ON THE NON-REGULARITY OF CERTAIN GENERALIZED LOTOTSKY TRANSFORMS

BY

C. F. KOCH

1. Introduction

The generalized Lototsky (or $[F, d_n]$) sequence to sequence transformation was defined by A. Jakimovski [3] as follows. Let $d_n = \rho_n e^{i\theta_n}$ be a fixed sequence of complex numbers ($d_n \neq -1$); then the elements P_{nk} of the $[F, d_n]$ matrix are defined by

$$P_{00} = 1, \qquad P_{0k} = 0, \quad k \neq 0$$

$$\prod_{k=1}^{n} (\theta + d_k)/(1 + d_k) = \sum_{k=0}^{\infty} P_{nk}\, \theta^k.$$

Necessary conditions that this matrix be regular are $\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$ and if $\lim_{n\to\infty} \theta_n$ exists then it must equal zero. V. F. Cowling and C. L. Miracle, [1] and [2], have shown that the additional condition

$$\sum_{n=1}^{\infty} \theta_n^2\, \rho_n^{-1} < +\infty$$

is sufficient to guarantee regularity. They also conjectured that the necessary conditions might alone be sufficient. That this is not the case is shown by an example due to A. Meir [4]. The purpose of this paper is to give some general conditions under which

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty \quad \text{and} \quad \lim_{n\to\infty} \theta_n = 0$$

but the $[F, d_n]$ transformation is not regular.

2. The main theorem

THEOREM 1. *Let* $\lim_{n\to\infty} \theta_n = 0$, $0 < \theta_n < \pi/2$ *for all* n. *If there exists a sequence* α_n *satisfying* $\theta_k \geq \alpha_n$ *for* $k \leq n$ *and*

$$\lim_{n\to\infty} \alpha_n^2 \sum_{k=1}^{n} \rho_k/(1 + \rho_k)^2 = +\infty$$

then the $[F, d_n]$ *transformation is not regular.*

Proof. We note that an acceptable choice for the sequence α_n is $\alpha_n = \min_{1\leq k\leq n} \theta_k$ and proceed with the proof.

Assume the $[F, d_n]$ matrix is regular. Let $\lambda_{kn} = \rho_k e^{i\beta_{kn}}$ where $\beta_{kn} = \theta_k - \alpha_n$ for $k \leq n$. Define the elements b_{nk} of the $[F, \lambda_{kn}]$ matrix by

$$b_{00} = 1, \qquad b_{0k} = 0, \quad k \neq 0$$

$$\prod_{j=1}^{n} (\theta + \lambda_{jn})/(1 + \lambda_{jn}) = \sum_{k=0} b_{nk}\, \theta^k.$$

The elements P_{nk} of the $[F, d_n]$ matrix are given by

Received July 19, 1965.

$$P_{nk} = \frac{1}{\prod_{j=1}^{n}(1 + d_j)} \sum_{s_1+s_2+\cdots+s_n+k=n} d_1^{s_1} d_2^{s_2} \cdots d_n^{s_n} \tag{2.1}$$

where $s_i = 0$ or 1 and the summation is taken over all possible values of the s_i such that $s_1 + s_2 \cdots + s_n = n - k$. Similarly the elements b_{nk} of the $[F, \lambda_{kn}]$ matrix are given by

$$b_{nk} = \frac{1}{\prod_{j=1}^{n}(1 + \lambda_{jn})} \sum_{s_1+s_2+\cdots+s_n+k=n} \lambda_{1n}^{s_1} \lambda_{2n}^{s_2} \cdots \lambda_{nn}^{s_n}.$$

Now since $\lambda_{kn} = \rho_k e^{i\beta_{kn}} = d_k e^{-i\alpha_n}$ we have

$$b_{nk} = \frac{e^{-i(n-k)\alpha_n}}{\prod_{j=1}^{n}(1 + \lambda_{jn})} \sum_{s_1+s_2+\cdots+s_n+k=n} d_1^{s_1} d_2^{s_2} \cdots d_n^{s_n}$$

so that

$$\left| \textstyle\sum_{s_1+s_2+\cdots+s_n+k=n} d_1^{s_1} d_2^{s_2} \cdots d_n^{s_n} \right| = | b_{nk} | \textstyle\prod_{j=1}^{n} | 1 + \lambda_{jn} |. \tag{2.2}$$

Inserting absolute values in (2.1), summing over k from 1 to n, and using (2.2) we obtain

$$\textstyle\sum_{k=0}^{n} | P_{nk} | = \prod_{j=1}^{n} | (1 + \lambda_{jn})/(1 + dj) | \sum_{k=0}^{n} |b_{nk}|$$

and since

$$\textstyle\sum_{k=0}^{n} | b_{nk} | \geq | \sum_{k=0}^{n} b_{nk} | = 1$$

we have

$$\textstyle\sum_{k=0}^{n} | P_{nk} | \geq \prod_{j=1}^{n} | (1 + \lambda_{jn})/(1 + dj) |. \tag{2.3}$$

Using $\lambda_{jn} = \rho_j(\cos \beta_{jn} + i \sin \beta_{jn})$ and $d_j = \rho_j(\cos \theta_j + i \sin \theta_j)$ we obtain

$$\left| \frac{1 + \lambda_{jn}}{1 + dj} \right| = \left(1 + \frac{2\rho_j(\cos \beta_{jn} - \cos \theta_j)}{1 + 2\rho_j \cos \theta_j + \rho_j^2} \right)^{1/2}.$$

Writing $\cos \beta_{jn} - \cos \theta_j$ as a product of sines and noting that $\alpha_n < \beta_{jn} + \theta_j < \pi$ we use the fact that $\sin \phi \geq (2/\pi)\phi$ for $0 \leq \phi \leq \pi/2$ to obtain

$$\left| \frac{1 + \lambda_{jn}}{1 + dj} \right| \geq \left(1 + \frac{4}{\pi^2} \frac{\rho_j \, \alpha_n(2\theta_j - \alpha_n)}{(1 + \rho_j)^2} \right)^{1/2}$$
$$\geq \left(1 + \frac{4\rho_j \, \alpha_n^2}{\pi^2(1 + \rho_j)^2} \right)^{1/2}$$

since $\theta_j \geq \alpha_n$ for $j \leq n$. Thus

$$\begin{aligned} \prod_{j=1}^{n} \left| \frac{1 + \lambda_{jn}}{1 + dj} \right| &\geq \prod_{j=1}^{n} \left(1 + \frac{4\rho_j \, \alpha_n^2}{\pi^2(1 + \rho_j)^2} \right)^{1/2} \\ &\geq \left(1 + \frac{4\alpha_n^2}{\pi^2} \sum_{j=1}^{n} \frac{\rho_j}{(1 + \rho_j)^2} \right)^{1/2}. \end{aligned} \tag{2.4}$$

But this estimate can be obtained for each n and since this last quantity diverges to plus infinity, $\sum_{k=0}^{n} | P_{nk} |$ is not uniformly bounded for all n and the assumed regularity of the $[F, d_n]$ matrix is contradicted.

It is clear that a similar theorem could be proven with the hypothesis $\pi/2 > \theta_n > 0$ for all n replaced by $-\pi/2 < \theta_n < 0$ for all n. See [1, Theorem 2.3].

3. A special case of Theorem 1

In the hypotheses of the previous theorem we have not precluded the possibility that the sequence ρ_n might tend to zero or plus infinity. If, however, we supply this restriction, the following sharper statement can be obtained.

THEOREM 2. *If, in addition to the hypotheses of Theorem* 1,

$$M \geq \rho_n \geq N > 0 \quad \text{and} \quad \theta_n = O(n^{-1/2} \log^{\varepsilon} (n + 1))$$

where $\varepsilon > 0$ *then the* $[F, d_n]$ *transformation is not regular.*

Proof. Assume the $[F, d_n]$ matrix to be regular. Then from (2.3) and (2.4) and the additional restriction on ρ_n we have

$$\sum_{k=0}^{n} | P_{nk} | \geq (1 + \sum_{j=1}^{n} 4N\alpha_n^2/\pi^2(1 + M)^2)^{1/2}.$$

Let $\alpha_n = Cn^{-1/2} \log^{\varepsilon'}(n + 1)$ where ε' satisfies $\varepsilon > \varepsilon' > 0$ and is to be specified later and C is a positive constant. Then

$$\begin{aligned} \sum_{k=0}^{n} | P_{nk} | &\geq \left(1 + \frac{4NC}{\pi^2(1 + M)^2} \log^{2\varepsilon'}(n + 1)\right)^{1/2} \\ &\geq \frac{2(NC)^{1/2}}{\pi(1 + M)} \log^{\varepsilon'}(n + 1). \end{aligned} \tag{3.1}$$

Now since $\theta_n = O(n^{-1/2} \log^{\varepsilon}(n + 1))$ we have $\lim_{k\to\infty} \theta_k = 0$ and

$$\theta_k \geq Cn^{-1/2} \log^{\varepsilon} (n + 1)$$

for $j \leq n$ and n sufficiently large with the possible exception of the first few terms since the sequence $n^{-1/2} \log^{\varepsilon} (n + 1)$ is ultimately monotone. We now specify $\varepsilon' < \varepsilon$ to be sufficiently small so that $\theta_k \geq Cn^{-1/2} \log^{\varepsilon'} (n + 1)$ for all $j \leq n$. Then from (3.1) we see that $\sum_{k=0}^{n} | P_{nk} |$ fails to be uniformly bounded for all n and the $[F, d_n]$ matrix is not regular.

If we choose $\rho_n = 1$ for all n and $\varepsilon = \frac{1}{2}$ we obtain Meir's example [4].

REFERENCES

1. V. F. COWLING AND C. L. MIRACLE, *Some results for the generalized Lototsky transform*, Canad. J. Math, vol. 14 (1962), pp. 418–435.
2. ———, *Corrections and remarks on some results for the generalized Lototsky transform*, Canad. J. Math., vol. 16 (1964), pp. 423–428.

3. A. Jakimovski, *A generalization of the Lototsky method of summability*, Michigan Math. J., vol. 6 (1959), pp. 277–290.
4. A. Meir, *On two problems concerning the generalized Lototsky transforms*, Canad. J. Math., vol. 16 (1964), pp. 339–342.

Kansas State University
Manhattan, Kansas