SPEED OF ENERGY PROPAGATION FOR PARABOLIC EQUATIONS

BY

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It is well known that in a system described by the heat equation an initial disturbance localized at the origin is felt everywhere at any positive value of time; that is, the disturbance propagates with infinite speed. Nevertheless, it seems physically clear that most of the energy introduced by the disturbance ought to spread over only a bounded region in a finite time; in some sense one-half the derivative of the diameter of this region as a function of time could be called the speed of energy propagation. In this note these ideas will be made precise and such a result established for the equation of heat conduction in n space variables. A weaker result of the same nature will then be established for the general second-order linear parabolic equation with Hölder-continuous coefficients.

THEOREM 1. *The heat equation*

$$u_t = \Delta u \tag{1}$$

in n space dimensions exhibits "almost finite speed of energy propagation" in the following sense: given any $\delta > 0$, then for the solution u of eq. (1) for impulse initial data,

$$u = \int_{E^n} K(x, t; \xi)\delta(\xi)\, d\xi,$$

where $\delta(\xi)$ is the "Dirac δ-function" and

$$K(x, t; \xi) = (4\pi t)^{-n/2} \exp\left(-\frac{1}{4t}\sum_{i=1}^{n}(x_i - \xi_i)^2\right)$$

is the heat kernel, there exists a function $a_\delta(t)$ having the form

$$a_\delta(t) = C\sqrt{t}$$

for some constant C (depending on n) such that

$$\int_{|x|>a_\delta(t)} u(x, t)\, dx \leq \delta \tag{2}$$

for $t \geq 0$.

Proof. Since $\int_{E^n} \delta(\xi)\, d\xi = 1$, the δ in eq. (2) represents the maximum fraction of the energy lying beyond $|x| = a_\delta(t)$. Clearly $a_\delta(t)$ satisifying (2) cannot be unique; for if $a_\delta(t)$ satisfies (2), so does any function $b_\delta(t)$ such that $b_\delta(t) \geq a_\delta(t)$ for all $t \geq 0$. Ideally, we would want the smallest

Received November 29, 1965.

[1] Research supported by NASA.

function $a_\delta(t)$ satisfying eq. (2). However, we shall here be satisfied with seeking a function having the required property; we shall show that it suffices to take $a_\delta(t)$ of the indicated form for a certain constant C. Clearly

$$u(x, t) = (4\pi t)^{-n/2} \exp\left(-\frac{1}{4t}\sum_{j=1}^{n} x_j^2\right) = (4\pi t)^{-n/2} e^{-r^2/4t},$$

where $r^2 = \sum_{j=1}^{n} x_j^2$. We are thus looking for $a_\delta(t)$ such that

$$(3) \quad (4\pi t)^{-n/2} \int_{r>a_\delta(t)} S_n r^{n-1} e^{-r^2/4t}\, dr = \pi^{-n/2} S_n \int_{y>a_\delta(t)/2\sqrt{t}} y^{n-1} e^{-y^2}\, dy \leq \delta,$$

where S_n is the surface area of the unit sphere in n-dimensional Euclidean space.

Before proceeding we determine the asymptotic expansion for

$$B(a) \equiv \int_a^\infty r^{n-1} e^{-r^2}\, dr.$$

Integrating by parts, we have

$$B(a) = \frac{1}{2} a^{n-2} e^{-a^2} + \frac{n-2}{2} \int_a^\infty r^{n-3} e^{-r^2}\, dr,$$

etc. This expansion is closely related to that for $\sqrt{\pi/2}\,(1 - \operatorname{erf} a)$, which it becomes for $n = 1$; that it is a valid asymptotic expansion for $B(a)$ is proved as for $\sqrt{\pi/2}\,(1 - \operatorname{erf} a)$ (Cf. [3, p. 37]). We thus have

$$| B(a) - \tfrac{1}{2} a^{n-2} e^{-a^2} | = o(a^{n-3} e^{-a^2})$$

as $a \to \infty$ (a real). Thus for $a \geq k_n$, where k_n is a constant depending on n,

$$(4) \qquad | B(a) | \leq a^{n-2} e^{-a^2}.$$

The condition expressed by (3) is just, in terms of the function B,

$$\pi^{-n/2} S_n\, B(a_\delta(t)/2\sqrt{t}) \leq \delta,$$

which will certainly be the case, by eq. (4), if

$$(a_\delta(t)/2\sqrt{t})^{n-2} e^{-a_\delta^2(t)/4t} \leq \pi^{n/2}\delta/S_n$$

and $a_\delta(t)/2\sqrt{t} \geq k_n$. Let K_n be the greatest value of α such that

$$\alpha^{n-2} e^{-\alpha^2} = \pi^{n/2}\delta/S_n\,;$$

then for $\alpha \geq K_n$ we have $\alpha^{n-2} e^{-\alpha^2} \leq \pi^{n/2}\delta/S_n$. We define $a_\delta(t)$ by

$$a_\delta(t) = 2 \max\{k_n, K_n\} \sqrt{t}\,;$$

it is clear that this function meets the requirements of the theorem.

We turn now to the differential equation

$$(5) \quad Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t) u_{x_i} + c(x, t) u - u_t = 0,$$

where the coefficients are defined and continuous in $E^n \times [0, T]$ for some $T > 0$. We assume that L is uniformly parabolic in $E^n \times [0, T]$; i.e., we require that there exist positive constants λ_0, λ_1 such that for any real n-vector ξ

$$\lambda_0 |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t)\xi_i \xi_j \leq \lambda_1 |\xi|^2$$

for $(x, t) \in E^n \times [0, T]$, where $|\xi|^2 = \sum_{i=1}^n \xi_i^2$. We assume also that the coefficients satisfy a Hölder condition with exponent α, $0 < \alpha \leq 1$:

$$|a_{ij}(x, t) - a_{ij}(x', t')| \leq A(|x - x'|^\alpha + |t - t'|^\alpha)$$

$$|b_i(x, t) - b_i(x', t)| \leq A|x - x'|^\alpha$$

$$|c(x, t) - c(x', t)| \leq A|x - x'|^\alpha,$$

provided $x, x' \in E^n$, $t, t' \in [0, T]$. Then it is known [1], [2] that a fundamental solution $\Gamma(x, t; \xi, \tau)$ exists and satisfies

$$|\Gamma(x, t; \xi, \tau)| \leq c_T(t - \tau)^{-n/2} \exp(-\lambda|x - \xi|^2/4(t - \tau)), \quad 0 < t \leq T,$$

where λ is a positive constant depending on A, λ_0, λ_1, and c_T is a constant depending on n *and on* T.

We shall prove

THEOREM 2. *Let $\delta > 0$ be arbitrary and let u be the solution of eq.* (5) *for impulse initial data (i.e., $u = \Gamma(x, t; 0, 0)$). Then for some constant C_T depending in general on n and T the function*

$$a_\delta(t) = C_T \sqrt{t}, \qquad 0 \leq t \leq T$$

satisfies

$$\int_{|x| > a_\delta(t)} u(x, t)\, dx \leq \delta, \qquad 0 \leq t \leq T. \tag{6}$$

Proof. This theorem is a simple consequence of Theorem 1, in view of the bound on $\Gamma(x, t; 0, 0)$. Indeed, setting $t' = t/\lambda$, we have

$$|u(x, t)| = |\Gamma(x, t; 0, 0)| \leq \text{const.}\ (t')^{-n/2} \exp(-x^2/4t'),$$

whence from Theorem 1 we conclude that there exists a constant C_T such that $a_\delta(t) = C_T\sqrt{t}$ satisfies inequality (6) for $t \in [0, T]$.

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