INTEGRAL REPRESENTATIONS OF SOLUTIONS OF THE GENERALIZED HEAT EQUATION

BY

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1. Introduction

In recent papers of F. M. Cholewinski and D. T. Haimo, [1], and of D. T. Haimo, [3], characterizations were derived for generalized temperature functions, defined for positive time, which may be represented by Poisson-Hankel-Stieltjes integral transforms. It is our aim here to explore the problem for generalized temperature functions considered over negative time. Although the representation theorems obtained can be proved by techniques analogous to those of the previous results, we use the more elegant approach of appealing to the Appell transform to reduce these cases to those dealt with earlier. In addition, we investigate some other generalized temperature functions which have integral representations. Some of the results are extensions, in part, of the work of D. V. Widder in [7].

2. Definitions and preliminary results

We need the following basic definitions.

DEFINITION 2.1. The generalized heat equation is

$$\Delta_x u(x, t) = \frac{\partial}{\partial t} u(x, t), \tag{2.1}$$

where $\Delta_x f(x) = f''(x) + (2\nu/x)f'(x)$, ν a fixed positive number.

DEFINITION 2.2. A generalized temperature function is a function of class C^2 which satisfies the generalized heat equation. We denote the class of such functions by H.

DEFINITION 2.3. The fundamental solution of the generalized heat equation is the function

$$G(x, y; t) = (1/2t)^{\nu+1/2} e^{-(x^2+y^2)/4t} \mathcal{I}(xy/2t), \tag{2.2}$$

where $\mathcal{I}(z) = 2^{\nu-1/2}\Gamma(\nu + 1/2)z^{1/2-\nu}I_{\nu-1/2}(z)$, $I_\alpha(z)$ being the Bessel function of imaginary argument of order α. We write $G(x; t)$ for $G(x, 0; t)$.

DEFINITION 2.4. If $V(x, t)$ is an arbitrary function of two variables, its Appell transform $V^A(x, t)$ is given by

$$V^A(x, t) = V^A_{x,t}(x, t) = G(x; t)V(x/t, -1/t). \tag{2.3}$$

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We next define a subclass H^* of H which plays an important role in our theory.

DEFINITION 2.5. A generalized temperature function $u(x, t)$ is a member of H^* for $a < t < b$, if and only if, for every t, t', $a < t' < t < b$,

$$u(x, t) = \int_0^\infty G(x, y; t - t')u(y, t')\, d\mu(y), \qquad d\mu(x) = \frac{2^{1/2-\nu}}{\Gamma(\nu + 1/2)} x^{2\nu}\, dx, \tag{2.4}$$

the integral converging absolutely. Functions in H^* are said to have the Huygens property.

As proved in Theorem 6.4 of [3], functions in H^* have a complex integral representation as well. Indeed, we have the following result.

LEMMA 2.6. *If $u(x, t) \in H^*$, $a < t < b$, then*

$$u(x, t) = \int_0^\infty G(ix, y; t' - t)u(iy, t')\, d\mu(y), \qquad a < t < t' < b. \tag{2.5}$$

A fundamental result is the invariance of membership in H^* under an Appell transformation. This is made explicit in the following lemma.

LEMMA 2.7. *If $u(x, t) \in H^*$ for $a < t < b$, then $u^A(x, t) \in H^*$ for $-1/a < t < -1/b$.*

Proof. That $u^A(x, t) \in H$ for $-1/a < t < -1/b$ may be verified, making use of the fact that

$$\Delta_x[f(x)g(x)] = f(x)\Delta_x g(x) + g(x)\Delta_x f(x) + 2f'(x)g'(x).$$

Now, consider

$$\begin{aligned}\int_0^\infty G(x, y; t - t')u^A(y, t')\, d\mu(y) &= \int_0^\infty G(x, y; t - t')G(y, t')u(y/t', -1/t')\, d\mu(y) \\ &= G(x; t)\int_0^\infty G(x/t, y; 1/t' - 1/t)u(y, -1/t')\, d\mu(y).\end{aligned}$$

Since $u(x, t) \in H^*$ for $a < t < b$, the integral on the right reduces to $u(x/t, -1/t)$ for $a < -1/t' < -1/t < b$. Hence

$$\begin{aligned}\int_0^\infty G(x, y; t - t')u^A(y, t')\, d\mu(y) &= G(x; t)u(x/t, -1/t), \qquad a < -1/t' < -1/t < 1/b, \\ &= u^A(x, t), \qquad -1/a < t' < t < -1/b\end{aligned}$$

and the lemma is proved.

We shall be concerned with the following integral transforms. See [1], [2].

DEFINITION 2.8. The Hankel transform $\varphi^\wedge(x)$ of a function φ defined on $(0, \infty)$ is given by

$$\varphi^\wedge(x) = \int_0^\infty \mathcal{J}(xy)\varphi(y)\, d\mu(y), \qquad 0 \leq x < \infty, \tag{2.6}$$

where

$$\mathcal{J}(x) = 2^{\nu-1/2}\Gamma(\nu + 1/2)x^{1/2-\nu}J_{\nu-1/2}(x),$$

$J_\alpha(x)$ being the ordinary Bessel function of order α, whenever the integral converges. We write

$$\varphi_i^\wedge(x) = \int_0^\infty \mathcal{J}(xy)\varphi(y)\, d\mu(y), \qquad 0 \leq x < \infty, \tag{2.7}$$

whenever the integral (2.7) converges.

DEFINITION 2.9. The Hankel-Stieltjes transform $\varphi^{\wedge s}(x)$ of a function α of bounded variation in every finite interval is given by

$$\varphi^{\wedge s}(x) = \int_0^\infty \mathcal{J}(xy)\, d\alpha(y), \qquad 0 \leq x < \infty, \tag{2.8}$$

whenever the integral converges. We write, also,

$$\varphi_i^{\wedge s}(x) = \int_0^\infty \mathcal{J}(xy)\, d\alpha(y), \qquad 0 \leq x < \infty \tag{2.9}$$

whenever the integral converges.

DEFINITION 2.10. The Poisson-Hankel transform of a function φ integrable in every finite interval is the function $\varphi^P(x, t)$ given by

$$\varphi^P(x, t) = \int_0^\infty G(x, y; t)\varphi(y)\, d\mu(y), \qquad 0 \leq x < \infty, \tag{2.10}$$

whenever the integral converges.

DEFINITION 2.11. The Poisson-Hankel-Stieltjes transform of a function α of bounded variation in every finite interval is the function $\varphi^{Ps}(x, t)$ given by

$$\varphi^{Ps}(x, t) = \int_0^\infty G(x, y; t)\, d\alpha(y), \qquad 0 \leq x < \infty, \tag{2.11}$$

whenever the integral converges.

We know, by Theorem 6.2 of [3], that within the interval of absolute convergence of the integral (2.11), $\varphi^{Ps}(x, t) \in H^*$. Thus we may readily establish the following result.

LEMMA 2.12. *If $\varphi(x) \in L$, $0 \leq x < \infty$, then $\varphi^P(x, t) \in H^*$ for $t > 0$, and*

$$\varphi^P(x, t) = [e^{-tx^2}\varphi^\wedge(x)]^\wedge. \tag{2.12}$$

Proof. Since $\varphi(x) \in L$ for $0 \leq x < \infty$,

$$\int_0^\infty | G(x, y; t)\varphi(y) | \, d\mu(y) \leq (1/2t)^{\nu+1/2} \int_0^\infty | \varphi(y) | \, d\mu(y) < \infty, \qquad t > 0,$$

so that the integral defining $\varphi^P(x, t)$ converges absolutely. Hence by Theorem (2.6) of [3], $\varphi^P(x, t) \in H^*$ for $t > 0$ and the first part of the lemma is established. Further, we have

$$\begin{aligned} \varphi^P(x, t) &= \int_0^\infty G(x, y; t)\varphi(y) \, d\mu(y) \\ &= \int_0^\infty \varphi(y) \, d\mu(y) \int_0^\infty e^{-tu^2} \mathcal{J}(xu)\mathcal{J}(yu) \, d\mu(u) \\ &= \int_0^\infty e^{-tu^2} \mathcal{J}(xu)\varphi^\wedge(u) \, d\mu(u), \end{aligned}$$

where the interchange in integration is valid by Fubini's theorem. But the final integral is the right hand side of (2.12) and the proof is complete.

We also have a companion result.

LEMMA 2.13. *If $\varphi(x) \in L$, $0 \leq x < \infty$, then $\varphi^P(ix, -t) \in H^*$ for $t < 0$, and*

$$\varphi^P(ix, -t) = [e^{tx^2}\varphi^\wedge(x)]_i^\wedge. \tag{2.13}$$

Proof. Since the integral defining $\varphi^P(x, t)$ converges absolutely for $t > 0$, we know that $\varphi^P(x, t) \in H^*$ for $t > 0$. Hence, by Lemma 2.6, for $0 < t < t' < \infty$,

$$\varphi^P(x, t) = \int_0^\infty G(ix, y; t' - t)\varphi^P(iy, t') \, d\mu(y),$$

the integral converging absolutely, and we have,

$$\varphi^P(ix, t) = \int_0^\infty G(x, y; t' - t)\varphi^P(iy, t') \, d\mu(y), \qquad 0 < t < t' < \infty,$$

so that $\varphi^P(ix, -t) \in H^*$ for $t < 0$. Also,

$$\begin{aligned} \varphi^P(ix, -t) &= \int_0^\infty G(ix, y; -t)\varphi(y) \, d\mu(y) \\ &= \int_0^\infty \varphi(y) \, d\mu(y) \int_0^\infty \mathcal{J}(yu)\mathcal{I}(xu)e^{tu^2} \, d\mu(u) \\ &= \int_0^\infty \mathcal{I}(xu)e^{tu^2}\varphi^\wedge(u) \, d\mu(u), \end{aligned}$$

the interchange of integration being valid for $t < 0$, and hence the result.

We complete this section with a formula giving the value of an integral transform of an Appell transform of a function of H^*.

THEOREM 2.14. *If $u(x, t) \in H^*$ for $| t | < \sigma$, then for any $t > 1/\sigma$,*

$$u(2x, 0)e^{tx^2} = [u^A(x, t)]_i^\wedge . \tag{2.14}$$

Proof. We have

$$\begin{aligned} e^{-tx^2} \int_0^\infty \mathcal{J}(xy)u^A(y, t)\, d\mu(y) &= e^{-tx^2} \int_0^\infty \mathcal{J}(xy)G(y; t)u(y/t, -1/t)\, d\mu(y) \\ &= e^{-tx^2} \int_0^\infty \mathcal{J}(xyt)G(y; 1/t)u(y, -1/t)\, d\mu(y) \\ &= \int_0^\infty G(2x, y; 1/t)u(y, -1/t)\, d\mu(y). \end{aligned}$$

But, since $u(x, t) \in H^*$ for $| t | < \sigma$, the last integral is simply $u(2x, 0)$, for $t > 1/\sigma$, and the proof is complete.

3. Integral representation of Appell transforms

Before deriving the main representation theorems of this section, we need to establish two preliminary lemmas. To this end, we make the following definition.

DEFINITION 3.1. An even entire function

$$f(x) = \sum_{n=0}^\infty a_n x^{2n} \tag{3.1}$$

belongs to the class $(1, \sigma)$, or has growth $(1, \sigma)$, if and only if

$$\limsup_{n\to\infty} n \mid a_n \mid^{1/n} \leq e\sigma. \tag{3.2}$$

LEMMA 3.2. *If $u(x, t) \in H^*$ for $| t | < \sigma$, then $u(-2ix, 0) \in (1, 1/\sigma)$.*

Proof. By Theorem 5.1 of [5], we have that

$$u(x, t) = \sum_{n=0}^\infty a_n P_{n,\nu}(x, t), \qquad | t | < \sigma,$$

where $P_{n,\nu}(x, t)$ is the generalized heat polynomial studied in [5]. Further, by Theorem 3.8 of [5], it follows that $u(x, 0)$ is an even function of growth $(1, 1/4\sigma)$. Since

$$u(x, 0) = \sum_{n=0}^\infty a_n x^{2n},$$

we find that, by (3.2),

$$\limsup_{n\to\infty} n \mid 2^{2n} a_n \mid^{1/n} \leq e/\sigma.$$

Hence it readily follows thas $u(-2ix, 0) \in (1, 1/\sigma)$ as required.

LEMMA 3.3. *For $s, t > 0$,*

$$G_{x,t}^A(x, y; s + t) = G_{y,s}^A(x, y; s + t). \tag{3.3}$$

Proof. The result is immediate from the definitions as each side is equal to

$$\left[\frac{1}{4(st - 1)}\right]^{\nu+1/2} \exp\left(-\frac{x^2 s + y^2 t}{4(st - 1)}\right) \mathcal{J}\left(\frac{xy}{2(st - 1)}\right). \tag{3.4}$$

We now are ready to establish a principal result.

THEOREM 3.4. *A necessary and sufficient condition that*

$$u(x, t) = [e^{-tx^2}\varphi(x)]^\wedge, \qquad t > 1/\sigma, \tag{3.5}$$

where $\varphi(x) \epsilon (1, 1/\sigma)$ *is even, is that there exist a function* $v(x, t) \epsilon H^*$ *for* $|t| < \sigma$, *and such that* $u(x, t) = v^A(x, t)$.

Proof. To prove sufficiency, we note that since $v(x, t) \epsilon H^*$ for $|t| < \sigma$, then for any σ', $0 < \sigma' < \sigma$,

$$v(x, t) = \int_0^\infty G(x, y; t + \sigma')v(y, -\sigma')\, d\mu(y), \tag{3.6}$$

the integral converging absolutely for $-\sigma' < t < \sigma$. Further,

$$u(x, t) = v^A(x, t) = G(x, t)v(x/t - 1/t),$$

so that by (3.6) and (3.3), we have

$$\begin{aligned} u(x, t) &= \int_0^\infty G(y; \sigma')G\left(x, \frac{y}{\sigma'}; t - 1/\sigma'\right) v(y, -\sigma')\, d\mu(y) \\ &= \int_0^\infty G(y; \sigma')v(y, -\sigma')\, d\mu(y) \int_0^\infty e^{-(t-1/\sigma')s^2} \mathcal{J}(xs)\mathcal{J}\left(\frac{ys}{\sigma'}\right) d\mu(s), \end{aligned}$$

or, if the order of integration may be reversed,

$$\begin{aligned} u(x, t) &= \int_0^\infty \mathcal{J}(xs)e^{-ts^2}\, d\mu(s) \int_0^\infty G(-2is, y; \sigma')v(y, -\sigma')\, d\mu(y) \\ &= \int_0^\infty \mathcal{J}(xs)e^{-ts^2}v(-2is, 0)\, d\mu(s), \qquad t > 1/\sigma'. \end{aligned}$$

If we set $\varphi(s) = v(-2is, 0)$, then by Lemma (3.2), $\varphi(x) \epsilon (1, 1/\sigma)$. It is clearly an even function and we then have, for all $t > 1/\sigma$,

$$u(x, t) = v^A(x, t) = [e^{-tx^2}\varphi(x)]^\wedge.$$

To justify the interchange in order of integration, we observe that

$$\begin{aligned} &\int_0^\infty |\mathcal{J}(xs)|e^{-ts^2}\, d\mu(s) \int_0^\infty |G(-2is, y; \sigma')||v(y, -\sigma')|\, d\mu(y) \\ &\qquad \leq \int_0^\infty e^{-ts^2+s^2/\sigma'}\, d\mu(s) \int_0^\infty G(y; \sigma')|v(y, -\sigma')|\, d\mu(y). \end{aligned}$$

The inner integral converges by (3.6) with $x = t = 0$. Further, the outer integral converges for $t > 1/\sigma'$, and the proof of sufficiency of the condition is complete.

To prove the necessity of the condition, assume (3.5) with $\varphi(x)$ an even function of growth $(1, 1/\sigma)$. Then, by Theorem 6.1 of [5], we have

$$u(x, t) = \textstyle\sum_{n=0}^\infty b_n W_{n,\nu}(x, t), \qquad 0 \leq 1/\sigma < t, \tag{3.7}$$

with

$$b_n = (-1)^n \frac{\varphi^{(2n)}(0)}{2^{2n}(2n)!}, \tag{3.8}$$

where $W_{n,\nu}(x, t)$ is the Appell transform of the generalized heat polynomial $P_{n,\nu}(x, t)$. We thus have

$$v(x, t) = \sum_{n=0}^{\infty} b_n P_{n,\nu}(x, t), \qquad -\sigma < t < 0, \tag{3.9}$$

where $u(x, t) = v^A(x, t)$. Since the series (3.9) always converges in a strip symmetric about the x-axis of the x-t plane, its convergence for $-\sigma < t < 0$ implies its convergence for $|t| < \sigma$. Now, an appeal to Theorem 5.1 of [5] yields the fact that $v(x, t) \in H^*$ for $|t| < \sigma$. Further, by (3.8) and (3.9), we have

$$\begin{aligned} v(x, 0) &= \sum_{n=0}^{\infty} \varphi^{(2n)}(0)/(2n)!\,(ix/2)^{2n} \\ &= \varphi(ix/2), \end{aligned}$$

or

$$v(-2is, 0) = \varphi(s), \tag{3.10}$$

which is the relation between v and φ established in the sufficiency proof.

COROLLARY 3.5. *If $v(x, t) \in H^*$ for $a < t < b$, and if $a < t_0 < b$, then*

$$v_{x,t}^A(x, t + t_0) = [e^{-tx^2}\varphi(x)]^\wedge, \qquad t > 1/\sigma, \tag{3.11}$$

where $\sigma = \text{Min}\,(t_0 - a, b - t_0)$, and $\varphi(x) = v(-2ix, t_0)$ is an even function of growth $(1, 1/\sigma)$.

Proof. The hypothesis clearly implies that $v(x, t + t_0) \in H^*$ for $|t| < \sigma$. We may thus apply the theorem and the result is immediate.

An example illustrating the corollary is given by

$$v(x, t) = G(x; t) \tag{3.12}$$

which is in H^* for $0 < t < \infty$. Hence we have

$$\begin{aligned} G_{x,t}^A(x, t + a) &= [e^{-tx^2}G(-2ix, a)]^\wedge, & t > 1/a, \\ &= (1/2a)^{\nu+1/2} \int_0^\infty \mathcal{G}(xy) e^{-y^2(t-1/a)}\, d\mu(y), & t > 1/a. \end{aligned} \tag{3.13}$$

Note that $\varphi(x) = G(-2ix, a)$ is an even function of growth $(1, 1/a)$, and that the integral (3.13) converges in no larger region than that predicted in (3.11).

Actually, the region of convergence of the integral (3.11) as given by the corollary is not the largest possible in every instance as the following example illustrates. Consider, for $\sigma, \delta > 0$,

$$v(x, t) = G(ix; \sigma + \delta - t) + G(x; t), \tag{3.14}$$

which is in H^* for $0 < t < \sigma + \delta$. Hence, by the corollary,

$$\begin{aligned} v^A_{x,t}(x, t + \sigma) &= [G(ix; \delta - t) + G(x; t + \sigma)]^A \\ &= \int_0^\infty \mathcal{J}(xy)e^{-ty^2}[G(2y; \delta) + G(2iy; \sigma)]\, d\mu(y) \end{aligned} \tag{3.15}$$

with the integral (3.15) converging for $t > [1/\text{Min}\ (\sigma, \delta)]$. If $\sigma > \delta$, the corollary thus predicts the convergence of the integral (3.15) for $t > 1/\delta$, whereas, actually, it is clear that the integral converges in the larger region $t > 1/\sigma$.

To take care of such cases, we introduce a theorem to indicate the conditions under which Corollary 3.5 may be strengthened to give the larger region of convergence.

THEOREM 3.6. *If* $v(x, t) \in H^*$ *for* $-\sigma < t < \delta$, $\sigma, \delta > 0$, *then*

$$v^A(x, t) = [e^{-tx^2}\varphi(x)]^\wedge, \qquad t > 1/\sigma, \tag{3.16}$$

where $\varphi(x) = v(2ix, 0)$ *is an even entire function and*

$$\limsup_{x\to\pm\infty} \frac{\log |\varphi(x)|}{x^2} \leq \frac{1}{\sigma}. \tag{3.17}$$

Proof. Note that if $\sigma > \delta$, then Corollary 3.5, with $t_0 = 0$, asserts that $\varphi(x)$ is an even function of growth $(1, 1/\delta)$ and that (3.16) holds with the integral converging for $t > 1/\delta$ rather than the larger region $t > 1/\sigma$ indicated in the present theorem.

To prove the theorem, we note that since $v(x, t) \in H^*$ for $-\sigma < t < \delta$, we have, for any σ', $0 < \sigma' < \sigma$,

$$v(x, t) = \int_0^\infty G(x, y; t + \sigma')v(y, -\sigma')\, d\mu(y),$$

the integral converging absolutely for $-\sigma' < t < \delta$. Then, as in the proof of Theorem 3.4

$$v^A(x, t) = \int_0^\infty e^{-ts^2}\mathcal{J}(xs)\varphi(s)\, d\mu(s), \qquad t > 1/\sigma',$$

or, since σ' may be taken arbitrarily close to σ,

$$v^A(x, t) = [e^{-tx^2}\varphi(x)]^\wedge, \qquad t > 1/\sigma,$$

where $\varphi(x) = v(-2ix, 0)$, but we no longer can conclude that $\varphi(x) \in (1, 1/\sigma)$. Instead, we have, since $v(x, t) \in H^*$ for $-\sigma < t < \delta$,

$$\varphi(x) = \int_0^\infty G(2ix, y; \sigma')v(y, -\sigma')\, d\mu(y),$$

so that

$$|\varphi(x)| \leq e^{x^2/\sigma'} \int_0^\infty G(y; \sigma')\, |v(y, -\sigma')|\, d\mu(y).$$

It is thus immediate that

$$\limsup_{x\to\pm\infty} \frac{\log |\varphi(x)|}{x^2} \le \frac{1}{\sigma'},$$

or, since σ' may be taken arbitrarily close to σ, (3.17) is established and the proof is complete.

It is of interest to note that the condition $\delta > 0$ cannot be improved, as indicated by the example

$$v(x, t) = G(ix; -t),$$

which is in H^* for $-\infty < t < 0$. Here

$$v^A(x, t) = 1/2^{2\nu+1}$$

which cannot have a Hankel representation since such functions must vanish at ∞.

Consider

$$u(x, t) = \int_0^\infty e^{-ty^2} \mathcal{G}(xy) e^{-y^4}\, d\mu(y), \qquad -\infty < t < \infty. \tag{3.18}$$

Here $u(x, t) = [e^{-tx^2}\varphi(x)]^\wedge$ where $\varphi(x) = e^{-x^4}$ is an even function which is not of growth $(1, \sigma)$ for any σ. However, there exists a function

$$v(x, t) = \int_0^\infty G(ix, y; -t) e^{-y^4/16}\, d\mu(y) \tag{3.19}$$

in H^* for $t < 0$ and such that $u(x, t) = v^A(x, t)$. A modification of the necessity part of Theorem 3.4, to include such an example, is given by the following result.

THEOREM 3.7. *If*

$$u(x, t) = \int_0^\infty e^{-ty^2} \mathcal{G}(xy) \varphi(y)\, d\mu(y), \qquad t > 1/\sigma \ge 0, \tag{3.20}$$

where $\varphi(x)$ is an even function for which

$$\limsup_{x\to\pm\infty} \frac{\log |\varphi(x)|}{x^2} \le \frac{1}{\sigma}, \tag{3.21}$$

then there exists a function $v(x, t) \in H^*$ for $-\sigma < t < 0$ and such that $u(x, t) = v^A(x, t)$.

Proof. Hypothesis (3.21) implies that for any σ', $0 < \sigma' < \sigma$,

$$\varphi(x) = O(e^{x^2/\sigma'}), \qquad x \to \pm\infty. \tag{3.22}$$

Now, if we set

$$u(x, t) = G(x, t) v(x/t, -1/t)$$

$$= \int_0^\infty e^{-ty^2} \mathcal{J}(xy)\varphi(y)\, d\mu(y),$$

then, formally,

$$v(x, t) = \int_0^\infty G(ix, y; -t)\varphi(y/2)\, d\mu(y). \tag{3.23}$$

But by (3.22), the integral defining $v(x, t)$ is dominated by

$$(-1/2t)^{\nu+1/2} e^{-x^2/4t} \int_0^\infty e^{y^2/4t} O(e^{y^2/4\sigma'})\, d\mu(y)$$

and consequently converges absolutely for $-\sigma' < t < 0$, or since σ' may be taken arbitrarily close to σ, for $-\sigma < t < 0$. Now

$$w(x, t) = \int_0^\infty G(x, y; t)\varphi(y/2)\, d\mu(y) \tag{3.24}$$

may be shown, in a similar way, to converge absolutely for $0 < t < \sigma$. It follows, by Theorem 6.2 of [3], that $w(x, t) \in H^*$ for $0 < t < \sigma$. Hence, by Lemma 2.6

$$w(x, t) = \int_0^\infty G(ix, y; t' - t) w(iy, t')\, d\mu(y), \qquad 0 < t < t' < \sigma,$$

or, by Theorem 5.3 of [1]

$$w(ix, -t) = \int_0^\infty G(x, y; t' + t) w(iy, t')\, d\mu(y), \qquad -\sigma' < -t' < t < 0.$$

It thus follows that $w(ix, -t) \in H^*$ for $-\sigma' < t < 0$ and consequently for $-\sigma < t < 0$. Since $w(ix, -t) = v(x, t)$, the theorem is established.

4. Temperatures in positive time

As noted in the introduction, in [1] and in [3], criteria were established for a class of generalized temperature functions, defined for positive time, to be represented by a Poisson-Hankel-Stieltjes transform. In this section, we find that, in addition, different representation formulas hold as well, if the class of generalized temperature functions considered is further restricted, in each case, by an additional condition.

THEOREM 4.1. *A necessary and sufficient condition that*

$$u(x, t) = [e^{-tx^2}\varphi(x)]^\wedge, \qquad 0 < t < \infty, \tag{4.1}$$

where $\varphi(x) = \alpha^{\wedge s}(s)$ *for a bounded, non-decreasing function* α, *is that, for* $0 < t < \infty$, $u(x, t) \in H$, $u(x, t) \geq 0$, *and for some* $t_0 > 0$,

$$\int_0^\infty u(x, t_0)\, d\mu(x) < \infty. \tag{4.2}$$

Proof. If we assume that, for $0 < t < \infty$, $u(x, t) \in H$ and $u(x, t) \geq 0$, then by Theorem 9.1 of [1], we have

$$u(x, t) = \int_0^\infty G(x, y; t)\, d\alpha(y), \qquad 0 < t < \infty, \tag{4.3}$$

for some $\alpha(y) \uparrow$. Further, if (4.2) is also assumed to hold, then, using (4.3), we find that

$$\begin{aligned} \int_0^\infty u(x, t_0)\, d\mu(x) &= \int_0^\infty d\mu(x) \int_0^\infty G(x, y; t_0)\, d\alpha(y) \\ &= \int_0^\infty d\alpha(y) \\ &< \infty. \end{aligned} \tag{4.4}$$

By virtue of (4.4), we may define

$$\varphi(x) = \int_0^\infty \mathcal{J}(xu)\, d\alpha(u), \qquad 0 \leq x < \infty. \tag{4.5}$$

We then have

$$\begin{aligned} \int_0^\infty e^{-ty^2}\mathcal{J}(xy)\varphi(y)\, d\mu(y) &= \int_0^\infty e^{-ty^2}\mathcal{J}(xy)\, d\mu(y) \int_0^\infty \mathcal{J}(yu)\, d\alpha(u) \\ &= \int_0^\infty G(x, u; t)\, d\alpha(u) \\ &= u(x, t), \end{aligned} \tag{4.6}$$

where the change in order of integration is valid by (4.4). Hence the condition is sufficient.

Conversely, assume that (4.1) holds, with $\varphi(x)$ given by (4.5) for some bounded, non-decreasing function α. Then, as in (4.6), we find that

$$\begin{aligned} u(x, t) &= \int_0^\infty e^{-ty^2}\mathcal{J}(xy)\varphi(y)\, d\mu(y) \\ &= \int_0^\infty G(x, y; t)\, d\alpha(u), \qquad 0 < t < \infty, \end{aligned}$$

so that an appeal to Theorem 9.1 of [1] confirms the fact that $u(x; t) \in H$ and $u(x, t) \geq 0$ for $0 < t < \infty$. Further, since

$$\int_0^\infty u(x, t_0)\, d\mu(x) = \int_0^\infty d\alpha(y)$$

and $\alpha(y)$ is a non-decreasing, bounded function, (4.2) holds for every $t_0 > 0$, and the proof is complete.

Note that the functions considered in this theorem form a proper subclass of the positive generalized temperature functions studied in [1]. An illustration is given by

$$u(x, t) = x^2 + 2(1 + 2\nu)t. \tag{4.7}$$

This is a positive generalized temperature function for which (4.2) fails to hold for any t_0. As predicted by Theorem 9.1 of [1], it has a Poisson-Hankel-Stieltjes representation

$$x^2 + 2(1 + 2\nu)t = \int_0^\infty G(x, y; t)\, d\alpha(y) \tag{4.8}$$

with

$$\alpha(y) = \frac{y^{2\nu+3}}{2^{\nu-1/2}(2\nu + 3)\Gamma(\nu + 1/2)}, \tag{4.9}$$

so that clearly $\alpha(y) \uparrow$. It does not have a representation of the form (4.1). On the other hand, for $a > 0$,

$$u(x, t) = G(x; t + a) \tag{4.10}$$

satisfies (4.2) for every $t_0 > 0$. Indeed, we have

$$G(x; t + a) = \int_0^\infty \mathcal{J}(xu)e^{-tu^2}\varphi(u)\, d\mu(u), \tag{4.11}$$

with

$$\varphi(x) = e^{-ax^2} = \int_0^\infty \mathcal{J}(xy)\, d\alpha(y) \tag{4.12}$$

where $d\alpha(y) = G(y; a)\, d\mu(y)$, so that α is bounded and nondecreasing.

5. Temperatures in negative time

In this section, we investigate the question of integral representation for generalized temperature functions considered over negative time. In the event that the functions themselves are positive, we have the following result.

THEOREM 5.1. *A necessary and sufficient condition that*

$$u(x, t) = \int_0^\infty e^{ty^2}\mathcal{J}(xy)\, d\alpha(y), \qquad -\infty < t < 0, \tag{5.1}$$

with $\alpha(y) \uparrow$, *is that, for* $-\infty < t < 0$, $u(x, t) \in H$ *and* $u(x, t) \geq 0$.

Proof. If (5.1) holds, with $\alpha(y) \uparrow$, then clearly $u(x, t) \geq 0$, and since the kernel of the integral (5.1) $\in H$ for each y, so is the integral by the validity of differentiation under the integral sign. Hence the condition is necessary.

Conversely, assuming that $u(x, t) \in H$ and $u(x, t) \geq 0$, for $-\infty < t < 0$, we have that

$$u^A(x, t) = G(x; y)u(x/t, -1/t) \tag{5.2}$$

is non-negative and in H for $0 < t < \infty$. We may thus apply Theorem 9.1 of [1] to get

$$u^A(x, t) = \int_0^\infty G(x, y; t)\, d\beta(y), \qquad 0 < t < \infty,$$

with $\beta(y) \uparrow$. Hence

$$u(x, t) = \int_0^\infty e^{ty^2} \mathcal{g}(xy)\, d\alpha(y), \qquad -\infty < t < 0,$$

where $\alpha(y) = \beta(2y)$ and the theorem is proved.

By applying the theorem to the function $u(x, t + c)$ we readily derive the following extension.

COROLLARY 5.2. *A necessary and sufficient condition that*

$$u(x, t) = \int_0^\infty e^{ty^2} \mathcal{g}(xy)\, d\alpha(y), \qquad -\infty < t < c, \tag{5.3}$$

with $\alpha(y) \uparrow$, *is that, for* $-\infty < t < c$, $u(x, t)$ *be a non-negative generalized temperature function.*

Theorem 5.1 is illustrated by the example

$$u(x, t) = G(ix; -t) = \int_0^\infty e^{ty^2} \mathcal{g}(xy)\, d\mu(y), \qquad -\infty < t < c. \tag{5.4}$$

The function (5.4) does not satisfy the condition

$$\int_0^\infty u(x, t_0) e^{x^2/4t_0}\, d\mu(x) < \infty \tag{5.5}$$

for any $t_0 < 0$. By adding such a restriction to the functions considered in Theorem 5.1, we obtain a subclass of temperature function which in addition to (5.1) have an alternative integral representation as given in the following result.

THEOREM 5.3. *A necessary and sufficient condition that*

$$u(x, t) = \int_0^\infty G(ix, y; -t)\varphi(y)\, d\mu(y), \qquad -\infty < t < 0, \tag{5.6}$$

where $\varphi(x) = \alpha^{\wedge s}(x)$, *for some non-decreasing, bounded function* α, *is that, for* $-\infty < t < 0$, $u(x, t) \in H$, $u(x, t) \geq 0$, *and for some* $t_0 < 0$,

$$\int_0^\infty u(x, t_0) e^{x^2/4t_0}\, d\mu(x) < \infty. \tag{5.7}$$

Proof. To establish the necessity of the condition, assume (5.6) with

$$\varphi(x) = \int_0^\infty \mathcal{g}(xy)\, d\alpha(y), \tag{5.8}$$

where $\alpha(y)$ is a non-decreasing bounded function. Then, substituting (5.8) in (5.6), we find that

$$u(x, t) = \int_0^\infty G(ix, y; -t)\, d\mu(y) \int_0^\infty \mathcal{J}(yz)\, d\alpha(z)$$

$$= \int_0^\infty e^{tz^2} \mathcal{J}(xz)\, d\alpha(z), \qquad -\infty < t < 0,$$

where the inversion of order of integration is valid, for $t < 0$, since

$$\int_0^\infty d\alpha(z) \int_0^\infty e^{y^2/4t}\, d\mu(y) < \infty, \qquad t < 0. \tag{5.10}$$

Since (5.9) holds, an appeal to Theorem 5.1 yields the fact that $u(x, t) \in H$ and $u(x, t) \geq 0$ for $-\infty < t < 0$. Further

$$\begin{aligned} \int_0^\infty u(x, t) e^{x^2/4t}\, d\mu(x) &= \int_0^\infty e^{tz^2}\, d\alpha(z) \int_0^\infty e^{x^2/4t} \mathcal{J}(xz)\, d\mu(x) \\ &= \int_0^\infty e^{tz^2} G(iz; -1/4t)\, d\alpha(z) \\ &= (-2t)^{\nu+1/2} \int_0^\infty d\alpha(z) < \infty, \end{aligned} \tag{5.11}$$

so that (5.7) holds and the condition is necessary.

Conversely, suppose that, for $-\infty < t < 0$, $u(x, t)$ is a non-negative generalized temperature function for which (5.7) holds for some $t_0 < 0$. By Theorem 5.1, we then have

$$u(x, t) = \int_0^\infty e^{ty^2} \mathcal{J}(xy)\, d\alpha(y), \qquad -\infty < t < 0, \tag{5.12}$$

for some $\alpha(y) \uparrow$. Since (5.7) holds for $t_0 < 0$, the left hand side of (5.11) is finite for t_0. Hence from the right hand side of (5.11), it follows that $\alpha(y)$ is bounded. Hence since $\alpha(y)$ is a bounded, monotonic increasing function, the integral $\int_0^\infty \mathcal{J}(xy)\, d\alpha(y)$ exists and defines a function of x. Let

$$\varphi(x) = \int_0^\infty \mathcal{J}(xy)\, d\alpha(y).$$

Then we derive the representation (5.6) by a computation as in the first part of the proof, and the theorem is established.

As an application of Theorem 5.1, we have the following result.

THEOREM 5.4. *If $u(x, t) \in H$ and $u(x, t) \geq 0$ for $-\infty < t \leq c$, and if*

$$\text{Max}_{|x| \leq r}\, u(x, t) = M(r),$$

then

$$\liminf_{r \to \infty} \frac{\log M(r)}{r} \leq 0$$

implies that $u(x, t)$ is constant for $-\infty < t \leq c$.

Proof. Without loss of generality, we may assume that $c = 0$. Now, suppose that $y = y_0 > 0$ is a point of increase of $\alpha(y)$ given in Theorem 5.1. Then, we have

$$u\,(x, 0) = \int_0^\infty \mathcal{G}(xy)\, d\alpha(y) \geq \int_{y_0-\delta}^{y_0+\delta} \mathcal{G}(xy)\, d\alpha(y) > k\mathcal{G}(x(y_0 - \delta)),$$

where

$$k = \alpha(y_0 + \delta) - \alpha(y_0 - \delta) > 0$$

and δ is such that $y_0 - \delta > 0$. Hence

$$M(r) \geq k\mathcal{G}(r(y_0 - \delta))$$

so that

$$\liminf_{r\to\infty} M(r)/r = \infty$$

contradicting the hypothesis. Hence $\alpha(y)$ has at most one point of increase at $y = 0$ and $u(x, t)$ is constant.

Note that any generalized temperature function which is uniformly bounded for $-\infty < t \leq c$ is necessarily constant.

Somewhat different criteria for functions in H will also yield a representation of the form (5.1) as indicated in the following theorem.

THEOREM 5.5. *A necessary and sufficient condition that*

$$u(x, t) = \int_0^\infty e^{ty^2}\mathcal{G}(xy)\, d\alpha(y), \qquad t < c < 0, \tag{5.13}$$

with

$$\int_0^\infty e^{cy^2}\, | d\alpha(y)| < \infty, \tag{5.14}$$

is that $u(x, t) \in H$ for $t < c < 0$, and that, for $t < c < 0$,

$$\int_0^\infty | u(x, t) | G(x; c - t)\, d\mu(x) < M. \tag{5.15}$$

Proof. If $u(x, t) \in H$ for $t < c < 0$, then $u^A(x, t) \in H$ for $0 < t < -1/c$. Further, if (5.15) holds for $t < c < 0$, then

$$\begin{aligned}\int_0^\infty &| u^A(x, t) | G(x; -1/c - t)\, d\mu(x) \\ &= (-c/2)^{\nu+1/2} \int_0^\infty | u(x, -1/t) | G(x; c + 1/t)\, d\mu(x) \\ &< \infty, \qquad 0 < t < -1/c.\end{aligned} \tag{5.16}$$

Hence, by Theorem 8.1 of [3], we have

$$u^A(x, t) = \int_0^\infty G(x, y; t)\, d\beta(y), \qquad 0 < t < -1/c, \tag{5.17}$$

with

(5.18) $$\int_0^\infty G(y; -1/c) \, | d\beta(y) | < \infty.$$

But (5.17) gives

$$u(x, t) = \int_0^\infty e^{y^2 t} \mathcal{G}(xy) \, d\beta(2y), \qquad t < c < 0,$$

or, taking $\alpha(y) = \beta(2y)$, we obtain

(5.19) $$u(x, t) = \int_0^\infty e^{y^2 t} \mathcal{G}(xy) \, d\alpha(y), \qquad t < c < 0,$$

with

$$\int_0^\infty G(2y; -1/c) \, | d\beta(2y) | = (-c/2)^{\nu+1/2} \int_0^\infty e^{cy^2} \, | d\alpha(y) | < \infty$$

so that the sufficiency is established.

To prove the necessity of the condition, note that if (5.13), (5.14) hold, then $u(x, t) \in H$ for $t < c < 0$, since differentiation under the integral sign is valid. Further, for $t < c < 0$,

$$\begin{aligned} \int_0^\infty | u(x, t) | \, G(x; c - t) \mathcal{G}\mu(x) &\le \int_0^\infty G(x; c - t) \, d\mu(x) \int_0^\infty e^{ty^2} \mathcal{G}(xy) \, | d\alpha(y) | \\ &= \int_0^\infty e^{ty^2} \, | d\alpha(y) | \int_0^\infty \mathcal{G}(xy) G(x; c - t) \, d\mu(x) \\ &= \int_0^\infty e^{cy^2} \, | d\alpha(y) | \\ &< \infty, \end{aligned}$$

and (5.15) holds so that the proof is complete.

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