THE INFLUENCE OF $\pi_1(Y)$ ON THE HOMOLOGY OF $M(X, Y)$

BY

MICHEAL N. DYER[1]

1. Introduction

Let $M(X, Y)$ denote the set of all continuous functions from X to Y where X is a locally finite CW complex of dimension k and Y is arc-connected. We give $M(X, Y)$ the compact-open topology. The purpose of this note is to compute the first $r - k - 1$ homology (and cohomology) groups of $M(X, Y)$ in terms of $\pi_1(Y)$, where $r > k + 1$ and $\pi_i(Y) = 0$ for $1 < i < r$. The main tool used is the exact couple $\mathcal{C}(X, Y, v)$, $v \in M(X, Y)$, of H. Federer, defined in [1]. We give a description of this exact couple in Section 2. In Section 3 we prove the main theorem and in Section 4 we compute the first few homology groups of $M(X, U)$, where U is the infinite unitary group [2], and $M(X, P^n)$, where P^n is the n-dimensional real projective space, for certain spaces X.

2. Description of $\mathcal{C}(X, Y, v)$

Let X^n = n-dimensional skeleton of X. Let U_j = the arc-component of $M(X^j, Y)$ containing $v_j = v \mid X^j$. Define the map

$$r : U_j \to U_{j-1}$$

by $r(f) = f \mid X^{j-1}$ $(f \in M(X^j, Y))$. Since X is locally finite, r is a fibering in the sense of Serre (see [2]). Let

$$F_j = r^{-1}(v_{j-1}) = \{f \in U_j \mid f \mid X^{j-1} = v_{j-1}\}.$$

F_j is a fiber of r.

Define

$$D = \sum\nolimits_{p,q\oplus} D_{p,q}$$

where $D_{p,q} = \pi_p(U_q, v_q)$ if $p, q \geq 0$, $D_{p,q} = 0$ otherwise, and

$$E = \sum\nolimits_{p,q\oplus} E_{p,q}$$

where $E_{p,q} = \pi_p(F_q, v_q)$ if $p, q \geq 0$ and $E_{p,q} = 0$ otherwise. Then the homotopy sequence of the fibering above becomes

$$\cdots \to E_{i,j} \xrightarrow{k} D_{i,j} \xrightarrow{i} D_{i,j-1} \xrightarrow{j} E_{i-1,j} \to \cdots,$$

where k is inclusion induced, $i = r_*$, and $j = \partial$. This makes $\{D, E, i, j, k\}$ an exact couple, denoted by $\mathcal{C}(X, Y, v)$.

We state the following theorem for future reference. The proof is in [1].

Received July 22, 1965.

[1] The results of this paper form a portion of the author's doctoral dissertation written at the University of California at Los Angeles under the direction of Professor Sze-Tsen Hu. The research was partially supported by a National Science Foundation Fellowship.

THEOREM 2.1. *If X is a locally finite CW complex and if Y is arc-connected and simple (= n-simple for all $n > 0$), then*

(a) $E_{p,q} \approx C^q(X, \pi_{p+q}(Y))$, *the group of q-dimensional cochains on X with coefficients in $\pi_{p+q}(Y)$, for $p \geq 1$. If $p = 0$, then $E_{0,q} \approx$ subgroup of $C^q(X, \pi_q(Y))$.*

(b) $E^2_{p,q} \approx H^q(X, \pi_{p+q}(Y))$ *for $p \geq 1$ and if $p = 0$, $E^2_{0,q} \approx$ subgroup of $H^q(X, \pi_q(Y))$.*

We filter $\pi_p(M(X, Y), v)$ as follows:

$$\pi_p(M(X, Y), v) \supset \pi_{p,0} \supset \pi_{p,1} \supset \cdots \supset \pi_{p,k-1} \supset 0,$$

where

$$\pi_{p,q} = \ker \{i^{(k-q)} : D_{p,k} \to D_{p,q}\},$$

$q < k = \dim X$ and $i^{(j)} = i \circ i \circ i \circ \cdots \circ i$ (j times). Then the usual proposition is true (see [1]).

PROPOSITION 2.2. $\pi_{p,q-1}/\pi_{p,q} \approx E^\infty_{p,q}$.

3. The main theorem

THEOREM 3.1. *Let X be a locally finite CW complex of dimension k. Let Y be 1-simple and arc-connected such that*

$$\pi_i(Y) = 0 \qquad (1 < i < r)$$

where $r > k + 1$. Let π denote $\pi_1(Y)$. Then there exist isomorphisms u_*, u^* such that for $0 \leq p < r - k$

$$u_* : H_p(M) \approx \textstyle\sum_{\oplus}^{\alpha \in A} \{H_p(\pi^B)\}_\alpha$$

$$u^* : H^p(M) \approx \textstyle\prod_{\alpha \in A} \{H^p(\pi^B)\}_\alpha$$

where $M = M(X, Y)$, $A = \pi_0(M)$, $B = \pi_0(X)$, and $\pi^B = \prod_{\beta \in B} (\pi)_\beta$, the group product.

COROLLARY 3.2. *Let X and Y be as in 3.1. In addition, let X be arc-connected. Then for $0 \leq p < r - k$,*

$$H_p(M) \approx \textstyle\sum_{\oplus}^{\alpha \in A} \{H_p(Y)\}_\alpha$$

$$H^p(M) \approx \textstyle\prod_{\alpha \in A} \{H^p(Y)\}_\alpha .$$

Proof. We first prove 3.1 for Y simple. Let $\{M_\alpha \mid \alpha \in A\}$ be the set of arc-components of $M(X, Y)$. Then for any $p > 0$

$$H_p(M) \approx \textstyle\sum_{\oplus}^{\alpha \in A} \{H_p(M_\alpha)\}$$

and

$$H^p(M) \approx \textstyle\prod_{\alpha \in A} \{H^p(M_\alpha)\}.$$

Thus we need to compute the homology and cohomology of M_α for each α in A.

Let us pick a base point $v_\alpha \in M_\alpha$ and consider $\mathcal{C}(X, Y, v_\alpha)$. For each α in A, we will show that

(a) $\pi_p(M, v_\alpha) = 0$ for $1 < p < r - k$
(b) $\pi_1(M, v_\alpha) \approx \pi^B$.

By hypothesis, $\pi_i(Y) = 0$ for $1 < p < r$. By 2.1(a)

$$E_{p,q}(X, Y, v_\alpha) \approx C^q(X, \pi_{p+q}(Y)) \qquad (p \geq 1, q > 0). \tag{1}$$

Thus if $q \leq k$, $E_{p,q} = 0$ for $1 < p < r - k$. 2.2 then implies that $\pi_i(M, v_\alpha) = 0$ for $1 < i < r - k$. This proves (a) for any α in A.

Since $E^2_{1,0} = E^\infty_{1,0}$ and $E_{1,j} = 0$ $(0 < j \leq k)$, then 2.2 implies

$$\pi_1(M, v_\alpha) \approx H^0(X, \pi_1(Y)) \approx \pi^B. \tag{2}$$

This proves (b) for any α in A.

Thus, by a well known theorem on the influence of $\pi_1(B)$ on the homology of any arc-connected space B, there exists isomorphisms u_*, u^* (see [2, Chapter IX]) such that

$$u_* : H_p(M_\alpha) \approx H_p(\pi^B)$$
$$u^* : H^p(M_\alpha) \approx H^p(\pi^B)$$

for $0 \leq p < r - k$. This proves Theorem 3.1 for Y simple.

In order to prove 3.1 for Y only 1-simple we must verify (a) and (b) above for each α in A. However, since $\pi_i(Y) = 0$ for $1 < i < r$, we have that Y is i-simple for $i = 1, 2, \cdots, r - 1$. This implies that (1) holds for $p + q < r$, which gives (a). By looking carefully at the proof of 2.1 in [1], one sees that 1-simplicity is all that is needed to give (2), and this gives (b). This proves 3.1. The corollary follows because $1 < p < r - k \leq r$ implies that $H_p(Y) \approx H_p(\pi) = H_p(\pi^B)$, since X is arc-connected.

3. Examples

(a) Let $Y = U$, the infinite unitary group, and

$$X = \vee_{i=1}^n (S^1)_i,$$

the one point union of n circles, where $n \leq \infty$. The dimension of X is one. Let

$$M(n) = M\{\vee_{i=1}^n (S^1)_i, U\}.$$

It can be shown that

$$\begin{aligned} \text{card}\,\{\pi_0[M(n)]\} &= \aleph_0 \quad \text{if } n < \infty \\ &= \mathfrak{c} \quad\ \text{if } n = \infty, \end{aligned}$$

where $\mathfrak{c}$ is the cardinality of the continuum. Since

$$\begin{aligned} \pi_i(U) &= Z \quad \text{if } i \text{ is odd} \\ &= 0 \quad \text{if } i \text{ is even}, \end{aligned}$$

we have $r = 3$. Thus $r - k = 2$. Therefore

$$H_0(M(n)) \approx H_1(M(n)) \approx \sum_{\oplus}^{\alpha \epsilon A} (Z)_\alpha$$

$$H^0(M(n)) \approx H^1(M(n)) \approx \prod_{\alpha \epsilon A} (Z)_\alpha$$

where

$$\begin{aligned} \text{card } A &= \aleph_0 \quad \text{if } n < \infty \\ &= \mathfrak{c} \quad \text{if } n = \infty. \end{aligned}$$

(b) Let $Y = P^r$ $(r > 2)$, the r-dim real projective space, and X be any connected finite CW complex such that X has only one vertex. Then $\pi_1(P^r) \approx Z_2$ and $\pi_i(P^r) = 0$ for $1 < i < r$. Suppose dim $X = k < r - 1$. A simple application of obstruction theory (see [2, Chapter VI]) shows that

$$\pi_0(M(X, P^r)) = [X, P^r] = Z_2 .$$

Thus

$$H_p(M(X, P^r)) \approx H_p(P^r) \oplus H_p(P^r) \quad (0 \leq p < r - k).$$

Similarly, if B is any $(r - 1)$-connected space which admits free action by a finite group π of homeomorphisms, and if

$$Y = B/\pi,$$

we have

$$\pi_0(M(X, Y)) \approx \pi$$

where X is as above such that dim $X = k < r - 1$. Then

$$H_p(M(X, Y)) \approx \sum_{\oplus}^{\alpha \epsilon \pi} (H_p(Y))_\alpha \quad (0 \leq p < r - k)\cdot$$

BIBLIOGRAPHY

1. HERBERT FEDERER, *A study of function spaces by spectral sequences*, Trans. Amer. Math. Soc., vol. 82 (1956), pp. 340–361.
2. S. T. HU, *Homotopy theory*, New York, Academic Press, 1959.
3. J. W. MILNOR, *Morse theory*, Annals of Mathematics Studies, no. 51, Princeton, 1963.

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA