A METRIC CHARACTERIZATION OF $C(X)$ AND ITS GENERALIZATION TO C^*-ALGEBRAS[1]

BY
BARNETT W. GLICKFELD

Introduction

The Gelfand-Neumark representation theorem states that any complex Banach *-algebra with identity A which satisfies

(1) $\| a \| \| a^* \| = \| aa^* \|$, (2) $\| a \| = \| a^* \|$, and (3) $(1 + aa^*)^{-1}$

exists, for all $a \in A$, is completely isomorphic to a C^*-algebra. In [5] Glimm and Kadison showed that it is sufficient to only assume (1). The problem discussed here is the weakening of condition (1).

A crucial point in the proof of the commutative Gelfand-Neumark theorem is the proof that each Hermitian element h of A has a real spectrum. This point can be dealt with by a simple argument based on the fact that $\| \exp ih \| = 1$ if h is Hermitian. The significance of the exponential function in the Lorch analytic function theory [6], and the development of a theory of Cauchy-Riemann equations for that theory, valid only in *-algebras [4], make it plausible that the formula $\| \exp ih \| = 1$ is of more than accidental importance.

In this paper, we prove that any complex Banach *-algebra with identity A satisfying (1a) $\| \exp ih \| = 1$ when h is a Hermitian element of A, is completely isomorphic to a C^*-algebra. From this result it is easy to see that each of the stronger conditions (1b) $\| aa^* \| = \| a \| \| a^* \|$ if $a \in A$ and $aa^* = a^*a$, and (1c) there is a neighborhood V of 1 in A and a function $\xi : V \to$ reals so that $\xi(1) = 1$, ξ is continuous at 1, and $\| a \| \| a^* \| \leq \xi(aa^*)$ whenever a, a^*, and aa^* all lie in V, also implies that A is completely isomorphic to a C^*- algebra. Thus whether or not A is C^* may be determined by either (as in (1a)) inspecting the Hermitian elements of A, (as in (1b)) testing the commutative *-subalgebras of A, or (as in (1c)) considering only a neighborhood of the identity in A.

En route to the commutative theorem, we show that condition (4) there is a positive constant M so that $\| \exp ih \| \leq M$, all Hermitian h, implies that A is topologically *-isomorphic to a C^*-algebra. A closely related result appears in Lumer [8, p. 77]. Another relevant theorem appears in Lumer [7] where it is shown that condition (1d) $\| aa^* \| = (1 + o(z))\| a \| \| a^* \|$ for $z = \| 1 - a \| \to 0$, implies that A is topologically *-isomorphic to a C^*-algebra, and moreover, that (1d) implies (1a). Thus (1d) implies that A is completely isomorphic to a C^*-algebra.

Received February 2, 1966.

[1] This work was partially supported by a National Science Foundation grant.

The author wishes to thank E. R. Lorch for several valuable discussions. We also wish to thank W. B. Arveson for calling our attention to an important lemma, and E. Berkson for correcting a minor slip in a proof.

An announcement of the commutative theory presented here (i.e. the results of Section 1) appears in the January 1964 A.M.S. Notices; the proofs given in Section 1 also appear in the author's 1964 Columbia dissertation [4]. The January 1966 A.M.S. Notices contain an announcement of the non-commutative theory.

We note here that most of the results, both commutative and non-commutative, presented in this paper have been independently discovered by E. Berkson in a paper [1] submitted to the Illinois Journal in January 1965. Berkson obtains the commutative theorem via the theory of scalar type operators.

A new proof of the commutative theorem has since been completed by Palmer in his Harvard dissertation [9].

Throughout this paper, A will denote a complex Banach *-algebra with identity, i.e. a complex Banach algebra with identity and an involution $^* : A \to A$ which satisfies (1) $(za + b)^* = \bar{z}a^* + b^*$, (2) $(ab)^* = b^*a^*$, and (3) $a^{**} = a$, for elements a, b of A and complex numbers z. An element a of A will be called Hermitian iff $a = a^*$, normal iff $aa^* = a^*a$, and unitary iff $aa^* = a^*a = 1$. The set of Hermitian elements of A will be denoted by H. If $a \in A$, $r(a)$ will denote the spectral radius of a, and exp a the infinite sum $\sum_{n \geq 0} a^n/n!$.

1. The commutative theory

In Section 1, A will always be assumed commutative.

LEMMA 1.1. *If h is a Hermitian element of A, then*

$$\| \exp ih \| \geq r(\exp ih) \geq 1.$$

Proof. Let $x + yi \in \sigma(h)$, x, y real. Since $\sigma(h)$ is closed under complex conjugation, $x - |y| i \in \sigma(h)$. By the spectral mapping theorem,

$$\exp(|y| + xi) = \exp(i(x - |y| i))$$

lies in $\sigma(\exp ih)$. But $|\exp(|y| + xi)| = \exp |y| \geq 1$.

For the remainder of the paper, assume that A also satisfies condition (4) i.e. that there is a positive constant M so that $\| \exp ih \| \leq M$ if $h \in H$. It follows from 1.1 that $M \geq 1$.

LEMMA 1.2. *If h is Hermitian, then $\sigma(h)$ is real.*

Proof. Let $x + yi \in \sigma(h)$, where x and y are real. By the proof of 1.1, $x - |y| i \in \sigma(th)$, thus $\exp(t|y| + itx)$ lies in the spectrum of $\exp ith$. Hence $M \geq r(\exp ith) \geq \exp t|y|$, all $t > 0$, so $y = 0$.

LEMMA 1.3. *The Gelfand representation is a *-homomorphism of A into* $C(\mathfrak{M})$.

Proof. 1.3 follows directly from 1.2.

LEMMA 1.4. *There exists an* $\varepsilon > 0$ *so that* $\varepsilon < 1$ *and* $\| h^2 \| \geq \varepsilon$ *when h is Hermitian and* $\| h \| = 1$.

Proof. Let ε be some number between 0 and 1, assume there is some Hermitian h so that $\| h \| = 1$ and $\| h^2 \| \leq \varepsilon$. Then for $n \geq 1$,

$$\| h^{2n} \| \leq \| h^2 \|^n \leq \varepsilon^n \quad \text{and} \quad \| h^{2n+1} \| \leq \| h^{2n} \| \| h \| \leq \varepsilon^n.$$

Set $\delta = \sqrt[3]{\varepsilon}$; then for $n \geq 1$,

$$\| h^{2n} \| \leq \delta^{2n} \quad \text{and} \quad \| h^{2n+1} \| \leq \delta^{2n+1}.$$

So if $k \geq 2$, $\| h^k \| \leq \delta^k$. Now for $t > 0$,

$$\begin{aligned} M \geq \| \exp ith \| &\geq -1 + \| th \| - \| \textstyle\sum_{k \geq 2} (ith)^k/k! \| \\ &\geq -1 + t - \textstyle\sum_{k \geq 2} t^k \| h \|^k/k! \\ &\geq -1 + t - \textstyle\sum_{k \geq 2} (t\delta)^k/k! \\ &\geq -\exp t\delta + t. \end{aligned}$$

Hence $M + \exp t \geq t$, all $t > 0$. Setting $t = M + 2$ yields the inequality $\exp (M + 2)\delta \geq 2$. Thus δ cannot come arbitrarily close to 0; since $\varepsilon = \delta^3$, neither can ε.

LEMMA 1.5. *There exists an* $\varepsilon > 0$ *so that* $\varepsilon < 1$ *and* $\| h^2 \| \geq \varepsilon \| h \|^2$ *when h is Hermitian.*

Proof. 1.5 follows directly from 1.4 via normalization.

LEMMA 1.6. *If* ε *is as in the statement of* 1.5, *then* $r(h) \geq \varepsilon \| h \|$ *when h is Hermitian.*

Proof. By induction on N and 2.5, $\| h^{2^N} \| \geq \varepsilon^{2^N - 1} \| h \|^{2^N}$ for $N > 0$. Taking 2^N-th roots of this inequality, letting $N \to \infty$ and applying the spectral radius formula, we obtain $r(h) \geq \varepsilon \| h \|$.

LEMMA 1.7. *If* ε *is as in the statement of* 1.5, *then*

$$r(a) \geq \varepsilon \| a \|/2 \quad \text{when} \quad a \in A.$$

Proof. Set $a = h_1 + ih_2$, where h_1 and h_2 are Hermitian. By 1.2, $r(a) \geq r(h_i)$, $i = 1, 2$. Thus

$$2r(a) \geq r(h_1) + r(h_2) \geq \varepsilon \| h_1 \| + \varepsilon \| h_2 \|$$

by 1.6. But $\varepsilon \| h_1 \| + \varepsilon \| h_2 \| \geq \varepsilon \| a \|$.

Theorem 1.8. *If A is a commutative Banach *-algebra with identity such that there is a positive constant M so that $\| \exp ih \| \leq M$ when h is Hermitian, then the Gelfand representation of A is a topological *-isomorphism of A onto $C(\mathfrak{M})$.*

Proof. By 1.3, $\hat{}$ is a *-homomorphism. By 1.7, A is semisimple, so $\hat{}$ is an isomorphism. An application of the Stone-Weierstrass theorem shows that $\hat{}(A)$ is dense in $C(\mathfrak{M})$; since by 1.7 $\hat{}(A)$ is a complete subalgebra of $C(\mathfrak{M})$, $\hat{}(A) = C(\mathfrak{M})$. The continuity of $\hat{}^{-1}$ also follows from 1.6.

For the remainder of the paper, assume that the M in condition (4) can be taken to be 1. By 1.1 this is equivalent to requiring that $\| \exp ih \| = 1$ whenever h is Hermitian.

We now state the central theorem of this paper.

THEOREM 1.9. *If A is a commutative Banach *-algebra with identity so that $\| \exp ih \| = 1$ when h is Hermitian, then the Gelfand representation of A is an isometric *-isomorphism of A onto $C(\mathfrak{M})$.*

Proof. By 1.8, it is sufficient to prove that $\hat{}$ is isometric. Define a new norm $|||\ \ |||$ on $C(\mathfrak{M})$ via $|||f||| = \| \hat{}^{-1}(f) \|$. By 1.7 and 1.8, $|||\ \ |||$ and the sup norm $\|\ \ \|$ are equivalent norms for $C(\mathfrak{M})$; since $\hat{}$ is norm-decreasing $|||f||| \geq \|f\|$ for $f \in C(\mathfrak{M})$. We must show that $|||f||| = \|f\|$, all $f \in C(\mathfrak{M})$.

LEMMA 1.10. *Let φ be a real-valued function in $C(\mathfrak{M})$; then*

$$\| \exp i\varphi \| = ||| \exp i\varphi ||| = 1.$$

Proof. Set $a = \hat{}^{-1}(\varphi)$; then $||| \exp i\varphi ||| = \| \exp ia \| = 1$.

LEMMA 1.11. *Let $f \in C(\mathfrak{M})$ so that $f(F) \neq 0$, all $F \in \mathfrak{M}$. Suppose further that there is some direction $\exp ix_0$ (x_0 a real number) in the complex plane so that there is no F in $\mathfrak{M}$ such that $f(F) = \rho \exp ix_0$ with $\rho > 0$. Then $|||f||| = \|f\|$.*

Proof. Without loss of generality we can assume that $\|f\| = 2$. Let φ be a real-valued element of $C(\mathfrak{M})$ such that $\varphi = \operatorname{Arg} f$. Set Y equal to the intersection of the complex circle of radius 1 and center 1 with the closed upper half plane. Define $R : [0, 2] \to Y$ by setting $R(s)$ equal to that unique point of Y satisfying $|R(s)| = s$. Set

$$u = \inf \{|f(F)| : F \in \mathfrak{M}\} > 0;$$

define $S : [u, 2] \to$ reals via $S(s) = \operatorname{Arg} R(s)$, where $0 \leq \operatorname{Arg} R(s) \leq \pi/2$. If $F \in \mathfrak{M}$,

$$R(|f(F)|)e^{-iS(|f(F)|)}e^{i\varphi(F)} = |R(|f(F)|)|\, e^{i\operatorname{Arg} f(F)},$$

which is just $f(F)$. Thus

$$(R \circ |f|)e^{i(\varphi-(S\circ|f|))} = f = (R \circ |f| - 1)e^{i(\varphi-(S\circ|f|))} + e^{i(\varphi-(S\circ|f|))}.$$

Hence

$$|||f||| \leq |||(R \circ |f| - 1)e^{i(\varphi - (S_\circ |f|))}||| + |||e^{i(\varphi-(S_\circ|f|))}|||$$
$$\leq |||R \circ |f| - 1||| + 1. \qquad \text{(by 1.10)}$$

But clearly $R \circ |f| - 1$ can be written in the form $\exp i\psi$, where ψ is a real-valued function of $C(\mathfrak{M})$. By 1.10, $|||R \circ |f| - 1||| = 1$, so $|||f||| \leq 2 = \|f\|$. Therefore $|||f||| = \|f\|$.

LEMMA 1.12. *Let $f \in C(\mathfrak{M})$. Suppose further that there is a direction $\exp ix_0$ (x_0 a real number) in the complex plane so that there is no $F \in \mathfrak{M}$ such that $f(F) = \rho \exp ix_0$, $\rho > 0$. Then $|||f||| = \|f\|$.*

Proof. For $n = 1, 2, \cdots$ set $f_n = f - (\exp ix_0)/n$. By 1.11 $|||f_n||| = \|f_n\|$, all n; since f_n converges uniformly to f,

$$|||f||| = \lim |||f_n||| = \lim \|f_n\| = \|f\|.$$

LEMMA 1.13. *Let $f \in C(\mathfrak{M})$, $\|f\| = 1$; let N be a non-negative integer. Suppose there is a direction $\exp ix_0$ in the complex plane so that there are no $F \in \mathfrak{M}$ and $\rho > 1 - 2^{-N}$ satisfying $f(F) = \rho \exp ix_0$. Then $|||f||| = \|f\| = 1$.*

Proof. By induction on N. Note that 1.12 deals with the case $N = 0$. We thus assume that $N > 0$ and that 1.13 is valid for $N - 1$. Choose ε so that $0 < \varepsilon < 1/2^{N+2}$. Choose $\delta > 0$ so that

$$|f(F)| \leq 1 - 2^N + \varepsilon \quad \text{when} \quad x_0 - \delta \leq \operatorname{Arg} f(F) \leq x_0 + \delta \quad \text{and} \quad \delta < \pi.$$

We now divide the closed unit disc D of the complex plane into 6 closed sectors, as indicated in Figure 1.

We define a continuous function $G : D \to D$ by defining it on each of the six sectors separately as follows: if $z \in \text{I}$,

$$G(z) = z/2(1 - 2^{-N}).$$

If $z \in \text{II}$,

$$G(z) = z/2|z|.$$

If $z \in \text{III}$, write z in the form

$$z = r \exp i(x_0 - t\delta),$$

where $0 \leq t \leq 1$ and $0 \leq r \leq 1 - 2^{-N}$. Then

$$G(z) = z(1 + 2^{1-N}(t - 1))/2(1 - 2^{-N}).$$

If $z \in \text{IV}$, write

$$z = r \exp i(x_0 - t\delta),$$

where $0 \leq t \leq 1$ and $1 - 2^{-N} \leq r \leq 1$. Set

$$G(z) = z(1 + 2(t - 1)(1 - r))/2r.$$

If $z \in \text{V}$, write

$$z = r \exp i(x_0 + t\delta),$$

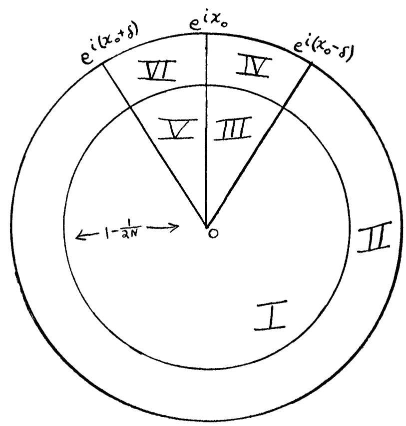

FIGURE 1

where $0 \leq t \leq 1$ and $0 \leq r \leq 1 - 2^{-N}$. Then define

$$G(z) = z(1 + 2^{1-N}(t - 1))/2(1 - 2^{-N}).$$

If $z \in \text{VI}$, write

$$z = r \exp i(x_0 + t\delta)$$

where $0 \leq t \leq 1$ and $1 - 2^{-N} \leq r \leq 1$. Set

$$G(z) = z(1 + 2(t - 1)(1 - r))/2r.$$

Among the relevant properties of $G(z)$ are: $G(z)$ is a continuous mapping of D into itself. If $z \in D$, $G(z)$ is a non-negative multiple of z, $|G(z)| \leq \frac{1}{2}$, $|G(z)| \leq |z|$, and

$$|z - G(z)| = |z| - |G(z)| \leq \tfrac{1}{2}.$$

Now define $g, h : \mathfrak{M} \rightarrow$ complex numbers via

$$g(F) = G(f(F)) \quad \text{and} \quad h(F) = f(F) - g(F).$$

Clearly $g, h \in C(\mathfrak{M})$ and $g + h = f$. By the above properties of G, $\| g \| \leq \frac{1}{2}$, $\| h \| \leq \frac{1}{2}$, and $g(F)$ and $h(F)$ are both non-negative multiples of $f(F)$, all F.

Suppose that $F \in \mathfrak{M}$, so that $g(F) \neq 0$ and $\operatorname{Arg} g(F) = x_0$. Then $\operatorname{Arg} f(F) = x_0$, by hypothesis $|f(F)| \leq 1 - 2^{-N}$. Thus $f(F)$ lies in sector III, so

$$|g(F)| \leq 2^{-1} - 2^{-N}.$$

Since $f = g + h$, $\| g \| = \frac{1}{2} = \| h \|$; so we can apply the induction hypothesis to $2g$; thus $\| g \| = ||| g ||| = \frac{1}{2}$.

Now suppose that $F \in \mathfrak{M}$ such that $h(F) \neq 0$ and $\operatorname{Arg} h(F) = x_0 - \delta$. Then $\operatorname{Arg} f(F) = x_0 - \delta$, by the choice of δ, $|f(F)| \leq 1 - 2^{-N} + \varepsilon$. If $f(F)$ lies in sector I, then

$$|h(F)| = |f(F)| - |G(f(F))| \leq 2^{-1} - 2^{-N}.$$

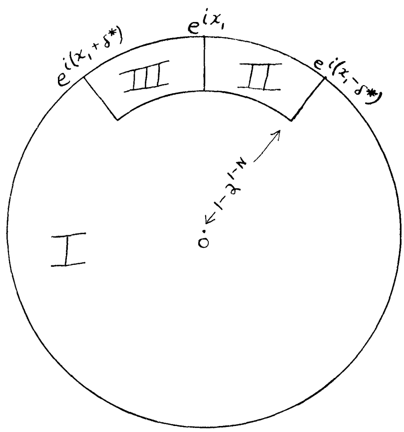

FIGURE 2

If $f(F) \epsilon$ sector II, then

$$| h(F) | = | f(F) | - | G(f(F)) | \leq 2^{-1} - 2^{-N} + \varepsilon.$$

Set $h_* = 2h$, then $\| h_* \| = 1$; furthermore there are no ρ and F such that

$$F \epsilon \mathfrak{M}, \quad \rho > 1 - 2^{-(N-1)} + 2\varepsilon \quad \text{and} \quad h_*(F) = \rho \exp i(x_0 - \delta).$$

We now focus our attention upon h_*. For convenience of notation, set $x_1 = x_0 - \delta$. Choose $\delta^* > 0$ so that $\delta^* < \pi$, and

$$| h_*(F) | \leq 1 - 2^{1-N} + 4\varepsilon$$

whenever $F \epsilon \mathfrak{M}$ and $x_1 - \delta^* \leq \operatorname{Arg} h_*(F) \leq x_1 + \delta^*$. Note that

$$1 - 2^{1-N} + 4\varepsilon < 1$$

by the construction of ε. Divide the closed unit disc into three closed sectors as indicated in Figure 2. We will define a continuous function $G^* : D \to D$ by defining it on each of the three sectors separately. We will not explicitly write out the formulas for G^*, but we will say what G^* does, and it will be clear that the formulas could be written out if necessary. On sector I, $G^*(z) = z$. On sectors II and III, $G^*(z)$ is a non-negative multiple of z

$$\text{and } 1 - 2^{1-N} \leq | G^*(z) | \leq | z |.$$

Furthermore, $G^*(t \exp ix_1) = (1 - 2^{1-N}) \exp ix_1$ when $1 - 2^{1-N} \leq t \leq 1$.

Now for $F \epsilon \mathfrak{M}$, set

$$h_1(F) = G^*(h_*(F)) \quad \text{and} \quad h_2(F) = h_*(F) - h_1(F);$$

$h_1, h_2 \epsilon C(\mathfrak{M})$ and $h_1 + h_2 = h_*$. Since $\| h_* \| = 1$, and $| h_*(F) | < 1$ when $h_*(F)$ lies in II or III, $\| h_1 \| = 1$. But if $h_1(F)$ lies on the ray through 0 and $\exp ix_1$, then $| h_1(F) | \leq 1 - 2^{-(N-1)}$ by the construction of h_1. Thus the inductive hypothesis can be applied: $||| h_1 ||| = \| h_1 \| = 1$. Therefore

$$|||h_*||| \leq |||h_1||| + |||h_2||| \leq 1 + |||h_2|||.$$

But $\| h_2 \| \leq 4\varepsilon$. Since $\| \quad \|$ and $||| \quad |||$ are equivalent norms, there is a positive constant v such that $||| \quad ||| \leq v\| \quad \|$. Thus $\| h_* \| \leq 1 + 4v\varepsilon$, so

$$|||f||| \leq |||g||| + |||h||| \leq 2^{-1} + 2^{-1}(1 + 4v\varepsilon).$$

Letting $\varepsilon \to 0$, we see that $|||f||| \leq 1$. Thus $|||f||| = \|f\| = 1$. 1.13 is proved.

LEMMA 1.14. *Let $f \in C(\mathfrak{M})$ such that $\|f\| = 1$; then $\|f\| = |||f|||$.*

Proof. Clearly f is the uniform limit of a sequence of functions f_n to which we can apply 1.13.

But now Theorem 1.9 is proved, as the restriction $\|f\| = 1$ of 1.14 is easily removed.

2. The non-commutative theory

We now remove the restriction that A be commutative. Still in effect is the requirement that $\| \exp ih \| = 1$ if h is Hermitian.

LEMMA 2.1. (Vidav [11]) *A can be renormed with the equivalent norm $||| \quad |||$ so that $(A, ||| \quad |||)$ is completely isomorphic to a C^*-algebra, and $|||h||| = \|h\|$ if h is Hermitian.*

LEMMA 2.2. *A can be renormed with the equivalent norm $||| \quad |||$ so that $(A, ||| \quad |||)$ is completely isomorphic to a C^*-algebra, and $\|a\| = |||a|||$ if a is normal.*

Proof. 2.2 follows easily from 1.9 and 2.1.

We must now pass from $\|a\| = |||a|||$, all normal a, to $\|a\| = |||a|||$, all $a \in A$.

LEMMA 2.3. (Russo and Dye [10]) *If A is completely isomorphic to a C^* algebra, and ϕ is a continuous linear mapping of A into a normed linear space X, then*

$$\|\phi\| = \sup\{\|\phi(a)\| : a \in A, a \text{ unitary}\}.$$

Now we can prove

THEOREM 2.4. *Let A be a Banach *-algebra with identity; suppose that $\| \exp ih \| = 1$ when h is Hermitian. Then A is completely isomorphic to a C^*-algebra.*

Proof. Let $||| \quad |||$ be as in 2.2.; let $1_A : (A, ||| \quad |||) \to (A, \| \quad \|)$ be defined by $1_A(a) = a$. By 2.2. $|||a||| = \|a\|$ for unitary a, so $\|a\| \leq |||a|||$ for all a by 2.3. But if $a_0 \in A$ and $\|a_0\| < |||a_0|||$ then

$$\|a_0a_0^*\| \leq \|a_0\| \, \|a_0^*\| < |||a_0||| \, |||a_0^*||| = |||a_0a_0^*||| = \|a_0a_0^*\|,$$

which is impossible. (The preceding argument is due to Bonsall [2].) Therefore $\| a \| = ||| a |||$, all $a \in A$; A is completely isomorphic to a C^*-algebra.

We conclude by proving the corollaries to 2.4 which are alluded to in the introduction.

COROLLARY 2.5. *Let A be a Banach *-algebra with identity. Suppose that $\| a \| \| a^* \| = \| aa^* \|$ when a is a normal element of A. Then A is completely isomorphic to a C^*-algebra.*

Proof. Set $S_N(a) = 1 + a + \cdots + a^N/N!$; S_N is the N-th partial sum of exp. If h is Hermitian, $S_N(ih)$ is normal and $S_N(ih)^* = S_N(-ih)$. Thus

$$\| S_N(ih) \| \| S_N(-ih) \| = \| S_N(ih)S_N(-ih) \|.$$

Letting $N \to \infty$ we see that

$$\| \exp ih \| \| \exp - ih \| = \| (\exp ih)(\exp - ih) \| = 1.$$

By 1.1 (which can easily be extended to non-commutative A) $\| \exp ih \| = 1$. By 2.4, A is completely isomorphic to a C^*-algebra.

COROLLARY 2.6. *Let A be a Banach *-algebra with identity. Suppose there is a neighborhood V of 1 in A and a function $\xi : V \to$ reals so that $\xi(1) = 1$, ξ is continuous at 1, and $\| a \| \| a^* \| \leq \xi(aa^*)$ whenever a, a^*, and aa^* all lie in V. Then A is completely isomorphic to a C^*-algebra.*

Proof. Again set $S_N(a) = 1 + a + \cdots + a^N/N!$. Choose an open neighborhood U of 0 so that $U = -U$ and $\exp iU$ is contained in the interior of V. Let h' be a Hermitian element which lies in U. Then $S_N(ih')^* = S_N(-ih')$. But eventually $S_N(ih')$ and $S_N(-ih')$ lie in V; since

$$\lim_{N\to\infty} S_N(ih')S_N(-ih') = (\exp ih')(\exp - ih') = 1,$$

eventually $S_N(ih')S_N(-ih')$ lies in V. So eventually

$$\| S_N(ih') \| \| S_N(-ih') \| \leq \xi(S_N(ih')S_N(-ih')),$$

letting $N \to \infty$ we see that

$$\| \exp ih' \| \| \exp - ih' \| \leq \xi(1) = 1.$$

But as in the proof of 2.5, an application of 1.1 shows that

$$\| \exp ih' \| = 1.$$

Now let h be an arbitrary Hermitian element of A. Choose a positive integer J so that $h' = h/J$ lies in U. Then

$$\| \exp ih \| = \| \exp iJh' \| \leq \| \exp ih' \|^J = 1,$$

so by 1.1 $\| \exp ih \| = 1$. Thus 2.4 can again be applied; A is completely ismorphic to a C^*-algebra.

BIBLIOGRAPHY

1. E. BERKSON, *Some characterizations of C^*-algebras*, Illinois J. Math., vol. 10 (1966), pp. 1–8.
2. F. BONSALL, *A minimal property of the norm in some Banach algebras*, J. London Math. Soc., vol. 29 (1954), pp. 156–164.
3. I. GELFAND AND M. NEUMARK, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Rec. Math. (Mat. Sbornik) N.S., vol. 12 (1943), pp. 197-213.
4. B. GLICKFELD, *Contributions to the theory of holomorphic functions in commutative Banach algebras with identity*, Columbia University dissertation, 1964.
5. J. GLIMM AND R. KADISON, *Unitary operators in C^*-algebras*, Pacific J. Math, vol. 10 (1960), pp. 547–556.
6. E. R. LORCH, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc., vol. 54 (1943), pp. 414–425.
7. G. LUMER, *Semi-inner product spaces*, Trans. Amer. Math. Soc., vol. 100 (1961), pp. 29–43.
8. ———, *Spectral operators, Hermitian operators, and bounded groups*, Acta Sci. Math., vol. 25 (1964), pp. 75–85.
9. T. PALMER, *Unbounded normal operators on Banach spaces*, Harvard University dissertation, 1965.
10. B. RUSSO AND H. DYE, *A note on unitary operators in C^*-algebras*, Duke Math. J., to appear.
11. I. VIDAV, *Eine metrische Kennzeichung der selbstadjungierten Operatoren*, Math. Zeitschr., vol. 66 (1956), pp. 121–128.

COLUMBIA UNIVERSITY
NEW YORK, NEW YORK
HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS