

A CHARACTERIZATION OF SELF-INJECTIVE RINGS

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Eilenberg and Nakayama [2] showed that if a ring R is left and right Noetherian, R is quasi-Frobenius if and only if R is self-injective. Self-injective rings have been considered by a number of other authors usually in connection with rings of quotients; see, for instance, Gentile [3], Lambek [5] and Wong and Johnson [6]. In this note we shall give a characterization of self-injective rings in terms of function topologies on modules as defined in S. U. Chase's paper [1].

Throughout this paper we shall assume that R is a ring with identity and with the discrete topology. The identity of R acts like identity on R -modules.

Let M be a left R -module with the discrete topology. We shall denote $\text{Hom}_R(M, R)$ by M^* ; M^* is called the dual of M . M^* is given the structure of a right R -module by $(fr)(x) = f(x)r$ where $f \in M^*$, $r \in R$ and $x \in M$.

If M and N are two R -modules and $\theta : M \rightarrow N$ is an R -homomorphism, applying $(*)$ gives $\theta^* : N^* \rightarrow M^*$ where for $f \in N^*$, $\theta^*(f)$ acts on elements of M like $f \circ \theta$. For further properties of $(*)$ see [4, Chapter 4].

The following definition is due to S. U. Chase.

DEFINITION 1. Let T be a right R -submodule of M^* . The T -topology on M is defined to be the weak topology induced on M by T . That is, it is the coarsest topology on M such that every element of T is a continuous homomorphism from M to R .

The T -topology on M is equivalent to the topology whose base for the neighborhood system of zero consists of all subsets of M of the form $\bigcap_{i=1}^n \text{Ker } T_i$, $T_i \in T$, $i = 1, \dots, n$. See [1, Prop. 1.2].

It is easy to see that the T -topology for M is Hausdorff if and only if for each $m \in M$, $m \neq 0$, there exists $f \in T$ such that $f(m) \neq 0$. In this situation we say that " T separates points of M ".

DEFINITION 2. We shall say T is separating if the T -topology (for M) is Hausdorff.

We shall denote by $C_T \text{Hom}_R(M, R)$ the right R -submodule of M^* consisting of all continuous R -homomorphisms from M to R where M has the separating T -topology. From the definition of T -topology it follows that $T \subseteq C_T \text{Hom}_R(M, R)$. We shall be interested in the case that

$$T = C_T \text{Hom}_R(M, R).$$

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The following theorem gives a characterization of self-injective rings.

THEOREM. *A necessary and sufficient condition that R be left self-injective is that $T = C_T \text{Hom}_R(M, R)$ for all left R -modules M and all separating right R -submodules T of M^* .*

Proof that the condition is sufficient. Suppose R is not left self-injective. Then there exists a left ideal L such that the exact sequence

$$0 \rightarrow L \xrightarrow{j} R \rightarrow \frac{R}{L} \rightarrow 0$$

induces the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_R\left(\frac{R}{L}, R\right) &\rightarrow \text{Hom}_R(R, R) \xrightarrow{j^*} \text{Hom}_R(L, R) \\ &\rightarrow \text{Ext}_R^1\left(\frac{R}{L}, R\right) \rightarrow 0 \quad \text{with } \text{Ext}_R^1\left(\frac{R}{L}, R\right) \neq 0. \end{aligned}$$

(See [4] for the notation "Ext".) That is, j^* is not an epimorphism, and we conclude that $\text{Hom}_R(L, R) \supsetneq j^*(\text{Hom}_R(R, R))$.

Let T be the right R -submodule of $\text{Hom}_R(L, R)$ generated by j . We note that T separates points of L , so L is Hausdorff in the T -topology. Since j is a monomorphism, L is discrete in the T -topology; that is,

$$C_T \text{Hom}_R(L, R) = \text{Hom}_R(L, R).$$

We shall show that $T = j^*(\text{Hom}_R(R, R))$. Since $T = jR$, for any $t \in T$ there exists an element r in R such that $t = jr$. Therefore $t(x) = jr(x) = j(x)r$ for all x in L . Now $j(x)r = (f_r \circ j)(x)$ where f_r is the right multiplication by r . Thus we have $t = f_r \circ j = j^*(f_r)$, $f_r \in \text{Hom}_R(R, R)$. This shows that $T \subseteq j^*(\text{Hom}_R(R, R))$. To show the reverse containment, we note that every homomorphism from R to R is given by the right multiplication of an element in R . Then it is easy to see that $j^*(\text{Hom}_R(R, R)) \subseteq T$. Therefore $T = j^*(\text{Hom}_R(R, R))$ and hence $C_T \text{Hom}_R(L, R) \supsetneq T$.

Proof that the condition is necessary. Suppose R is left self-injective. As we noted above, it is always true that $T \subseteq C_T \text{Hom}_R(M, R)$. We wish to show that $C_T \text{Hom}_R(M, R) \subseteq T$ for all left R -modules M and right R -submodules T of M^* .

Let $f \in C_T \text{Hom}_R(M, R)$. Since R is discrete and f is continuous, $\text{Ker } f$ is open in M . Then there exist $t_1, t_2, \dots, t_n \in T$ such that

$$\bigcap_{i=1}^n \text{Ker } t_i \subseteq \text{Ker } f.$$

Let $K = \bigcap_{i=1}^n \text{Ker } t_i$, $F_n = R \oplus R \oplus \dots \oplus R$, (n copies of R). Define

$$\theta : M \rightarrow F_n$$

by $\theta(m) = (t_1(m), t_2(m), \dots, t_n(m))$, $m \in M$. Then $\text{Ker } \theta = K$.

Consider the exact commutative diagram

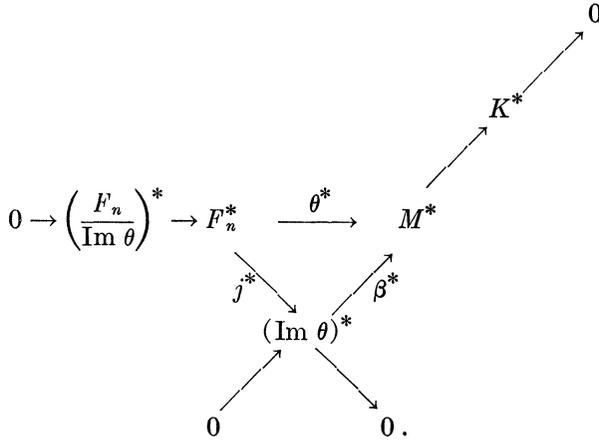
$$\begin{array}{ccccccc}
 0 & \rightarrow & K & \rightarrow & M & \xrightarrow{\theta} & F_n \rightarrow \frac{F_n}{\text{Im } \theta} \rightarrow 0 \\
 & & & & & \searrow & \nearrow \\
 & & & & & \beta & j \\
 & & & & & \searrow & \nearrow \\
 & & & & & \text{Im } \theta & \\
 & & & & & \nearrow & \searrow \\
 & & & & & 0 & \\
 & & & & & \nearrow & \searrow \\
 & & & & & 0 &
 \end{array}$$

where β is θ with restricted image and $j =$ the injection of $\text{Im } \theta$ into F_n .

(*) the above diagram and we have the exact commutative diagram

$$\begin{array}{ccccccc}
 & & & & & & \nearrow \\
 & & & & & & \text{Ext}_R^1(\text{Im } \theta, R) \\
 & & & & & \nearrow & \\
 & & & & & K^* & \\
 & & & & & \nearrow & \\
 0 & \rightarrow & \left(\frac{F_n}{\text{Im } \theta}\right)^* & \rightarrow & F_n^* & \xrightarrow{\theta^*} & M^* \\
 & & & & & \searrow & \nearrow \\
 & & & & & j^* & \beta^* \\
 & & & & & \searrow & \nearrow \\
 & & & & & (\text{Im } \theta)^* & \\
 & & & & & \nearrow & \searrow \\
 & & & & & 0 & \text{Ext}_R^1\left(\frac{F_n}{\text{Im } \theta}, R\right) \\
 & & & & & & \searrow \\
 & & & & & & 0
 \end{array}$$

Since R is left self-injective, $\text{Ext}_R^1(\cdot, R) = 0$ and the above diagram reduces to the following exact commutative diagram:



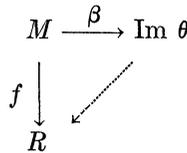
Now we shall show $\text{Im } \theta^*$ is contained in the right R -submodule of M^* generated by t_1, t_2, \dots, t_n . Let $g \in \theta^*(F_n^*)$, say $g = \theta^*(\alpha)$, $\alpha \in F_n^* = \text{Hom}_R(F_n, R)$. Let α_i be the restriction of α on the i^{th} summand of F_n . Since each α_i is given by the right multiplication by an element in R , say r_i , and $\alpha = \sum_{i=1}^n \alpha_i$; for $m \in M$ we have

$$\begin{aligned}
 g(m) &= [\theta^*(\alpha)](m) = \alpha[\theta(m)] = \alpha(t_1(m), t_2(m), \dots, t_n(m)) \\
 &= \sum_{i=1}^n t_i(m)r_i = \sum_{i=1}^n [t_i r_i](m).
 \end{aligned}$$

This shows that $g = \sum_{i=1}^n t_i r_i$, $r_i \in R$. Hence $\text{Im } \theta^*$ is contained in the right R -submodule of M^* generated by t_1, t_2, \dots, t_n .

Now by exactness and commutativity of the above diagram $\beta^*[(\text{Im } \theta)^*] = \text{Im } \theta^*$. Therefore $\beta^*[(\text{Im } \theta)^*] \subseteq T$.

Considering the following diagram



where $\text{Ker } f \cong K = \text{Ker } \theta = \text{Ker } \beta$, we see that there exists $f' \in (\text{Im } \theta)^*$ such that $f' \circ \beta = f$. Therefore it follows that $f \in \beta^*[(\text{Im } \theta)^*]$ and since we have shown that $\beta^*[(\text{Im } \theta)^*]$ is contained in T , $f \in T$. This completes the proof of the theorem.

The following lemma [1, Lemma 1.6] is an easy consequence of Definition 1.

LEMMA. *Let M be a topological R -module, and N be an R -module with the S -topology, where S is an R -submodule of N^* . Then a homomorphism $f : M \rightarrow N$ is continuous if and only if $f^*(S)$ is contained in the submodule of M^* consisting of all continuous homomorphisms from M to R .*

COROLLARY. *Let R be a left self-injective ring. Let M and N be R -modules with the separating T -, S -topologies respectively, where $T \subseteq M^*$, $S \subseteq N^*$. Then a homomorphism $f: M \rightarrow N$ is continuous if and only if $f^*(S) \subseteq T$.*

The proof is a direct consequence of the theorem and the lemma.

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