ALMOST-GAUSSIAN DOMAINS

BY
GERALD L. KERR[1]

1. Let $\mathfrak{o}$ be a Krull domain with quotient field K. Let I be the collection of all rank 1 prime ideals of $\mathfrak{o}$ and for each $p \in I$, let v_p be the corresponding p-adic valuation on K. Finally, let $\mathcal{S}$ be the set of all group homomorphisms g from the multiplicative group K^* of non-zero elements of K into the additive group of real numbers and such that g is non-negative on $K^* \cap \mathfrak{o}$. The first part of this paper is devoted to proving the following three theorems which are basic to the statement of our main result:

THEOREM (A). *Given an element g of $\mathcal{S}$, there exists a real-valued function $\bar{G}$ defined on I such that*

(i) $\bar{G}(p) \geq 0$ *for each* $p \in I$.
(ii) $g(x) = \sum_{p \in I} \bar{G}(p)v_p(x)$ *for each* $x \in K^*$.

(Note that the choice of $\bar{G}$ depends on the choice of g.)

THEOREM (B). *Let $g \in \mathcal{S}$ and let $\bar{G}$ be a function satisfying conditions* (i) *and* (ii) *of Theorem* (A) *relative to g. For each element $p \in I$, let*

$$G'(p) = \inf \{g(x)/v_p(x); x \in p, x \neq 0\}.$$

Then $\bar{G}(p) \leq G'(p)$ for each $p \in I$. Moreover, given one element $q \in I$, the function $\bar{G}$ can be selected so that $\bar{G}(q) = G'(q)$.

THEOREM (C). *Let g be an element of $\mathcal{S}$. The following statements are equivalent:*

(i) $g(x) = \sum_{p \in I} G'(p)v_p(x)$ *for each* $x \in K^*$.
(ii) *There is but one function $\bar{G}$, corresponding to g, which satisfies conditions* (i) *and* (ii) *of Theorem* (A).
(iii) *Given $q \in I$ and $\varepsilon > 0$, there exists an element $x \in q$ such that*

$$\sum_{p \in I, p \neq q} G'(p)v_p(x) \leq \varepsilon v_q(x).$$

The above results were originally obtained by Samuel [5] for the class of all *integer*-valued valuations w on K whose corresponding valuation ring R_w dominates $\mathfrak{o}$, where $\mathfrak{o}$ was assumed to be a normal local domain. We have obtained these more general results by making use of a different ex-

Received March 12, 1964.
[1] Work on this paper was supported in part by an NSF Grant and in part by an NSF Cooperative Graduate Fellowship.

tension theorem for linear functionals than that employed by Samuel. Otherwise, our methods parallel those of Samuel.

In this paper, an element g of $\mathcal{S}$ is said to be *perfect* in case the three equivalent conditions of Theorem (C) are satisfied. Thus, if $\mathfrak{o}$ is a normal local domain and w is an integer-valued valuation whose corresponding valuation ring dominates $\mathfrak{o}$, then $\mathfrak{o}$ is almost-gaussian (presque-factoriel) relative to w, as defined by Samuel [5], if and only if w is perfect. The change in language is due to the fact that $\mathcal{S}$ always contains many perfect elements (for example, each v_p is perfect) so that, in general, the fact that a given element f of $\mathcal{S}$ is perfect may give no additional information about the domain $\mathfrak{o}$ itself. We now state our main result.

THEOREM (D). *Let f and g be elements of $\mathcal{S}$ and assume f is perfect. Let*

$$l(f, g) = \inf \{f(x)/g(x); x \in K^* \cap \mathfrak{o}, g(x) > 0\},$$

where g is assumed not to be the zero homomorphism. If $l(f, g) \neq 0$, then also g is perfect.

Let $\mathfrak{o}$ be a normal local domain and assume that for any two divisors v, w of second kind relative to $\mathfrak{o}$ the relation $l(v, w) \neq 0$ holds. In view of Theorem (D), it would be proper to call such a domain "almost-gaussian" (without reference to any particular divisor of second kind) in case there exists a divisor w of second kind relative $\mathfrak{o}$ which is perfect. A class of two-dimensional normal local domains which are, in fact, "almost-gaussian" in the above sense of the word, has been described in [2].

The number $l(f, g)$ (see Theorem (D)) is called the linking number of f over g on $K^* \cap \mathfrak{o}$. In Section 3 such linking numbers are studied briefly in a purely abstract setting. That is, let T be an arbitrary non-empty set and let f, g be non-negative functions defined on T into the set of real numbers with ∞ adjoined. A number $l_T(f, g)$ (possibly infinite) is defined such that if f and g are as in Theorem (D), then

$$l_{K^* \cap \mathfrak{o}}(f, g) = \inf \{f(x)/g(x); x \in K^* \cap \mathfrak{o}, g(x) > 0\}$$

and if g is trivial, then $l_{K^* \cap \mathfrak{o}}(f, g) = \infty$. Throughout this paper the number $l_{K^* \cap \mathfrak{o}}(f, g)$ will be denoted simply by $l(f, g)$. More importantly, suppose T is a noetherian ring, A and B proper ideals of T such that Rad A = Rad B and the intersection of all positive integral powers of A is the zero ideal. Let $l_A(B)$ be the number obtained by comparing high powers of A and B introduced earlier by Samuel [4]. Let $\bar{v}_A$ and $\bar{v}_B$ be the homogeneous pseudo-valuations defined by A and B, respectively [3]. We show here that $l_A(B) = l_T(\bar{v}_A, \bar{v}_B)$.

Finally, in Section 4 we show that if f and g are perfect elements of $\mathcal{S}$ such that $l(f, g) \neq 0$ or $l(g, f) \neq 0$, then $f + g$ is perfect. Moreover, a partial ordering is defined on $\mathcal{S}$ which is such that each non-trivial perfect element of $\mathcal{S}$

can be embedded in a natural way in a distributive lattice of non-trivial perfect elements.

We express our appreciation to Professor H. T. Muhly for his generous assistance and encouragement during the preparation of this paper.

2. Throughout this paper, R will denote the set of real numbers. The following lemma is a modification of a well-known extension theorem for linear functionals on a partially ordered vector space and is probably also known. However, since no specific reference could be located, we give a proof here.

LEMMA 2.1. *Let $\mathcal{E}$ be a vector space over R, C a convex cone in $\mathcal{E}$ whose vertex is the neutral element of $\mathcal{E}$ relative to vector addition. Let $\mathfrak{F}$ be a vector subspace of $\mathcal{E}$ and let G be a linear functional defined on $\mathfrak{F}$ which takes non-negative values on $C \cap \mathfrak{F}$. Assume further that for each $Y \in \mathcal{E}$ there exist elements $x, x' \in \mathfrak{F}$ such that $x' - y \in C$ and $y - x \in C$. Then there exists a linear functional $\bar{G}$ defined on $\mathcal{E}$ which takes non-negative values on C and which extends G. Moreover, if $x_0 \in \mathcal{E}$ and $x_0 \notin \mathfrak{F}$, let*

$$S(\mathfrak{F}, x_0) = \{x \in \mathfrak{F}; x_0 - x \in C\},$$

$$T(\mathfrak{F}, x_0) = \{x' \in \mathfrak{F}; x' - x_0 \in C\}.$$

Then the numbers

$$\alpha = \sup \{G(x); x \in S(\mathfrak{F}, x_0)\}$$

and

$$\beta = \inf \{G(x'); x' \in T(\mathfrak{F}, x_0)\}$$

are defined and $\alpha \leq \beta$. For any γ such that $\alpha \leq \gamma \leq \beta$, $\bar{G}$ can be chosen so that $\bar{G}(x_0) = \gamma$.

Proof. Let $\mathfrak{F}_1 = \mathfrak{F} + Rx_0$. If $x \in S(\mathfrak{F}, x_0)$ and $x' \in T(\mathfrak{F}, x_0)$, then

$$(x' - x_0) + (x_0 - x) = x' - x \in C.$$

Hence, $G(x) \leq G(x')$, so α and β are defined, $\alpha \leq \beta$. Let γ be any real number such that $\alpha \leq \gamma \leq \beta$. For each element $x + tx_0$ ($x \in \mathfrak{F}$, $t \in R$) of $\mathfrak{F}_1$, define $G_1(x + tx_0)$ to be $G(x) + t\gamma$. It is easy to verify that G_1 is a linear functional on $\mathfrak{F}_1$ which is non-negative on $C \cap \mathfrak{F}_1$ and which extends G. Let N be the collection of all ordered pairs of the form $(\mathfrak{M}, H)$ where $\mathfrak{M}$ is a vector subspace of $\mathcal{E}$ which contains $\mathfrak{F}_1$ and H is a linear functional on $\mathfrak{M}$ which is non-negative on $C \cap \mathfrak{M}$ and which extends G_1. A partial ordering, under which N is inductive, will be defined as follows: $(\mathfrak{M}, H) \prec (\mathfrak{M}', H')$ in case $\mathfrak{M}'$ contains $\mathfrak{M}$ and H' extends H. Let $(\mathfrak{M}_0, H_0)$ be a maximal element of N. If $\mathfrak{M}_0 \neq \mathcal{E}$, then there exists $y_0 \in \mathcal{E}$, $y_0 \notin \mathfrak{M}_0$ and a linear functional H_1 defined on $\mathfrak{M}_1 = \mathfrak{M}_0 + Ry_0$ such that $(\mathfrak{M}_1, H_1) \in N$ and

$$(\mathfrak{M}_0, H_0) \prec (\mathfrak{M}_1, H_1).$$

This contradicts the maximality of $(\mathfrak{M}_0, H_0)$, so $\mathfrak{M}_0 = \mathcal{E}$ and H_0 is the required linear functional, Q.E.D.

In order to prove Theorem (A) and Theorem (B) we proceed initially as did Samuel [5]. Let $\mathcal{E}$ be the vector space over R with base I. For each $x \in K^*$, let $(x) = \sum_{p \in I} v_{p(x)} \cdot p$ and let $\mathcal{H} = \{(x); x \in K^*\}$. Clearly, $\mathcal{H}$ is a subgroup of $\mathcal{E}$ under vector addition. For each $(x) \in \mathcal{H}$, let $G_0((x))$ be defined to be $g(x)$. Since g assumes the value zero at each unit of $\mathfrak{o}$, G_0 is well defined and is, moreover, a group homomorphism from $\mathcal{H}$ into the group of real numbers under addition. Let $\mathfrak{F}$ be the vector subspace of $\mathcal{E}$ generated by $\mathcal{H}$. The function G_0 can be extended uniquely by linearity to a linear functional G on $\mathfrak{F}$. Let $\mathfrak{F}^+ = \{X \in \mathfrak{F}; G(X) \geq 0\}$, let

$$P = \{(\textstyle\sum_{p \in I} \alpha_p \cdot p) \in \mathcal{E}; \alpha_p \geq 0, p \in I\}$$

and let $C = P + \mathfrak{F}^+$. Then C is a convex cone in $\mathcal{E}$ whose vertex is the neutral element of $\mathcal{E}$. Since $I \subseteq C$, Theorem (A) and Theorem (B), (ii) will be proved once it has been shown that G can be extended to a linear functional $\bar{G}$ defined on $\mathcal{E}$ which is nonnegative on C and that, given $q \in I$, $\bar{G}$ can be chosen so that $\bar{G}(q) = G'(q)$.

Suppose first that $\bar{G}$ is any function satisfying conditions (i) and (ii) of Theorem (A). It is easy to verify that condition (i) of Theorem (B) holds. Let $q \in I$ and assume that $q \in \mathfrak{F}$. Then by [5], Theorem 2, (a), there exists $y \in K^*$ such that $(y) = v_q(y) \cdot q$. Thus, $G'(q) \leq g(y)/v_q(y) = \bar{G}(q)$. But already (Theorem (B), (i)) $\bar{G}(q) \leq G'(q)$, so $\bar{G}(q) = G'(q)$. The problem of proving that $\bar{G}$ exists has thus been reduced to the problem of verifying the hypotheses of Lemma 2.1. It will first be shown that

$$C \cap \mathfrak{F} \subseteq \mathfrak{F}^+.$$

In order to do this, it is enough to show that $P \cap \mathfrak{F} \subseteq \mathfrak{F}^+$, so let $Y \in P \cap \mathfrak{F}$ be given. Let ε be an arbitrary positive real number. By using directly the techniques employed by Samuel [6] in the proof of §1, Lemma 1, (2), it can be shown that there exists $x \in \mathfrak{o}$ and a positive integer n such that $| G(Y) - n^{-1}g(x) | \leq \varepsilon n^{-1}$. Since $g(x) \geq 0$ and ε was chosen arbitrarily, it follows that $G(Y) \geq 0$. Now let $Y = \sum_{p \in I} \alpha_p \cdot p$ be an arbitrary element of $\mathcal{E}$. Elements $X, X' \in \mathfrak{F}$ must be constructed such that $X' - Y \in C$ and $Y - X \in C$. Let $J = \{p \in I; \alpha_p \neq 0\}$. If J is empty, there is nothing to prove, so assume this is not the case. For each $p \in J$, select $x_p \in p$ such that $v_p(x_p) \geq | \alpha_p |$ and let $x = \prod_{p \in J} x_p$. Clearly, $(x) - Y \in P \subseteq C$ and $Y - (x^{-1}) \in P \subseteq C$. We have shown that there exists a linear function $\bar{G}$ which is non-negative on C and which extends G (Lemma 2.1). Finally, suppose $q \in I$ and $q \notin \mathfrak{F}$. Let

$$S(\mathfrak{F}, q) = \{X \in \mathfrak{F}; q - X \in C\}$$

and let

$$T(\mathfrak{F}, q) = \{X' \in \mathfrak{F}; X' - q \in C\}.$$

If $X \in S(\mathfrak{F}, q)$, let $\bar{G}$ be any linear functional on $\mathcal{E}$, non-negative on C, which extends G. Then $G(X) = \bar{G}(X) \leqq \bar{G}(q) \leqq G'(q)$. Thus, $\alpha \leq G'(q)$, where $\alpha = \sup \{G(X);\ X \in S(\mathfrak{F}, q)\}$. On the other hand, let

$$X' = \sum_{i \in J} \alpha_i(x_i)$$

(J is a finite set) be an element of $T(\mathfrak{F}, q)$. Then $X' - q = Y_1 + Y_2$ where $Y_1 \in \mathfrak{F}^+$ and $Y_2 \in P$. Let $Y_1 = \sum_{j \in J'} \beta_j(Y_j)$ and let $Y_2 = \sum_{p \in I} \gamma_p \cdot p$ (J' is a finite set and $\gamma_p = 0$ for almost all $p \in I$). Thus, for $p \neq q$,

$$\sum_{i \in J} \alpha_i\, v_p(x_i) = \sum_{j \in J'} \beta_j\, v_p(y_j) + \gamma_p$$

and

$$\sum_{i \in J} \alpha_i\, v_q(x_i) = \sum_{j \in J'} \beta_j\, v_q(y_j) + \gamma_q + 1.$$

Let $\varepsilon > 0$ be given. A positive integer n and $x, y \in K^*$ will be constructed so that $x/y \in q$, $g(x/y)/v_q(x/y) \leq g(x/y)/n$ and, moreover, such that

$$|\,(g(x/y)/n) - (\sum_{i \in J} \alpha_i\, g(x_i) - \sum_{j \in J'} \beta_j\, g(y_j))\,| \leq 2\varepsilon.$$

Since by hypothesis $\sum_{j \in J'} \beta_j\, g(y_i) \geq 0$, this will show that $G(X') \geq G'(q)$ and, hence, that the number

$$\beta = \inf \{G(X');\, X' \in T(\mathfrak{F}, x_0)\}$$

is not less than $G'(q)$. Corresponding to the choice of ε, choose $\delta > 0$ such that whenever $|\,\alpha_i' - \alpha_i\,| \leq \delta$, $i \in J$, and $|\,\beta_j' - \beta_j\,| \leq \delta$, $j \in J'$, then

$$|\sum_{i \in J} \alpha_i'\, g(x_i) - \sum_{i \in J} \alpha_i\, g(x_i)| \leq \varepsilon$$

and

$$|\sum_{j \in J'} \beta_j'\, g(y_j) - \sum_{j \in J'} \beta_j\, g(y_j)| \leq \varepsilon.$$

There exist integers $n > 0$, a_i, b_j and $d_p \geq 0$ such that for all $i \in J$, $j \in J'$, and $p \in I$ the relations

$$|\,a_i\, n^{-1} - \alpha_i\,| \leq \delta n^{-1},$$

$$|\,b_j\, n^{-1} - \beta_j\,| \leq \delta n^{-1},$$

$$|\,d_p\, n^{-1} - \gamma_p\,| \leq 3^{-1} n^{-1}$$

hold (see [1, VII, §1, n° 1, Prop. 2]). Without loss of generality it can be assumed that δ satisfies the following additional conditions:

(a) $\delta \sum_{i \in J} |\,v_p(x_i)| < 3^{-1}$ for all $p \in I$.
(b) $\delta \sum_{j \in J'} |\,v_p\,(y_j) < 3^{-1}$ for all $p \in I$.

If $p \neq q$, then

$$n^{-1}\,|\sum_{i \in J} a_i\, v_p(x_i) - \sum_{j \in J'} b_j\, v_p(y_j) - d_p\,| < n^{-1}.$$

Since the number inside the absolute-value signs is an integer, it follows that

$$\sum_{i \in J} a_i v_p(x_i) = \sum_{j \in J'} b_j v_p(y_j) + d_p .$$

In a similar fashion it can be shown that

$$\sum_{i \in J} a_i v_q(x_i) = \sum_{j \in J'} b_j v_q(y_j) + d_q + n.$$

Let $x = \prod_{i \in J} x_i^{a_i}$ and let $y = \prod_{j \in J'} y_j^{b_j}$. It is easy to verify that n, x and y have all the desired properties, Q.E.D.

In view of Theorem (B), the arguments previously used by Samuel [5] now apply directly to prove Theorem (C).

3. Throughout this section, R^* will denote the set $R \cup \{\infty\}$ and the following conventions will be adopted.

(1) If $a \in R$, then $a < \infty$.
(2) If $a \in R^*$, then $a + \infty = \infty + a = \infty$.
(3) If $a \in R^*$ and $a > 0$, then $a \cdot \infty = \infty \cdot a = \infty$.
(4) $0 \cdot \infty = \infty \cdot 0 = 0$.

Let T be an arbitrary non-empty set. Let $\mathfrak{F}$ be the collection of all non-negative R^*-valued functions on T. An element $f \in \mathfrak{F}$ is said to be *trivial* in case for each $x \in T$, $f(x) = 0$ or $f(x) = \infty$. Let f, g be arbitrary elements of $\mathfrak{F}$. Let

$$L_T(f, g) = \{r \in R^*; f(x) \geq rg(x) \text{ for all } x \in T\}.$$

If $L_T(f, g)$ is bounded in R, let $l_T(f, g) = \sup L_T(f, g)$. Otherwise, let $l_T(f, g) = \infty$. It is easy to verify that $f(x) \geq l_T(f, g)g(x)$ for all $x \in T$.

DEFINITION 3.1. *The number $l_T(f, g)$ is called the linking number of f over g on T.* (Note that when $f, g \in \mathcal{S}$,

$$l_{K^* \cap \mathfrak{o}}(f, g) = \inf \{f(x)/g(x); x \in K^* \cap \mathfrak{o}, g(x) > 0\}).$$

PROPOSITION 3.1 *Let f, g and h be elements of $\mathfrak{F}$.*

(i) *If f is trivial, then $l_T(f, f) = \infty$. If f is non-trivial, then $l_T(f, f) = 1$.*
(ii) $l_T(f, g)l_T(g, h) \leq l_T(f, h)$.
(iii) *If f is non-trivial or g is non-trivial, then*

$$l_T(f, g)l_T(g, f) \leq 1.$$

(iv) *Let $f + g$ denote the point-wise sum of f and g. Then*

$$l_T(f, h) + l_T(g, h) \leq l_T(f + g, h).$$

(A case when equality holds will be given in the next section in Theorem 4.2.)

(v) *Let f and g be non-trivial. There exists a real number $\alpha (\alpha \neq 0, \alpha \neq \infty)$ such that $f(x) = \alpha g(x)$ for all $x \in T$ if and only if $l_T(f, g)l_T(g, f) = 1$.*

Proof. Clear.

Let S be a commutative ring with identity. By a pseudo-valuation on S we shall mean an R^*-valued non-negative function v, defined on S, which has the following properties:

(1) $v(1) = 0, v(0) = \infty$.
(2) $v(x \cdot y) \geq v(x) + v(y)$.
(3) $v(x - y) \geq \min\{v(x), v(y)\}$.

A pseudo-valuation v is said to be *homogeneous* in case for each $x \in S$ and each positive integer n, $v(x^n) = nv(x)$.

Let v be an arbitrary pseudo-valuation on S. Rees [3] has shown that $\lim_{\omega \to \infty} v(x^n)/n = \bar{v}(x)$ exists for each $x \in S$ and that $\bar{v}$ is a homogeneous pseudo-valuation on S. Moreover, $\bar{v}(x) \geq v(x)$ for each $x \in S$.

PROPOSITION 3.2. *If v, w are pseudo-valuations on S, then*

$$l_S(v, w) \leq l_S(\bar{v}, w) = l_S(\bar{v}, \bar{w}).$$

Proof. Since $v(x) \leq \bar{v}(x)$ for each $x \in S$, $l_S(v, w) \leq l_S(\bar{v}, w)$. On the other hand, since $w(x) \leq \bar{w}(x)$ for each $x \in S$, $l_S(\bar{v}, \bar{w}) \leq l_S(\bar{v}, w)$. If $x \in S$,

$$\bar{v}(x) = \bar{v}(x^n)/n \geq l_S(\bar{v}, w)w(x^n)/n.$$

Consequently, $\bar{v}(x) \geq l_S(\bar{v}, w)\bar{w}(x)$ so that $l_S(\bar{v}, w) \leq l_S(\bar{v}, \bar{w})$, Q.E.D.

Let A be an ideal of S and let v be a pseudo-valuation on S. The number $\inf\{v(\mathrm{a}); x \in A\}$ will be denoted by $v(A)$. For each $x \in S$, let $v_A(x) = \infty$ in case $x \in \bigcap_{n>0} A^n$ and if $x \notin \bigcap_{n>0} A^n$, let $v_A(x)$ be that integer t (≥ 0) such that $x \in A^t$ but $x \notin A^{t+1}$. Clearly, v_A is a pseudo-valuation on S. Let S be a noetherian ring, A and B proper ideals of S such that (1) $\operatorname{Rad} A = \operatorname{Rad} B$, and (2) $\bigcap_{n>0} A^n = 0$ (hence $\bigcap_{n>0} B^n = 0$). Samuel [4] has shown that $\lim_{n \to \infty} v_A(B^n)/n$ exists and has denoted this number by $l_A(B)$. It has been observed by Rees [3] that (1′) $\bar{v}_A(x) \geq l_A(B)\bar{v}_B(x)$ for all $x \in S$ and (2′) $l_A(B) = \bar{v}_A(B)$. The following proposition shows that $l_A(B) = l_S(\bar{v}_A, \bar{v}_B)$.

PROPOSITION 3.3. *Let S be a commutative ring with identity and let A be an ideal of S. Let v be an arbitrary pseudo-valuation on S. Then*

(i) $l_S(v, v_A) = v(A)$.
(ii) $l_S(\bar{v}, \bar{v}_A) = \bar{v}(A)$.

Proof. Statement (ii) follows from (i) and Proposition 3.2. If first will be shown that $v(x) \geq v(A)v_A(x)$ for all $x \in S$. Suppose $x \in A^n$, $n \geq 0$. Then $v(x) \geq v(A^n) \geq nv(A)$. It follows from this that $v(x) \geq v(A)v_A(x)$ and, therefore, $v(A) \leq l_S(v, v_A)$. On the otherhand,

$$v(A) \geq l_S(v, v_A)v_A(A) \geq l_S(v, v_A),$$

Q.E.D.

4. Let the notation be as in Section 1. Since for each $g \in \mathcal{S}$, $G'(p)$ is precisely equal to $l(g, v_p)$, the symbol $l(g, v_p)$ will henceforth replace the less suggestive symbol $G'(p)$.

DEFINITION 4.1 *An element $g \in \mathcal{S}$ is said to be perfect in case* $g(x) = \sum_{p \in I} l(g, v_p)v_p(x)$ *for each* $x \in K^*$.

DEFINITION 4.2. *Let $g \in \mathcal{S}$. Any function $\bar{G}$ satisfying* (i) *and* (ii) *of Theorem* (A) *is called a representation function for g.*

LEMMA 4.1. *Let f be a perfect element of $\mathcal{S}$. Let g be a non-trivial element of $\mathcal{S}$ and let $\bar{G}$ be any representation function for g. Then*

$$l(f, g) = \inf\{l(f, v_p)/\bar{G}(p); p \in I, \bar{G}(p) > 0\}$$
$$= \inf\{l(f, v_p)/l(g, v_p); p \in I, l(g, v_p) > 0\}.$$

Proof. Let

$$r = \inf\{l(f, v_p)/\bar{G}(p); p \in I, \bar{G}(p) > 0\}$$

and let

$$r' = \inf\{l(f, v_p)/l(g, v_p); p \in I, l(g, v_p) > 0\}.$$

Since $\bar{G}(p) \leq l(g, v_p)$ for each $p \in I$, $r' \leq r$. If $x \in K^* \cap \mathfrak{o}$, then since $r\bar{G}(p) \leq l(f, v_p)$ for all $p \in I$, it follows that $r \cdot g(x) \leq f(x)$. Thus, $r' \leq r \leq l(f, g)$. On the other hand, $l(f, g)l(g, v_p) \leq l(f, v_p)$ for each $p \in I$, so $l(f, g) \leq r' \leq r$. Hence, $r' = r = l(f, g)$, Q.E.D.

LEMMA 4.2. *Let $f \in \mathcal{S}$ and let $\bar{F}$ be a representation function for f. Let q be any element of I. The following are equivalent*:

(i) *For each $\varepsilon > 0$, there exists $x \in q$ such that*

$$\sum_{p \in I, p \neq q} \bar{F}(p)v_p(x) \leq \varepsilon v_p(x).$$

(ii) $\bar{F}(q) = l(f, v_q)$.

We wish to point out that condition (i) is similar to, but slightly weaker than Samuel's condition that q be almost-principal relative to f. (See [5, §2].)

Proof. Suppose (i) holds. Let $\varepsilon > 0$ be given and select $x \in q$ such that $\sum_{p \in I, p \neq q} \bar{F}(p)v_p(x) \leq \varepsilon v_q(x)$. Then $f(x) \leq (\bar{F}(q) + \varepsilon)v_q(x)$ so that $l(f, v_q) \leq \bar{F}(q) + \varepsilon$. Since ε was chosen arbitrarily, $l(f, v_q) \leq \bar{F}(q)$. Hence, equality holds (Theorem (B), (i)). Conversely, let $\varepsilon > 0$ be given. Choose $x \in q$ such that $f(x)/v_q(x) \leq l(f, v_q) + \varepsilon$. Since by hypothesis $\bar{F}(q) = l(f, v_q)$, it is immediate that $\sum_{p \in I, p \neq q} \bar{F}(p)v_p(x) \leq \varepsilon v_q(x)$, Q.E.D.

THEOREM 4.1. *Let f be a perfect element of $\mathcal{S}$. If $g \in \mathcal{S}$ and $l(f, g) \neq 0$, then g is perfect.*

Proof. If $l(f, q) = \infty$, then g is trivial and is already perfect, so assume

$l(f, g) \neq \infty$. For each $x \in K^*$, let $g'(x) = l(f, g)g(x)$. It suffices to show that g' is perfect. Since $g'(x) \leq f(x)$ for all $x \in K^* \cap \mathfrak{o}$, $1 \leq l(f, g')$. Let $\bar{G}'$ be any representation function for g'. It follows from Lemma 4.1 that $\bar{G}'(p) \leq l(f, v_p)$ for each $p \in I$. Since f is perfect, condition (iii) of Theorem (C) is satisfied relative to f. Hence, for each $q \in I$ and each $\varepsilon > 0$, there exists $x \in q$ such that $\sum_{p \in I, p \neq q} \bar{G}'(p) v_p(x) \leq \varepsilon v_q(x)$. By Lemma 4.2, $\bar{G}'(q) = l(g', v_q)$, Q.E.D.

THEOREM 4.2. *Let f and g be perfect elements of $\mathcal{S}$ and let $f + g$ denote the point-wise sum of f and g (clearly, $f + g \in \mathcal{S}$). If $l(f, g) \neq 0$ or $l(g, f) \neq 0$, then $f + g$ is perfect.*

Proof. Since $f + g = g + f$, it can be assumed that $l(f, g) \neq 0$. If $l(f, g) = \infty$, g is trivial and there is nothing to prove, so assume $l(f, g) \neq \infty$. It will be shown that $l(f, f + g) \neq 0$. From Lemma 4.1 and the fact that $l(g, v_p) = 0$ whenever $l(f, v_p) = 0$ (since $l(f, g) \neq 0$), it follows that

$$l(f, f + g) = \inf \{l(f, v_p)/(l(f, v_p) + l(g, v_p)); p \in I, l(f, v_p) > 0\}.$$

For each $p \in I$ such that $l(f, v_p) \neq 0$,

$$l(f, v_p)/(l(f, v_p) + l(g, v_p)) = (\tfrac{1}{2}l(f, v_p) + \tfrac{1}{2}l(f, v_p))/(l(f, v_p) + l(g, v_p)) \geq \min \{\tfrac{1}{2}, \tfrac{1}{2}l(f, g)\} > 0,$$

where $\frac{1}{2}l(f, v_p)/l(g, v_p) = \infty$ in case $l(g, v_p) = 0$. Thus, $l(f, f + g) \neq 0$ so $f + g$ is perfect, Q.E.D.

We shall explore briefly a few lattice properties of $\mathcal{S}$. Let f, g be elements of $\mathcal{S}$. A partial ordering is defined on $\mathcal{S}$ as follows: $g \prec f$ in case $1 \leq l(f, g)$ (i.e., $g(x) \leq f(x)$ for all $x \in K^* \cap \mathfrak{o}$). As a consequence of Lemma 4.1, when f is perfect, $g \prec f$ if and only if $l(g, v_p) \leq l(f, v_p)$ for each $p \in I$. For each pair f, g of perfect elements of $\mathcal{S}$, define $(f \cap g)(x)$ to be

$$\sum_{p \in I} (\min \{l(f, v_p), l(g, v_p)\}) v_p(x)$$

and define $(f \cup g)(x)$ to be

$$\sum_{p \in I} (\max \{l(f, v_p), l(g, v_p)\}) v_p(x).$$

Then $f \cap g$, $f \cup g$ are elements of $\mathcal{S}$ and $f \cap g$ is perfect due to the fact that $l(f, f \cap g) \geq 1 > 0$. Since $l(f + g, f \cup g) \geq 1 > 0$, $f \cup g$ is perfect when $l(f, g) \neq 0$ or $l(g, f) \neq 0$. It is clear that (relative to the partial ordering $\prec$ restricted to the set $\mathcal{S}'$ of perfect elements of $\mathcal{S}$) $f \cap g = \mathrm{GLB}\{f, g\}$ and when $f \cup g$ is perfect, $f \cup g = \mathrm{LUB}\{f, g\}$.

LEMMA 4.3. *Let f, g and h be non-trivial perfect elements of $\mathcal{S}$. Then*

(i) $l(f \cup g, h) \geq \max \{l(f, h), l(g, h)\}$.
(ii) $l(h, f \cup g) = \min \{l(h, f), l(h, g)\}$.
(iii) $l(f \cap g, h) = \min \{l(f, h), l(g, h)\}$.
(iv) $l(h, f \cap g) \geq \max \{l(h, f), l(h, g)\}$.

Proof. Clear.

Let f be any non-trivial perfect element of $\mathcal{S}$ and let

$$L(f) = \{g \in \mathcal{S}; l(f, g) \neq 0 \quad \text{and} \quad l(g, f) \neq 0\}.$$

Clearly, $f \in L(f)$, so $L(f)$ is non-empty. If $g \in L(f)$, then g is perfect due to the fact that $l(f, g) \neq 0$. On the other hand, $l(g, f) \neq 0$ implies g is non-trivial. If $g, g' \in L(f)$, then $l(g, g') \geq l(g, f)l(f, g') > 0$. Thus, $g \cup g'$ is again perfect. From Lemma 4.3 it follows that $g \cap g'$ and $g \cup g'$ are in $L(f)$ whenever $g, g' \in L(f)$.

THEOREM 4.3. *Let $\mathcal{S}'$ be the collection of all nontrivial perfect elements of $\mathcal{S}$. Let $\mathcal{L} = \{L(f); f \in \mathcal{S}'\}$. Then $\mathcal{L}$ is a partition of $\mathcal{S}'$ and each element $L(f)$ of $\mathcal{L}$ is a distributive lattice under the operations $\cup$ and $\cap$.*

Proof. Clear.

REFERENCES

1. N. BOURBAKI, *Topologie générale*, Éléments de Mathématique, Livre III, Paris, Hermann, 1947.
2. H. T. MUHLY, *On a problem of Samuel concerning divisors of second kind*, to appear.
3. D. REES, *Valuations associated with a local ring*, Proc. London Math. Soc. (3), vol. 5 (1955), pp. 107–128.
4. P. SAMUEL, *Some asymptotic properties of powers of ideals*, Ann. of Math. (2), vol. 56 (1952), pp. 11–21.
5. ———, *Multiplicités de certaines composantes singulières*, Illinois J. Math., vol. 3 (1959), pp. 318–327.
6. ———, *Sur l'image reciproque d'un diviseur*, J. Reine Angew. Math., vol. 204 (1960), pp. 1–10.

UNIVERSITY OF IOWA
IOWA CITY, IOWA