \to A_1 \vee A_2$. Consider the two elements of $\pi(\Sigma(\Omega A_1 \times \Omega A_2), A_1 \vee A_2)$ which are the compositions of the following evident maps ($i = 1, 2$)

$$\Sigma(\Omega A_1 \times \Omega A_2) = \Sigma\Omega(A_1 \times A_2) \to A_1 \times A_2 \to A_i \to A_1 \vee A_2 .$$

By definition, the group commutator of these two elements is $\bar{c}$. The relationship between the maps h, k and $\bar{c}$ is exhibited in the following lemma of Ganea, Hilton and Peterson.

LEMMA 4.1. [6, p. 134] *If A_1 and A_2 are any 1-connected spaces and*

$$a : A_1 \flat A_2 \to \Sigma(\Omega A_1 \times \Omega A_2) \quad \textit{and} \quad b : \Sigma(\Omega A_1 \times \Omega A_2) \to A_1 \flat A_2$$

are defined by $a = sh^{-1}$ and $b = hk^{-1}\Sigma q$, where h^{-1} and k^{-1} are the homotopy inverses of h and k, then $ba = \iota$ and $ib = \bar{c}$.

In dealing with cohomology groups and Eilenberg-MacLane spaces $K(G, m)$ it is convenient to identify $H^m(A; G)$ with $\pi(A, K(G, m))$ and $K(G, m)$ with $\Omega K(G, m + 1)$, for any space A. The natural isomorphism

$$\tau : H^{m+1}(\Sigma A; G) \to H^m(A; G)$$

of cohomology groups is then just the adjoint isomorphism

$$\tau : \pi(\Sigma A, K(G, m + 1)) \to \pi(A, \Omega K(G, m + 1)).$$

Now let $A_i = K(G_i, m_i + 1)$, where m_i is an integer > 0 ($i = 1, 2$). For an inversive H'-space X, the flat product assigns to $\alpha_i \in H^{m_i+1}(X; G_i)$ an element $\{\alpha_1, \alpha_2\} \in \pi(X, A_1 \flat A_2)$. Since $A_1 \flat A_2$ is not an Eilenberg-MacLane space even though A_1 and A_2 are, we do not yet have a cohomology product. However we shall define $\theta : A_1 \flat A_2 \to K(G_1 \otimes G_2, m_1 + m_2 + 1)$ which will determine our cohomology product. Consider the isomorphisms

$$H^{m+1}(\Sigma(\Omega A_1 \# \Omega A_2); G) \xrightarrow[\approx]{\tau} H^m(\Omega A_1 \# \Omega A_2; G) \xrightarrow[\approx]{\eta} \mathrm{Hom}(H_m(\Omega A_1 \# A_2), G)$$

$$= \mathrm{Hom}(G, G),$$

where $m = m_1 + m_2$, $G = G_1 \otimes G_2$ and η is the homomorphism of the universal coefficient theorem for cohomology. There exists a map

$$l : \Sigma(\Omega A_1 \# \Omega A_2) \to K(G, m+1)$$

such that $\eta\tau(l) = \mathrm{id}$, the identity automorphism in $\mathrm{Hom}(G, G)$. Now with $A_i = K(G_i, m_i + 1)$, $G = G_1 \otimes G_2$ and $m = m_1 + m_2$ we define

$$\theta : A_1 \flat A_2 \to K(G, m+1)$$

to be the following composition:

$$(4.2)\quad A_1 \flat A_2 \xrightarrow{h^{-1}} C\Omega A_1 \times \Omega A_2 \cup \Omega A_1 \times C\Omega A_2 \xrightarrow{k} \Sigma(\Omega A_1 \# \Omega A_2) \xrightarrow{l} K(G, m+1).$$

Definition 4.3. The *cohomology flat product* of $\alpha_1 \in H^{m_1+1}(X; G_1)$ and $\alpha_2 \in H^{m_2+1}(X; G_2)$ is the element

$$\langle \alpha_1, \alpha_2 \rangle = \theta \circ \{\alpha_1, \alpha_2\} \quad \text{in} \quad H^{m_1+m_2+1}(X; G_1 \otimes G_2),$$

where X is an inversive H'-space and $m_1, m_2 > 0$. The extension to n-fold products is immediate. If $\alpha_i \in H^{m_i+1}(X; G_i)$, $i = 1, \cdots, n$, then $\langle \alpha_1, \cdots, \alpha_n \rangle$ in $H^{m_1+\cdots+m_n+1}(X; G_1 \otimes \cdots \otimes G_n)$ is inductively defined by

$$\langle \alpha_1, \cdots, \alpha_n \rangle = \langle \langle \alpha_1, \cdots, \alpha_{n-1} \rangle, \alpha_n \rangle.$$

Theorem 4.4. *If $\alpha_i \in H^{m_i+1}(\Sigma Y; G_i)$, $i = 1, \cdots, n$, then*

$$\tau\langle \alpha_1, \cdots, \alpha_n \rangle = \tau\alpha_1 \smile \cdots \smile \tau\alpha_n,$$

where $\smile$ denotes cup product and $\tau : H^{r+1}(\Sigma Y;\ ) \to H^r(Y;)$ is the adjoint isomorphism.

Remark 4.5. Theorem 4.4 may clearly be rephrased as follows: For any space Y and any $\beta_i \in H^{m_i}(Y; G_i)$ where $i = 1, \cdots, n$,

$$\beta_1 \smile \cdots \smile \beta_n = \tau\langle \tau^{-1}(\beta_1), \cdots, \tau^{-1}(\beta_n) \rangle.$$

Proof. Let us first prove Theorem 4.4 in the case $n = 2$, $Y = \Omega A_1 \times \Omega A_2$ and $\tau\alpha_i$ is the projection q_i of $\Omega A_1 \times \Omega A_2$ onto ΩA_i. The general case will follow. The proof may be more easily visualized by considering the diagram

$$\begin{tikzcd}
C\Omega A_1 \times \Omega A_2 \cup \Omega A_1 \times C\Omega A_2 \arrow[rr, "h"] & & A_1 \flat A_2 \arrow[r, "\theta"] & K(G, m+1) \\
& A_1 \vee A_2 \arrow[ur, "i"] & \Sigma(\Omega A_1 \times \Omega A_2) \arrow[l, "\bar{c}"'] \arrow[u, "b"'] \arrow[d, "\Sigma q"] & \\
& & \Sigma(\Omega A_1 \# \Omega A_2) \arrow[uull, "k^{-1}"] \arrow[uur, "l"'] &
\end{tikzcd}$$

in which all of the four triangles are commutative.

We must show

$$\tau(\theta\{\tau' q_1, \tau' q_2\}) = \smile, \tag{4.6}$$

where $\smile$ stands for $q_1 \smile q_2$ in $H^m(\Omega A_1 \times \Omega A_2 ; G)$ and τ' denotes the inverse of τ. In the short exact sequence

$$0 \rightarrow H^m(\Omega A_1 \# \Omega A_2 ; G) \xrightarrow{q^*} H^m(\Omega A_1 \times \Omega A_2 ; G) \xrightarrow{j^*} H^m(\Omega A_1 \vee \Omega A_2 ; G) \rightarrow 0$$

we have $j^*(\smile) = 0$. Therefore there is a unique element

$$\smile' \in H^m(\Omega A_1 \# \Omega A_2 ; G)$$

such that $q^*(\smile') = \smile$. Now $\Omega A_i = K(G_i, m_i)$, $G = G_1 \otimes G_2$, $m = m_1 + m_2$ and the element $\smile$ is the universal cup product element. Therefore $\eta(\smile') = \text{id}$ in the diagram

$$H^m(\Omega A_1 \# \Omega A_2 ; G) \underset{\approx}{\xrightarrow{\eta}} \text{Hom}(H_m(\Omega A_1 \# \Omega A_2), G) = \text{Hom}(G, G).$$

Since the map $l : \Sigma(\Omega A_1 \# \Omega A_2) \rightarrow K(G, m + 1)$ is defined by the equation $\eta\tau(l) = \text{id}$ it follows that $\tau(l) = \smile'$. Hence $q^*\tau(l) = q^*(\smile') = \smile$ and so

$$\tau(l\Sigma q) = \smile.$$

Therefore, in order to establish (4.6), it suffices to prove

$$l\Sigma q = \theta\{\tau' q_1, \tau' q_2\}, \tag{4.7}$$

where $\tau' = \tau^{-1}$. We look more closely at the element $\{\tau' q_1, \tau' q_2\}$. By definition $i_*\{\tau' q_1, \tau' q_2\}$ is the commutator $(l_1 \tau' q_1, l_2 \tau' q_2)$, where

$$l_i : A_i \rightarrow A_1 \vee A_2$$

and

$$i_* : \pi(\Sigma(\Omega A_1 \times \Omega A_2), A_1 \flat A_2) \rightarrow \pi(\Sigma(\Omega A_1 \times \Omega A_2), A_1 \vee A_2).$$

But $(l_1 \tau' q_1, l_2 \tau' q_2) = (l_1 e_1 \Sigma q_1, l_2 e_2 \Sigma q_2)$, where $e_i : \Sigma\Omega A_i \rightarrow A_i$. However, from the definition of $\bar{c} : \Sigma(\Omega A_1 \times \Omega A_2) \rightarrow A_1 \vee A_2$ preceding Lemma 4.1, $\bar{c} = (l_1 e_1 \Sigma q_1, l_2 e_2 \Sigma q_2)$. Therefore by Lemma 4.1,

$$i_*\{\tau' q_1, \tau' q_2\} = \bar{c} = i_*(b).$$

Since i_* is one-one (Lemma 3.2),

$$\{\tau' q_1, \tau' q_2\} = b. \tag{4.8}$$

By (4.7) and (4.8) we see that it suffices to prove

$$l\Sigma q = \theta b. \tag{4.9}$$

By definition $\theta = lkh^{-1}$ (4.2) and $b = hk^{-1}\Sigma q$ (Lemma 4.1), and so (4.9) is verified. This proves (4.6) and demonstrates the theorem in the case $n = 2$, $Y = \Omega A_1 \times \Omega A_2$ and $\tau\alpha_i = q_i$.

Theorem 4.4 for $n = 2$ is now easily established. If Y is any space and $\alpha_i \in H^{m_i+1}(\Sigma Y; G_i)$ are any elements, $i = 1, 2$, then the $\tau\alpha_i : Y \to \Omega A_i$ determine a map $\alpha : Y \to \Omega A_1 \times \Omega A_2$ such that $q_i \alpha = \tau\alpha_i$. Then

$$\begin{aligned} \tau\alpha_1 \smile \tau\alpha_2 &= \alpha^*(q_1 \smile q_2) \\ &= \alpha^*\tau\langle\tau' q_1, \tau' q_2\rangle \\ &= \tau(\Sigma\alpha^*\langle\tau' q_1, \tau' q_2\rangle) \\ &= \tau\langle\Sigma\alpha^*\tau' q_1, \Sigma\alpha^*\tau' q_2\rangle \quad \text{(by Proposition 3.5)} \\ &= \tau\langle\alpha_1, \alpha_2\rangle. \end{aligned}$$

This proves the theorem for $n = 2$. A simple inductive argument now yields Theorem 4.4 for arbitrary n.

Remark 4.10. The cohomology flat product is defined by choosing a certain θ in $\pi(A_1 \flat A_2, K(G, m + 1))$. We observe here that θ is uniquely determined by Theorem 4.4. That is, if θ' is any element in

$$\pi(A_1 \flat A_2, K(G, m + 1))$$

such that

$$\tau(\theta' \circ \{\alpha_1, \alpha_2\}) = \tau\alpha_1 \smile \tau\alpha_2$$

for all $\alpha_i \in H^{m_i+1}(\Sigma Y; G_i)$, then $\theta = \theta'$. For

$$\tau(\theta'\{\tau' q_1, \tau' q_2\}) = \smile = \tau(\theta\{\tau' q_1, \tau' q_2\}),$$

and so by (4.8), $\tau(\theta' b) = \tau(\theta b)$. Therefore we have $\theta' b = \theta b$. But, by Lemma 4.1, b admits a right homotopy inverse, and so $\theta' = \theta$.

5. Applications

In this section we present a few applications of the preceding material. Our first application concerns the relationship between n-fold cup products and the co-commutator map of weight n. We begin by extending previous definitions and results from $n = 2$ to arbitrary n.

For any n spaces $A_1, \cdots, A_n$ we inductively define a map

$$i_n : A_1 \flat \cdots \flat A_n \to A_1 \vee \cdots \vee A_n,$$

where

$$A_1 \flat \cdots \flat A_n = (A_1 \flat \cdots \flat A_{n-1}) \flat A_n,$$

$$A_1 \vee \cdots \vee A_n = (A_1 \vee \cdots \vee A_{n-1}) \vee A_n.$$

For $n = 2$, let $i_2 = i : A_1 \flat A_2 \to A_1 \vee A_2$. For $n > 2$, i_n is the composition

$$(A_1 \flat \cdots \flat A_{n-1}) \flat A_n \xrightarrow{i}$$

$$(A_1 \flat \cdots \flat A_{n-1}) \vee A_n \xrightarrow{i_{n-1} \vee \iota} (A_1 \vee \cdots \vee A_{n-1}) \vee A_n.$$

The following proposition extends Lemma 3.2.

PROPOSITION 5.1. *If $A_1, \cdots, A_n$ are 1-connected spaces and X is an H'-space, then $i_{n*} : \pi(X, A_1 \flat \cdots \flat A_n) \to \pi(X, A_1 \vee \cdots \vee A_n)$ is one-one.*

Proof. As in Lemma 3.2 it suffices to prove this result for $X = \Sigma Y$, since every H'-space is dominated by a suspension [5]. Thus it is sufficient to show that

$$\Omega i_{n*} : \pi(Y, \Omega(A_1 \flat \cdots \flat A_n)) \to \pi(Y, \Omega(A_1 \vee \cdots \vee A_n))$$

is one-one. We do this by proving by induction on n that the map

$$\Omega i_n : \Omega(A_1 \flat \cdots \flat A_n) \to \Omega(A_1 \vee \cdots \vee A_n)$$

has a left homotopy inverse.

For the case $n = 2$, see the proof of Lemma 3.2. Now assume that Ωi_{n-1} has a left homotopy inverse. Since Ωi_n is the composition

$$\Omega((A_1 \flat \cdots \flat A_{n-1}) \flat A_n) \xrightarrow{\Omega i}$$
$$\Omega((A_1 \flat \cdots \flat A_{n-1}) \vee A_n) \xrightarrow{\Omega(i_{n-1} \vee \iota)} \Omega((A_1 \vee \cdots \vee A_{n-1}) \vee A_n)$$

and since Ωi has a left homotopy inverse, it only remains to prove that $\Omega(i_{n-1} \vee \iota)$ has a left homotopy inverse.

It is well known that for any spaces B_1 and B_2 there is a natural homotopy equivalence

$$\Omega(B_1 \vee B_2) \simeq \Omega B_1 \times \Omega B_2 \times \Omega(B_1 \flat B_2) \qquad [7, \text{p. } 112].$$

If B_1 and B_2 are 1-connected, we have seen in §4 that

$$kh^{-1} : B_1 \flat B_2 \to \Sigma(\Omega B_1 \# \Omega B_2)$$

is a homotopy equivalence. Thus there is a natural homotopy equivalence

$$\Omega(B_1 \vee B_2) \simeq \Omega B_1 \times \Omega B_2 \times \Omega\Sigma(\Omega B_1 \# \Omega B_2).$$

We use this last fact to show that $\Omega(i_{n-1} \vee \iota)$ has a left homotopy inverse. Let $B_1 = A_1 \flat \cdots \flat A_{n-1}$, $B_1' = A_1 \vee \cdots \vee A_{n-1}$ and $B_2 = A_n$ and consider the diagram

$$\begin{array}{ccc} \Omega(B_1 \vee B_2) & \simeq & \Omega B_1 \times \Omega B_2 \times \Omega\Sigma(\Omega B_1 \# \Omega B_2) \\ \Omega(i_{n-1} \vee \iota)\Big\downarrow & & \Big\downarrow \Omega i_{n-1} \times \iota \times \Omega\Sigma(\Omega i_{n-1} \# \iota) \\ \Omega(B_1' \vee B_2) & \simeq & \Omega B_1' \times \Omega B_2 \times \Omega\Sigma(\Omega B_1' \# \Omega B_2). \end{array}$$

Since by the inductive assumption Ωi_{n-1} has a left homotopy inverse, so does $\Omega(i_{n-1} \vee \iota)$. This completes the proof of Proposition 5.1.

Next let X be an associative, inversive H'-space and let us write ${}^nX$ for $X \vee \cdots \vee X$ (n summands). Then a co-commutator map $\psi_n = \psi_n(X)$ of weight n in $\pi(X, {}^nX)$ [4, p. 103] is inductively defined as follows: ψ_2 is the ordinary group commutator of the two inclusions $X \to X \vee X$. ψ_n is the commutator of ψ_{n-1} followed by the inclusion ${}^{n-1}X \to {}^nX$ and of the inclusion $X \to {}^{n-1}X \vee X = {}^nX$.

PROPOSITION 5.2. *If ι is the identity map of an associative, inversive H'-space X and $i_{n*} : \pi(X, X \flat \cdots \flat X) \to \pi(X, X \vee \cdots \vee X)$, then*

$$i_{n*}\{\iota, \cdots, \iota\} = \psi_n(X),$$

where $\{\iota, \cdots, \iota\}$ is the flat product of ι with itself n times and $\psi_n(X)$ is the co-commutator map of weight n.

Proof. It follows immediately from the definitions that $i_*\{\iota, \iota\} = \psi_2(X)$ A simple induction then establishes Proposition 5.2.

PROPOSITION 5.3. *Let X be a 1-connected, associative, inversive H'-space and let $\alpha_i \in H^{m_i+1}(X; G_i)$ be any elements, $m_i > 0$ $(i = 1, \cdots, n)$.*

(a) *If $\psi_n(X) = 0$, then $\langle \alpha_1, \cdots, \alpha_n \rangle = 0$.*
(b) *If $N \cdot \psi_n(X) = 0$, then $N \cdot \langle \alpha_1, \cdots, \alpha_n \rangle = 0$.*[2]

Proof. A straightforward inductive construction based on definition (4.2) shows that there is a map

$$\theta_n : A_1 \flat \cdots \flat A_n \to K(G_1 \otimes \cdots \otimes G_n, m_1 + \cdots + m_n + 1),$$

where $A_i = K(G_i, m_i + 1)$, such that

$$\theta_n \circ \{\alpha_1, \cdots, \alpha_n\} = \langle \alpha_1, \cdots, \alpha_n \rangle$$

for all $\alpha_i \in H^{m_i+1}(X; G_i)$. Thus by Proposition 3.5

$$\theta_{n*}(\alpha_1 \flat \cdots \flat \alpha_n)_*\{\iota, \cdots, \iota\} = \langle \alpha_1, \cdots, \alpha_n \rangle.$$

This equation, the equation stated in Proposition 5.2, and the fact that $i_{n*} : \pi(X, X \flat \cdots \flat X) \to \pi(X, X \vee \cdots \vee X)$ is one-one (Proposition 5.1) establish the proposition.

Theorem 4.4 and Proposition 5.3 now yield

COROLLARY 5.4. *Let Y be any (path-connected) space and let $\beta_i \in H^{m_i}(Y; G_i)$ be any elements, $m_i > 0$ $(i = 1, \cdots, n)$.*

(a) *If $\psi_n(\Sigma Y) = 0$, then the n-fold cup product $\beta_1 \smile \cdots \smile \beta_n = 0$.*
(b) *If $N \cdot \psi_n(\Sigma Y) = 0$, then $N \cdot (\beta_1 \smile \cdots \smile \beta_n) = 0$.*

Corollary 5.4 (a) generalizes the result of Berstein and Ganea [4, Theorem 5.8], $\smile$-long $Y \leqq$ conil ΣY. We note that another proof of 5.4 (a) appears in [6, Theorem 4.4]. Corollary 5.4 (b) generalizes a result of Arkowitz and Curjel [3].

Our final application deals with a modified distributive law. Let B and C

[2] We are here writing group operations additively.

be any spaces and $\alpha \in \pi(\Omega B, \Omega C)$ any element. We write all group operations additively. Then there is an element $\gamma = -\alpha q_2 - \alpha q_1 + \alpha(q_1 + q_2)$ in $\pi(\Omega B \times \Omega B, \Omega C)$, where $q_1, q_2 : \Omega B \times \Omega B \rightarrow \Omega B$ are the projections. If

$$j^* : \pi(\Omega B \times \Omega B, \Omega C) \rightarrow \pi(\Omega B \vee \Omega B, \Omega C),$$

then clearly $j^*(\gamma) = 0$. Consequently there is a unique element $H(\alpha)$ in $\pi(\Omega B \# \Omega B, \Omega C)$ such that $q^* H(\alpha) = \gamma$, where $q : \Omega B \times \Omega B \rightarrow \Omega B \# \Omega B$ s the identi fication map. Thus we have a function

$$H : \pi(\Omega B, \Omega C) \rightarrow \pi(\Omega B \# \Omega B, \Omega C).$$

Now we define the *dual Hopf invariant* $\mathcal{H} : \pi(\Omega B, \Omega C) \rightarrow \pi(\Omega(B \flat B), \Omega C)$ to be the following composition:

$$\pi(\Omega B, \Omega C) \xrightarrow{H} \pi(\Omega B \# \Omega B, \Omega C) \xrightarrow{\tau^{-1}} \pi(\Sigma(\Omega B \# \Omega B), C) \xrightarrow{(kh^{-1})^*}$$

$$\pi(B \flat B, C) \xrightarrow{\Omega} \pi(\Omega(B \flat B), \Omega C).$$

PROPOSITION 5.5 (Modified Distributivity). *For any spaces A, B, C and any elements β_1, $\beta_2 \in \pi(A, \Omega B)$ and $\alpha \in \pi(\Omega B, \Omega C)$,*

$$\alpha \circ (\beta_1 + \beta_2) = \alpha \circ \beta_1 + \alpha \circ \beta_2 + \mathcal{H}(\alpha) \circ \tau\{\tau^{-1}(\beta_1), \tau^{-1}(\beta_2)\}.$$

The proof of this proposition requires a straightforward but rather long computation, and hence is omitted. However we do make a few remarks. First we note that the dual of Proposition 5.5 is well known. The duality between $\mathcal{H}$ and the Hopf invariant is best seen by taking B to be an Eilenberg-MacLane space. Secondly, the element $\tau\{\tau^{-1}(\beta_1), \tau^{-1}(\beta_2)\}$ is just $[\beta_1, \beta_2]$, the dual product of β_1 and β_2 as defined in [1] (see also Remark 3.4). Finally, we observe that actual distributivity holds in Proposition 5.5 whenever $\mathcal{H}(\alpha) \circ \tau\{\tau^{-1}(\beta_1), \tau^{-1}(\beta_2)\} = 0$. Since $\{\tau^{-1}(\beta_1), \tau^{-1}(\beta_2)\}$ is defined by means of a commutator, this occurs, for instance, whenever the group $\pi(\Sigma A, B \vee B)$ is abelian.

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