# A RIGIDITY THEOREM FOR SUBALGEBRAS OF LIE AND ASSOCIATIVE ALGEBRAS

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## Introduction

Let $L$ be a finite-dimensional Lie algebra over an algebraically closed field and let $\Gamma_n(L)$ be the Grassmann variety of $n$-dimensional subspaces of $L$; $\Gamma_n(L)$ is a projective algebraic variety. We denote by $\mathfrak{a}$ the (Zariski) closed subset of $\Gamma_n(L)$ consisting of all $n$-dimensional subalgebras. The algebraic group $\mathrm{Aut}(L)$ of automorphisms of the Lie algebra $L$ acts in an obvious manner as an algebraic transformation group on $\Gamma_n(L)$ and $\mathfrak{a}$ is stable under the action of $\mathrm{Aut}(L)$. Let $M$ be an $n$-dimensional subalgebra of $L$ and let $m$ denote the point of $\Gamma_n(L)$ corresponding to $M$. If $G$ is a subgroup of $\mathrm{Aut}(L)$, then we say that $M$ is a rigid subalgebra of $L$ with respect to $G$ if the orbit $G(m)$ is a (Zariski) open subset of $\mathfrak{a}$. Intuitively, this definition says that every small deformation of the subalgebra $M$ is trivial. It follows from the definition that there are only a finite number of conjugacy classes (under $G$) of rigid subalgebras of $L$. The following theorem gives a sufficient "infinitesimal" condition that a subalgebra $M$ be rigid.

THEOREMS 9.3, 11.4. *Let $L$ be a Lie algebra over an algebraically closed field (resp. over the field* **C** *of complex numbers) and let $G$ be an algebraic subgroup (resp. complex Lie subgroup) of* $\mathrm{Aut}(L)$ *with Lie algebra* $\mathfrak{g}$. *Let $M$ be a* subalgebra of $L$ *such that every crossed homomorphism of $M$ into the $M$-module $L/M$ is induced by a derivation $D \in \mathfrak{g}$ of $L$. Then $M$ is a rigid subalgebra of $L$ with respect to $G$.*

The idea of the proof is quite simple. Roughly speaking, we show that the tangent space of $\mathfrak{a}$ at $m$ is included in the space of crossed homomorphisms. Similarly, the tangent space to $G(m)$ at $m$ includes the space of crossed homomorphisms induced by elements of $\mathfrak{g}$. A simple result on algebraic transformation groups completes the proof in the algebraic case. In the complex case we also need an auxiliary result concerning linear Lie groups acting on algebraic sets.

The most natural subgroup of $\mathrm{Aut}(L)$ to consider is the group of inner automorphisms of $L$ (if it exists). In this case the infinitesimal condition for rigidity can be expressed by Lie algebra cohomology.

COROLLARY. *Let $L$ be a complex Lie algebra (resp. the Lie algebra of a linear algebraic group over an algebraically closed field) and let $G$ be the group of inner automorphisms of $L$.*

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(a) *Let $M$ be a subalgebra of $L$ such that $H^1(M, L/M) = 0$. Then $M$ is rigid with respect to $G$.*

(b) *There exists only a finite number of conjugacy classes (under $G$) of subalgebras $M'$ of $L$ such that $H^1(M', L/M') = 0$.*

The above results also hold for real Lie algebras, with only minor modifications.

The corollary above shows that a semi-simple subalgebra of a real or complex Lie algebra $L$ is always rigid. A Cartan subalgebra of $L$ is also rigid. A number of other examples of rigid subalgebras are given in 12.

The above results are easily carried over to subalgebras of associative algebras. In this case, of course, Lie algebra cohomology is replaced by the cohomology of associative algebras.

We also prove similar rigidity theorems for ideals (of a Lie or associative algebra) and for submodules.

We would like to add that a similar rigidity theorem holds for subgroups of Lie groups. In this case, Lie algebra cohomology is replaced by the (differentiable) cohomology of groups. Details will appear in a forthcoming paper by the author.

## 0. Preliminaries

All vector spaces and algebras over a field $k$ will be assumed finite dimensional. We shall use the notation $V = \oplus_{j \in J} W_j$ for both internal and external direct sums of vector spaces. In particular, we shall consider $W_j$ as a subspace of $V$ without further comment. If $V$ and $W$ are vector spaces, then $\mathrm{Hom}(V, W)$ denotes the vector space of all linear mappings of $V$ into $W$. This notation will be used even if $V$ or $W$ admit additional structures, such as that of a Lie algebra.

Let $V = W_1 \oplus W_2$ and let

$$H = \{T \in \mathrm{Hom}(V, V) \mid T(W_2) = \{0\} \quad \text{and} \quad T(V) \subset W_2\};$$

$H$ is a vector subspace of $\mathrm{Hom}(V, V)$ and is canonically isomorphic to $\mathrm{Hom}(W_1, W_2)$. Throughout this paper, if we are given a direct sum decomposition $V = W_1 \oplus W_2$, we shall identify $\mathrm{Hom}(W_1, W_2)$ with the subspace $H$ of $\mathrm{Hom}(V, V)$ by means of the canonical isomorphism.

Our basic reference for algebraic geometry will be [7]. In particular, an algebraic variety is always considered as a topological space, with the Zariski topology; subsets of an algebraic variety are given the induced topology. Algebraic groups (always over an algebraically closed field) are as in [6]. The closed subgroups of an algebraic group are just the algebraic subgroups. For details concerning the Lie algebra of a linear algebraic group, we refer to [3].

**R** (resp. **C**) denotes the field of real (resp. complex) numbers.

## 1. Linear coordinates on the Grassmannian variety

Let $V$ be a vector space over a field $k$ and let $n \leq \dim V$ be a positive integer. We denote by $\Gamma_n(V)$ the Grassmannian variety of $n$-dimensional subspaces of $V$. If $k$ is algebraically closed, then $\Gamma_n(V)$ has a natural structure of projective algebraic variety. (See, e.g., [6, Exposé 5, p. 5–12].) If $W$ is a subspace of $V$ of codimension $n$, we shall denote by $\Gamma_W$ the subset of $\Gamma_n(V)$ consisting of all $n$-dimensional subspaces of $V$ which are transversal to $W$. If $k$ is algebraically closed, then $\Gamma_W$ is an open subvariety of $\Gamma_n(V)$.

Let $V = U \oplus W$ with $\dim U = n$, let $P$ be the projection operator on $V$ with kernel $U$ and image $W$, and let $Q$ be the complementary projection (i.e., $Q = I - P$). If $T \in \mathrm{Hom}(U, W)$, then let $\varphi(T)$ denote the image of the operator $Q + T$; $\varphi(T)$ is an $n$-dimensional subspace of $V$ which is transversal to $W$, hence a point of $\Gamma_W$. Moreover, a straightforward argument shows that the map $T \to \varphi(T)$ is a bijection of $\mathrm{Hom}(U, W)$ onto $\Gamma_W$. If the base field $k$ is algebraically closed, it can be shown that $\varphi$ is an isomorphism of $\mathrm{Hom}(U, W)$ (considered as an algebraic variety) onto the open subvariety $\Gamma_W$ of $\Gamma_n(V)$. We omit the details of the proof.

## 2. The action of $GL(V)$ on $\Gamma_n(V)$.

We continue with the notation of 1. The group $GL(V)$ of automorphisms of the vector space $V$ acts in an obvious way on $\Gamma_n(V)$. Precisely, if $g \in GL(V)$ and if $W \in \Gamma_n(V)$, then $g \cdot W$ is the $n$-dimensional subspace $g(W)$ of $V$. If the base field $k$ is algebraically closed, then $\Gamma_n(V)$ is an algebraic transformation space for the algebraic group $GL(V)$ (see [6, Exposé 5, p. 5–13] for definitions). Let $g \in GL(V)$ be such that $U' = g(U)$ is transversal to $W$ and that $W' = g(W)$ is transversal to $U$. Let $P' = gPg^{-1}$ and let $Q' = gQg^{-1}$; $P'$ is the projection with kernel $U'$ and image $W'$ and $Q'$ is the complementary projection. It follows from the conditions imposed on $g$ that $P + Q'$ and $P' + Q$ are invertible operators.

LEMMA 2.1. *Let* $B = PQ'(P + Q')^{-1}$. *Then*: (i) $B(V) \subset W$; (ii) $B(W) = \{0\}$; *and* (iii) *the image of* $Q + B$ *is the subspace* $g(U)$.

Conditions (i) and (ii) imply that $B$ is a point of $\mathrm{Hom}(U, W)$ (which we have identified with a subspace of $\mathrm{Hom}(V, V)$ as described in 0). Condition (iii) implies that $B$ is the unique point of $\mathrm{Hom}(U, W)$ which corresponds to the subspace $g(U)$, i.e., that $\varphi(B) = g(U)$.

*Proof.* Condition (i) follows immediately from the definition of $B$. Furthermore, the definition of $B$ implies that $BP = (P - B)Q'$; this implies that $BPP' = 0$. This gives $BP(P' + Q) = 0$. But, since $P' + Q$ is invertible, this implies that $BP = 0$, hence that $B(W) = \{0\}$.

To prove (iii), it suffices to show that $P'(Q + B) = 0$. For this will imply that the image of $Q + B$ is included in $U'$. Since the dimension of this image is equal to that of $U'$, condition (iii) will follow. But, multiplying by the invertible operator $P + Q'$, we see that the equation $P'(Q + B) = 0$ is equivalent to

$$P'PQ' + P'Q(P + Q') = 0.$$

But we have

$$P'PQ' + P'Q(P + Q') = P'(P + Q)Q' + P'QP = P'Q' + P'QP = 0.$$

This finishes the proof.

## 3. The algebraic set of subalgebras of a Lie algebra

If $V$ and $W$ are vector spaces over a field $k$, we denote by $\mathrm{Alt}^n(V, W)$ the vector space of all alternating multilinear maps of the $n$-fold Cartesian product $V \times \cdots \times V$ into $W$. We note that $\mathrm{Alt}^1(V, W) = \mathrm{Hom}(V, W)$ and $\mathrm{Alt}^0(V, W) = W$. We set $\mathrm{Alt}(V, W) = \oplus_{n \geq 0} \mathrm{Alt}^n(V, W)$.

Let $L$ be a Lie algebra. If $S, T \in \mathrm{Alt}^1(V, L)$, we define

$$[S, T] \in \mathrm{Alt}^2(V, L)$$

to be the map

$$(x, y) \to [Sx, Ty] + [Tx, Sy].$$

We note that $[S, T] = [T, S]$. The map $(S, T) \to [S, T]$ is a bilinear mapping of $\mathrm{Alt}^1(V, L) \times \mathrm{Alt}^1(V, L)$ into $\mathrm{Alt}^2(V, L)$. Although we shall not be concerned with the fact here, we remark that this map can be extended to a bilinear map of $\mathrm{Alt}(V, L) \times \mathrm{Alt}(V, L)$ into $\mathrm{Alt}(V, L)$ which defines on $\mathrm{Alt}(V, L)$ the structure of a graded Lie algebra. For details, see [13].

Our reason for introducing the product $(S, T) \to [S, T]$ is that it gives us a convenient way of expressing the conditions that a subspace of a Lie algebra $L$ over a field $k$ be a subalgebra. Let $M$ be an $n$-dimensional subalgebra of $L$ and let $W$ be a subspace of $L$ of codimension $n$ which is transversal to $M$; thus $L = M \oplus W$. Let $P$ denote the projection operator on $L$ with kernel $M$ and image $W$ and let $Q$ be the complementary projection.

LEMMA 3.1. *Let characteristic $k \neq 2$, let $T \in \mathrm{Hom}(M, W)$, and let $V$ be the image of $Q + T$. Then $V$ is a subalgebra of $L$ if and only if $T$ satisfies the following equation:*

$$(P - T) \circ [Q + T, Q + T] = 0. \tag{3.2}$$

*Proof.* We observe that $Q + T$ is the projection operator with kernel $W$ and image $V$ and $P - T$ is the complementary projection.

Assume that (3.2) is satisfied and let $x, y \in V$. Then

$$x = (Q + T)(x) \quad \text{and} \quad y = (Q + T)(y).$$

Hence $2[x, y] = [Q + T, Q + T](x, y)$. Thus $(P - T)[x, y] = 0$, and it follows that $[x, y] \in V$. Thus $V$ is a subalgebra. The converse is proved similarly.

Let $\varphi : \mathrm{Hom}(M, W) \to \Gamma_W$ be defined as in 1. Let $\mathfrak{A}$ denote the subset of $\Gamma_n(L)$ consisting of all $n$-dimensional subalgebras of $L$. Then, according to Lemma 3.1, $\varphi(T) \in \mathfrak{A}$ if and only if $T$ satisfies equation (3.2). If the base field $k$ is algebraically closed, then it follows easily from this observation that $A$ is a closed subset of the algebraic variety $\Gamma_n(L)$.

## 4. The case of characteristic 2

If characteristic $k = 2$ and if $T \in \mathrm{Alt}^1(V, L)$, then $[T, T] = 0$. In this case it is convenient to introduce a quadratic mapping

$$q : \mathrm{Alt}^1(V, L) \to \mathrm{Alt}^2(V, L)$$

which corresponds to the map $T \to (\frac{1}{2}([T, T]$ in case the characteristic is distinct from 2. If $T \in \mathrm{Alt}^1(V, L)$ then $q(T)$ is the map $(x, y) \to [Tx, Ty]$. The map $q$ has the following properties:

(i) if $\lambda \in k$ and if $T \in \mathrm{Alt}^1(V, L)$, then $q(\lambda T) = \lambda^2 q(T)$; and
(ii) if $S, T \in \mathrm{Alt}^1(V, L)$, then

$$q(S + T) = q(S) + [S, T] + q(T).$$

We note that, in the case of characteristic 2, equation (3.2) should be replaced by

$$(P - T) \circ q(Q + T) = 0.$$

In order to avoid the repeated consideration of special cases, we shall henceforth assume that the base field of all algebras considered is of characteristic $\neq 2$. The arguments given can be immediately carried over to the case of characteristic 2 by systematically replacing all squares $[T, T]$ by $q(T)$ and using (i) and (ii) above. We omit further details.

## 5. The tangent space of an algebraic variety

Let $k$ be algebraically closed and let $Y$ be an algebraic variety over $k$. Let $y \in Y$, let $\mathcal{O}(y)$ denote the local ring of $y$ on $Y$, and let $\rho$ denote the homomorphism $f \to f(y)$ of $\mathcal{O}(y)$ into $k$. A tangent vector to $Y$ at $y$ is a $\rho$-derivation of $\mathcal{O}(y)$ into $k$, that is, a linear mapping $D$ of $\mathcal{O}(y)$ into $k$ such that

$$D(fg) = f(y)Dg + g(y)Df.$$

We denote by $T(Y, y)$ the tangent space of $Y$ at $y$. It is a finite-dimensional vector space over $k$. If $f : Y \to Y'$ is a morphism of algebraic varieties, then there is an induced linear mapping $df_y$ of $T(Y, y)$ into $T(Y', f(y))$; $df_y$ is called the derived mapping of $f$ at $y$. If $f' : Y' \to Y''$ is another morphism, then

$$d(f' \circ f)_y = df'_{f(y)} \circ df_y .$$

If $f : V \to W$ is a polynomial mapping of vector spaces, then (with the obvious identifications) the derived mapping of $f$ at $y$ is just the differential of $f$ at $y$, as defined in [3, p. 35]. If Y is a subvariety of a variety Y′ and if $f : Y \to Y'$ denotes the inclusion mapping, then $df_y$ is an injection. In particular, if $Y$ is a subvariety of a vector space $V$ and if $f : Y \to V$ is the injection morphism, then the image of $df_y$ can be identified with the linear subspace of $V$ consisting of all $x \in V$ which satisfy the following condition: for every polynomial function $P$ on $V$ which vanishes on $Y$, the differential $dP_y$ vanishes at $x$.

For a detailed discussion and proofs of the above results, we refer the reader to [7, Chapter VI]. We note that we have slightly rephrased the definition of the tangent space given in [7].

## 6. The tangent space of $\mathfrak{A}$

In this section we use the following notation: $L$ is a Lie algebra over the algebraically closed field $k$; $M$ is a subalgebra of $L$ of dimension $n$ and $W$ is a subspace of $L$ of codimension $n$ which is transversal to $M$; $P$ is the projection operator with image $W$ and kernel $M$ and $Q$ is the complementary projection; the map $\varphi : \mathrm{Hom}(M, W) \to \Gamma_W$ is defined as in 1; $\mathfrak{A}$ denotes the closed subset of $\Gamma_n(V)$ consisting of all $n$-dimensional subalgebras; in order to avoid confusing notation, we shall denote by $m$ the point of $\Gamma_n(V)$ corresponding to the subspace $M$.

In this section we shall compute what it is tempting to call the tangent space to $\mathfrak{A}$ at $m$; we note, however, that since $\mathfrak{A}$ is not in general irreducible, the tangent space $T(\mathfrak{A}, m)$ is not defined according to the definition we have chosen.

The vector space $\mathrm{Alt}^2(M, W)$ is naturally isomorphic to the linear subspace of $\mathrm{Alt}^2(L, L)$ consisting of all $S \in \mathrm{Alt}^2(L, L)$ which satisfy the following conditions: (i) $S(L \times L) \subset W$; and (ii) $S(x, y) = 0$ if either $x$ or $y$ is in $W$. Henceforth, we shall identify $\mathrm{Alt}^2(M, W)$ with the above-described subspace of $\mathrm{Alt}^2(L, L)$ by means of the natural isomorphism.

Let $\psi : \mathrm{Hom}(M, V) \to \mathrm{Alt}^2(L, L)$ be defined by

$$\psi(T) = (P - T) \circ [Q - T, Q - T].$$

The image of $\psi$ is included in $\mathrm{Alt}^2(M, W)$. Henceforth, we consider $\psi$ as a mapping of $\mathrm{Alt}^1(M, W) = \mathrm{Hom}(M, W)$ into $\mathrm{Alt}^2(M, W)$. Since $M$ is a subalgebra, it follows from Lemma 3.1 that $P \circ [Q, Q] = 0$. Thus

$$\psi(T) = 2P \circ [Q, T] + P \circ [T, T] - T \circ [Q, Q] - 2T \circ [Q, T] - T \circ [T,T].$$

An elementary computation shows that the derived mapping $d\psi_{(0)}$ is the map

$$T' \to 2P \circ [Q, T'] - T' \circ [Q, Q].$$

We denote by $Z$ the kernel of $d\psi_{(0)}$.

Let $\mathfrak{A}' = \varphi^{-1}(\mathfrak{A})$. According to Lemma 3.1, $T \in \mathfrak{A}'$ if and only if $\psi(T) = 0$. Let $Y'$ be an irreducible component of $\mathfrak{A}'$ passing through 0. Then, identifying $T(Y', 0)$ with a subspace of $\mathrm{Hom}(M, W)$, we see immediately from the remarks made in 3 that $T(Y', 0) \subset Z$.

The quotient space $L/M$ admits a natural structure of $M$-module defined as follows: if $x \in M$ and $(y + M) \in L/M$, then $x \cdot (y + M) = [x, y] + M$. The graded vector space $\mathrm{Alt}(M, L/M)$ is the underlying graded vector space of the complex $C(M, L/M) = \oplus_{j \geq 0} C^j(M, L/M)$ used to compute the cohomology space of the Lie algebra $M$ with coefficients in the $M$-module $L/M$ (see e.g. [1, p. 282]). Let $\pi : L \to L/M$ denote the canonical mapping. Then $\pi'$, the restriction of $\pi$ to $W$, is an isomorphism of $W$ and $L/M$. The map $\pi'$ induces an isomorphism $\theta : \operatorname{Alt}(M, W) \to C(M, L/M)$ of graded vector spaces defined as follows: if $f \in \operatorname{Alt}^j(M, W)$, then $\theta(f) = \pi' \circ f$.

Let $\delta$ denote the coboundary operator in the complex $C(M, L/M)$. Then, if $T \in \operatorname{Alt}^1(M, W)$ and if $x, y \in M$,

$$\begin{aligned}\theta \circ d\psi_{(0)}(T)\cdot(x, y) &= 2(\pi[x, Ty] + \pi[Tx, y] - \pi(T[x, y])) \\ &= 2\delta(\pi \circ T)\cdot(x, y) = (2\delta \circ \theta)(T)\cdot(x, y).\end{aligned}$$

Thus $\theta \circ d\psi_{(0)} = 2(\delta \circ \theta)$. It follows immediately that $\theta$ maps $Z$ isomorphically onto $Z^1(M, L/M)$, the space of 1-cocycles of the complex $C(M, L/M)$.

We summarize the results of this section in the following proposition:

PROPOSITION 6.1. *Let* $\psi : \operatorname{Hom}(M, W) \to \operatorname{Alt}^2(M, W)$ *be defined by*

$$\psi(T) = (P - T) \circ [Q + T, Q + T].$$

*Then* $\psi^{-1}(0) = \varphi^{-1}(\mathfrak{A})$. *Let* $\operatorname{Hom}(M, W)$ *and* $C^1(L, L/M)$ *be identified by means of the isomorphism* $\theta$. *Then the kernel of the differential* $d\psi_{(0)}$ *is precisely* $Z^1(M, L/M)$. *If* $Y'$ *is an irreducible component of* $\mathfrak{A}' = \varphi^{-1}\mathfrak{A})$ *which contains* 0, *then* $T(Y', 0) \subset Z^1(M, L/M)$.

## 7. The tangent space to $G(m)$

We continue with the notation of 6. Let $G$ be a closed irreducible subgroup of the algebraic group $\operatorname{Aut}(L)$ of all automorphisms of the Lie algebra $L$ and let $\mathfrak{g}$ be the Lie algebra of $G$. Then $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{D}(L)$ of all derivations of $L$ [3, p. 143]. As usual, we identify $\mathfrak{g}$ with the tangent space $T(G, e)$ to $G$ at the identity element $e$.

Let $G'$ denote the set of all $g \in G$ such that $g(M)$ is transversal to $W$ and such that $g(W)$ is transversal to $M$. It is easy to see that $G'$ is an open subvariety of $G$ which contains $e$; thus $T(G', e)$ can be identified with $T(G, e)$. Let $\beta : G' \to \operatorname{Hom}(M, W)$ be the map $g \to PgQg^{-1}(P + gQg^{-1})^{-1}$. We observe that $\beta$ is well defined because of the conditions imposed on $g$. It follows from Lemma 2.1 that $\varphi \circ \beta$ is the map $g \to g\cdot m$ of $G'$ into $\Gamma_n(L)$. A straightforward computation shows that $d\beta_e$ is the map $D \to PDQ$ of $\mathfrak{g} = T(G, e)$ into $\operatorname{Hom}(M, W)$. (The basic point to remember in the computation is that $PeQ = 0$.)

By definition, a crossed homomorphism of $M$ into the $M$-module $L/M$ is a linear mapping $f : M \to L/M$ such that, for $x, y \in M$,

$$f([x, y]) = x\cdot f(y) - y\cdot f(x);$$

thus the set of crossed homomorphisms of $M$ into $L/M$ is just $Z^1(M, L/M)$, the space of 1-cocycles of the complex $C(M, L/M)$. If $D$ is a derivation of $L$, then the map $x \to \pi(D(x))$ of $M$ into $L/M$ is a crossed homomorphism; hence each derivation of $L$ induces a crossed homomorphism of $M$ into $L/M$.

If $D$ is a derivation of $L$, then $PDQ \in \operatorname{Hom}(M, W)$. It can be easily checked that $\theta(PDQ) \in C^1(M, L/M)$ is the crossed homomorphism of $M$ into

$L/M$ induced by $D$. Thus the image of the linear map $\theta \circ d\beta_e$ of $\mathfrak{g}$ into $C^1(M, L/M)$ is precisely the set of crossed homomorphisms of $L$ into $M$ induced by the elements of $\mathfrak{g}$.

We summarize these results as follows:

PROPOSITION 7.1. *Let* $\operatorname{Hom}(M, L/M)$ *be identified with* $C^1(M, L/M)$ *by means of the isomorphism* $\theta$. *Then the image of the derived mapping*

$$d\beta_e \colon \mathfrak{g} \to C^1(M, L/M)$$

*is the space of crossed homomorphisms of* $M$ *into* $L/M$ *induced by the elements of* $\mathfrak{g}$.

## 8. A proposition on algebraic transformation spaces

Let the algebraic variety $X$ be an algebraic transformation space for the algebraic group $G$ and let $x \in X$. Then, according to [6, Exposé 5, Lemma 4, pp. 5–13], the orbit $G(x)$ is relatively open in its closure. If $G$ is irreducible, then $G(x)$ is irreducible, and hence is a subvariety of $X$.

PROPOSITION 8.1. *Let the algebraic variety* $X$ *be an algebraic transformation space for the irreducible algebraic group* $G$. *Let* $x \in X$ *be such that* $T(X, x) = T(G(x), x)$. *Then the orbit* $G(x)$ *is an open subset of* $X$.

*Proof.* Since $G$ acts transitively on the orbit $G(x)$, it follows that every point of $G(x)$ is simple. Thus

$$\dim G(x) = \dim T(G(x), x) = \dim T(X, x) \geq \dim X.$$

Hence, $\dim G(x) = \dim X$. The conclusion now follows from [7, Propsition 1, p. 97].

## 9. The rigidity theorem

DEFINITION 9.1. Let $L$ be a Lie algebra over an algebraically closed field $k$, let $M$ be an $n$-dimensional subalgebra of $L$, and let $m$ denote the point of $\Gamma_n(L)$ corresponding to the subspace $M$. Let $\mathfrak{A}$ be the closed subset of $\Gamma_n(L)$ consisting of all $n$-dimensional subalgebras of $L$. Let $G$ be a closed subgroup of $\operatorname{Aut}(L)$. Then $M$ is a *rigid subalgebra* of $L$ with respect to $G$ if the orbit $G(m)$ is an open subset of $\mathfrak{A}$.

We say that two subalgebras $M_1$ and $M_2$ of a Lie algebra $L$ are *conjugate* under a subgroup $G$ of $\operatorname{Aut}(L)$ if $M_1$ and $M_2$ lie on the same orbit under the action of $G$ on $\Gamma_n(L)$.

PROPOSITION 9.2. *Let* $L$ *and* $G$ *be as above. Then there exists only a finite number of conjugacy classes (under* $G$*) of rigid subalgebras of* $L$ *with respect to* $G$.

This follows from the fact that $\mathfrak{A}$ is a Noetherian space (or a Zariski space in the terminology of [7]).

THEOREM 9.3. *Let* $L$ *be a Lie algebra over an algebraically closed field, let* $G$

*be a closed subgroup of* Aut($L$) *and let* $\mathfrak{g}$ *be the Lie algebra of* $G$. *Let* $M$ *be a subalgebra of* $L$ *such that every crossed homomorphism of* $M$ *into* $L/M$ *is induced by a derivation* $D \in \mathfrak{g}$ *of* $L$. *Then* $M$ *is a rigid subalgebra of* $L$ *with respect to* $G$.

*Proof.* We may assume that $G$ is irreducible. Thus the orbit $G(m)$ is irreducible. Let $Y$ be an irreducible component of $\mathfrak{A}$ which contains $G(m)$. Let $\varphi : \text{Hom}(M, W) \to \Gamma_n(L)$ be defined as in 1, and let $Y' = \varphi^{-1}(Y)$; $Y'$ is an irreducible component of Hom$(M, W)$ passing through 0. To simplify notation, we identify Hom$(M, W)$ with $C^1(M, L/M)$ by means of the isomorphism $\theta$. By Proposition 6.1, $T(Y', 0)$ is contained in $Z^1(M, L/M)$. Let $\beta : G' \to C^1(M, L/M)$ be defined as in 7; by 7.1, the image of $d\beta_e$ is precisely the vector space $B$ of crossed homomorphisms of $M$ into $L/M$ induced by derivations $D \in \mathfrak{g}$. The hypothesis states that $B = Z^1(M, L/M)$. The derived mapping $d\varphi_{(0)}$ maps $T(Y', 0)$ isomorphically onto $T(Y, m)$. Furthermore, since $\varphi \circ \beta$ is the map $g \to g \cdot m$ of $G'$ into $\Gamma_n(L)$, it follows that $d\varphi_{(0)}$ maps $B$ into a subspace of $T(G(m), m)$. Let

$$Z' = d\varphi_{(0)}(Z^1(M, L/M)) = d\varphi_{(0)}(B).$$

Then

$$Z' \supset T(Y, m) \supset T(G(m), m) \supset Z'.$$

Hence $T(Y, m) = T(G(m), m)$. It follows from Proposition 8.1 that $G(m)$ is an open subset of $Y$. It follows easily that $Y$ is the only irreducible component of $\mathfrak{A}$ which meets $G(m)$; thus $G(m)$ is an open subset of $\mathfrak{A}$. This proves 9.3.

Let $L$ be the Lie algebra of a connected linear algebraic group $G$ and let Ad : $G \to$ Aut($L$) be the adjoint representation of $G$ (defined as in [3, definition 2, p. 141]). Then Ad($G$) is a closed subgroup of Aut($L$) [6, Exposé 3, Théorème 4, p. 3–04]; it is called the group of inner automorphisms of $L$. If $G$ is a linear algebraic group with identity component $G_0$, and if $L$ is the Lie algebra of $G$ (and hence of $G_0$), then Ad($G_0$) is defined to be the group of inner automorphisms of $L$. We note that the group of inner automorphisms of $L$ is not determined by the Lie algebra structure of $L$ alone, but depends on the representation of $L$ as the Lie algebra of a linear algebraic group.

Let $L$ be a Lie algebra. If $x \in L$, we denote by ad($x$) the linear map $y \to [x, y]$. The map $x \to \text{ad}(x)$ is a homomorphism of $L$ into the Lie algebra $\mathfrak{D}(L)$ of derivations of $L$. The image ad($L$) is the Lie algebra of *inner derivations* of $L$. If $L$ is the Lie algebra of a connected linear algebraic group $G$, then ad is the differential of the adjoint representation of $G$ [3, Proposition 7, p. 142]. It follows that the Lie algebra of the group Ad($G$) of inner automorphisms of $L$ contains the Lie algebra ad($L$) of inner derivations of $L$. If the base field is of characteristic 0, it follows from [3, Théorème 12, p. 172] that ad($L$) is the Lie algebra of Ad($G$). In this case it follows further from [3, Corollaire 1, p. 156] that the group of inner automorphisms of $L$ depends on $L$ alone, and is independent of the choice of $G$.

If $M$ is a subalgebra of the Lie algebra $L$, then the space of crossed homomorphisms of $M$ into $L/M$ induced by the elements of $\mathrm{ad}(L)$ is precisely $B^1(M, L/M)$, the space of 1-coboundaries of $C^1(M, L/M)$. Thus we obtain the following corollary of 9.3:

COROLLARY 9.4. *Let $L$ be the Lie algebra of a linear algebraic group (over an algebraically closed field) and let $G$ be the group of inner automorphisms of $L$.*

(a) *Let $M$ be a subalgebra of $L$ such that $H^1(M, L/M) = 0$. Then $M$ is a rigid subalgebra of $L$ with respect to $G$.*

(b) *There exists only a finite number of conjugacy classes (under $G$) of subalgebras $M'$ of $L$ such that $H^1(M', L/M') = 0$.*

## 10. Linear Lie groups acting on algebraic sets

In 10 and 11 we shall consider algebraic sets $X$ in $\mathbf{R}^n$, $\mathbf{C}^n$, $P_n(\mathbf{R})$, and $P_n(\mathbf{C})$. We shall have occasion to consider two distinct topologies on $X$, the topology induced on $X$ by the usual (Hausdorff) topology of the ambient space, and the Zariski topology of $X$ induced by the Zariski topology of the ambient space. In order to avoid confusion, references to topological concepts in the Zariski topology will be given the prefix "Zariski" (in Sections 10 and 11 only), e.g., open subsets in the Zariski topology are Zariski-open.

The following result is proved in [12]:

10.1. *Let $X$ be an irreducible algebraic set in $\mathbf{C}^n$ (resp. $\mathbf{R}^n$) and let $G$ be a complex Lie subgroup (resp. Lie subgroup) of $GL(n, \mathbf{C})$ (resp. $GL(n, \mathbf{R})$) such that $X$ is stable under the action of $G$. Let $x \in X$ be such that the orbit $G(x)$ is an open subset of $X$. Then $G(x)$ is a Zariski-open subset of $X$ (resp. is one component of a Zariski-open subset of $X$).*

For completeness, we give the proof. We consider the case $X \subset \mathbf{C}^n$. Let $\mathfrak{g}$ denote the Lie algebra of $G$ and let $q$ be the dimension of $X$ as an algebraic variety. Then the hypothesis implies that the dimension of $G(x)$, as a complex submanifold of $\mathbf{C}^n$, is equal to $q$. The (complex) tangent space of $G(x)$ at $x$ is $\mathfrak{g}(x) = \{T(x) \mid T \in \mathfrak{g}\}$; thus $\dim \mathfrak{g}(x) = q$. But (and this is the key point of the proof), an elementary argument shows that the set

$$U = \{y \in X \mid \dim \mathfrak{g}(y) \geq q\}$$

is a Zariski-open subset of $X$. It follows easily that the orbits of $G$ on $U$ partition $U$ into disjoint open sets. Since $U$ is connected [14, p. 163], the conclusion follows. If $X \subset \mathbf{R}^n$, $U$ is not necessarily connected, although it has only a finite number of components [16]; otherwise, the argument is the same.

The action of $GL(n, \mathbf{C})$ on $\mathbf{C}^n$ induces an action of $GL(n, \mathbf{C})$ on $P_{n-1}(\mathbf{C})$. Similarly for $GL(n, \mathbf{R})$. As an elementary corollary of 10.1, we obtain:

10.2. *The result of 10.1 is equally valid when $X$ is an irreducible algebraic set in $P_{n-1}(\mathbf{C})$ (resp. $P_{n-1}(\mathbf{R})$).*

*Proof.* We consider the case $X \subset P_{n-1}(\mathbf{C})$. Let

$$p : (\mathbf{C}^n - \{0\}) \to P_{n-1}(\mathbf{C})$$

be the canonical projection. Let

$$G_1 = \{\lambda g \in GL(n, \mathbf{C}) \mid g \in G \quad \text{and} \quad \lambda \in \mathbf{C}, \lambda \neq 0\}.$$

Then the proof of 10.2 follows easily by considering the action of $G_1$ on $p^{-1}(X)$ and applying 10.1.

## 11. The rigidity theorem for real and complex Lie algebras

The results of 10 allow us to extend Theorem 9.3 for the case of Lie algebras over either $\mathbf{C}$ or $\mathbf{R}$.

DEFINITION 11.1. Let $L$ be a Lie algebra over $\mathbf{C}$ (resp. $\mathbf{R}$), let $G$ be a complex Lie subgroup (resp. Lie subgroup) of $\mathrm{Aut}(L)$, and let $\mathfrak{A}$ denote the closed subset of $\Gamma_n(L)$ consisting of all $n$-dimensional subalgebras of $L$. Let $M$ be an $n$-dimensional subalgebra of $L$ and let $m \in \mathfrak{A}$ denote the corresponding point of $\Gamma_n(L)$. Then $M$ is a *rigid subalgebra* of $L$ with respect to $G$ if the orbit $G(m)$ is an open subset of $\mathfrak{A}$ (in the usual (Hausdorff) topology of $\mathfrak{A}$).

The following proposition shows that if $L$ is a complex Lie algebra and if the complex Lie subgroup $G$ of $\mathrm{Aut}(L)$ is an algebraic subgroup, then the definition of rigidity given above coincides with that of 9.1.

PROPOSITION 11.2. *Let $L$ be a Lie algebra over* $\mathbf{C}$ (*resp.* $\mathbf{R}$), *and let $G$ and $\mathfrak{A}$ be as in* 11.1. *Let $M$ be a rigid subalgebra of $L$ and let $m \in \mathfrak{A}$ correspond to $M$. Then the oribit $G(m)$ is a Zariski-open subset* (*resp. one component of a Zariski-open subset*) *of $\mathfrak{A}$.*

The proof of 11.2 follows easily from 10.2. We use the fact that the action of $G$ on $\Gamma_n(L)$ is induced by a linear action of $G$ on the ambient projective space for $\Gamma_n(L)$.

COROLLARY 11.3. *Let $L$ and $G$ be as in* 11.1. *Then there exists only a finite number of conjugacy classes* (*under $G$*) *of rigid subalgebras of $L$.*

The proof is similar to that of 9.2. If $L$ is a real Lie algebra, we must also use the fact that a real algebraic set has only a finite number of components [16].

THEOREM 11.4. *Let $L$ be a Lie algebra over* $\mathbf{C}$ (*resp.* $\mathbf{R}$), *let $G$ be a complex Lie subgroup* (*resp. Lie subgroup*) *of* $\mathrm{Aut}(L)$, *and let $\mathfrak{g}$ be the Lie algebra of $G$. Let $M$ be a subalgebra of $L$ such that every crossed homomorphism of $M$ into $L/M$ is induced by a derivation $D \in \mathfrak{g}$. Then $M$ is a rigid subalgebra of $L$ with respect to $G$.*

*Proof.* Let $n = \dim M$ and let $m \in \mathfrak{A}$ correspond to $M$. Let $W$ be a linear subspace of $L$ which is transversal to $M$. Let $\varphi : \mathrm{Hom}(M, W) \to \Gamma_n(L)$ be defined as in 1; $\varphi$ is a diffeomorphism of $\mathrm{Hom}(M, W)$ onto the open submanifold $\Gamma_W$ of $\Gamma_n(L)$. Let $\psi : \mathrm{Hom}(M, W) \to \mathrm{Alt}^2(M, W)$ be defined as in 6. Let $\mathfrak{A}' = \varphi^{-1}(\mathfrak{A})$. Then, by 6.1, $\mathfrak{A}' = \psi^{-1}(0)$. Let the open subset $G'$ of $G$

and the map $\beta : G' \to \mathrm{Hom}(M, W)$ be defined as in 7; then $\varphi(\beta(g)) = g \cdot m$ for $g \in G'$. We identify $\mathrm{Hom}(M, W)$ with $C^1(M, L/M)$ by means of the isomorphism $\theta$. By 6.1, the kernel of the differential $d\psi_{(0)}$ is precisely the set $Z^1(M, L/M)$ of crossed homomorphisms of $M$ into $L/M$ (since $\psi$ is a polynomial mapping, its (algebraic) differential is identical to its differential as a differentiable map). It is shown in 7 that the image $B$ of the differential $d\beta_e$ is the set of crossed homomorphisms induced by derivations $D \in \mathfrak{g}$. The hypothesis of 11.4 states that $B = Z^1(M, L/M)$. Then [15, Lemma 1, p. 149] implies that there exists a neighborhood $N$ of 0 in $\mathrm{Hom}(M, W)$ such that $N \cap \mathfrak{A}'$ is included in $\beta(G')$. This implies that $G(m)$ is an open subset of $\mathfrak{A}$, hence that $M$ is a rigid subalgebra of $L$. This proves 11.4.

Let $L$ be a Lie algebra over either **R** or **C**. Then the Lie algebra of the Lie group $\mathrm{Aut}(L)$ is the Lie algebra $\mathfrak{D}(L)$ of all derivations of $L$ [2]. If $\mathfrak{g}$ is a subalgebra of $\mathfrak{D}(L)$, then there exists a unique analytic subgroup $G$ of $\mathrm{Aut}(L)$ with Lie algebra $\mathfrak{g}$; in the complex case, $G$ is a complex Lie subgroup. If $\mathfrak{g}$ is the subalgebra of all inner derivations of $L$, then the corresponding analytic group $G$ is called the group of *inner automorphisms* of $L$; the group of inner automorphisms of $L$ is not, in general, an algebraic subgroup of $\mathrm{Aut}(L)$. If $L$ is the Lie algebra of a connected Lie group $H$, then the group $G$ of inner automorphisms of $L$ is the group of automorphisms of $L$ induced by inner automorphisms of $H$.

COROLLARY 11.5. *Let $L$ be a Lie algebra over* **C** *or* **R** *and let $G$ be the Lie group of inner automorphisms of $L$.*

(a) *Let $M$ be a subalgebra of $L$ such that $H^1(M, L/M) = 0$. Then $M$ is rigid with respect to $G$.*

(b) *There exists only a finite number of conjugacy classes (under $G$) of subalgebras $M'$ of $L$ such that $H^1(M', L/M') = 0$.*

## 12. Applications

(a) **Semi-simple and reductive subalgebras.** If $L$ is a semi-simple Lie algebra over a field of characteristic 0, then it is known that $H^1(L, V) = 0$ for every $L$-module $V$ (see [8]). Thus we obtain:

PROPOSITION 12.1. *Let $L$ be a Lie algebra over either* **R** *or* **C** *and let $M$ be a semi-simple subalgebra of $L$. Then $M$ is a rigid subalgebra of $L$ with respect to the group $G$ of inner automorphisms of $L$.*

We observe that Proposition 12.1 gives a relatively elementary proof of the known fact that a given Lie algebra $L$ over either **R** or **C** admits only a finite number of conjugacy classes of semi-simple subalgebras. The usual proof of this result depends upon the classification of semi-simple Lie algebras and the Levi-Whitehead theorem.

If $L$ is a Lie algebra and if $V$ is an $L$-module, then $v \in V$ is an *invariant* if $x \cdot v = 0$ for every $x \in L$. If $L$ is a reductive Lie algebra over a field of characteristic 0 and if $V$ is a semi-simple $L$-module such that the space of invariants of $V$ reduces to $\{0\}$, then $H^1(L, V) = 0$. (See [11, Theorem 10, p. 599].) If $M$ is a subalgebra of a Lie algebra $L$, then a necessary and sufficient condition that the space of invariants of the $M$-module $L/M$ reduce to $\{0\}$ is that $M$ be equal to its own normalizer in $L$. Thus we obtain:

PROPOSITION 12.2. *Let $L$ be a Lie algebra over either* **R** *or* **C** *and let $M$ be a reductive subalgebra of $L$ which is equal to its own normalizer in $L$. Assume further that $L/M$ is a semi-simple $M$-module. Then $M$ is a rigid subalgebra of $L$ with respect to the group $G$ of inner automorphisms of $L$.*

(b) **Further examples of rigid subalgebras.** Let $L$ be a Lie algebra and let $V$ be an $L$-module. The cochain complex $C(L, V)$ admits a natural structure of $L$-module. If $x \in L$, then we denote by $\rho_x$ the corresponding operator on $C(L, V)$. Corresponding to each $x \in L$, there is also a homogeneous linear mapping $i_x$ of degree $-1$ of $C(L, V)$ into itself defined as follows: if $f \in C^{n+1}(L, V)$, then $i_x \cdot f$ is the map

$$(x_1, \cdots, x_n) \to f(x, x_1, \cdots, x_n).$$

Let $\delta$ be the coboundary operator on $C(L, V)$. Then the operators $\rho_x$ and $i_x$ are related by the formula

$$i_x \circ \delta + \delta \circ i_x = \rho_x . \tag{12.3}$$

For a more detailed discussion of the above material, see [11].

The following (known) result gives a convenient criterion for the vanishing of certain cohomology spaces.

12.4. *Let $x \in L$ be such that the restriction of $\rho_x$ to $C^n(L, V)$ is non-singular. Then $H^n(L, V) = 0$.*

*Proof.* It follows from 12.3 that $Z^n(L, V)$ is stable under $\rho_x$; thus $\rho_x$ maps $Z^n(L, V)$ isomorphically onto itself. But 12.3 implies further that the restriction of $\rho_x$ to $Z^n(L, V)$ agrees with $\delta \circ i_x$. Thus $Z^n(L, V)$ is included in the image of $\delta$ and $H^n(L, V) = 0$.

Let $V$ be a vector space over a field $k$ and let $T \in \operatorname{Hom}(V, V)$. For each $\lambda \in k$, let

$$V(T, \lambda) = \{x \in V \mid (T - \lambda I)^n(x) = 0 \quad \text{for some} \quad n\}$$

(Here, $I : V \to V$ denotes the identity map.) If $k$ is algebraically closed, it is known that $V = \oplus_{\lambda \in k} V(T, \lambda)$. We remark that $V(T, \lambda) = \{0\}$ if $\lambda$ is not an eigenvalue of $T$.

Let $L$ be a Lie algebra and let $x \in L$. For simplicity, we shall denote $L(\operatorname{ad}(x), \lambda)$ by $L(x, \lambda)$. It is known that $[L(x, \lambda), L(x, \mu)] \subset L(x, \lambda + \mu)$ (see e.g. [9, Lemma 3.2, p. 138]). In particular, $L(x, 0)$ is a subalgebra.

PROPOSITION 12.5. *Let $L$ be a Lie algebra over a field $k$, let $x \in L$, and let $\Delta$ be the set of eigenvalues of* $\operatorname{ad}(x)$. *Let $M$ be a subalgebra of $L$ which is of the* *form* $M = \oplus_{\lambda \in \Delta'} L(x, \lambda)$, *where* $\Delta'$ *is a subset of* $\Delta$ *which contains* 0. *Then* $H^1(M, L/M) = 0$.

*Proof.* It suffices to prove the proposition for $k$ algebraically closed; the proof for the general case follows by extension of the base field of $L$ to an algebraically closed field. Let $\Delta''$ denote the complement of $\Delta'$ in $\Delta$ and let $W = \oplus_{\lambda \in \Delta''} L(x, \lambda)$. Then $L = M \oplus W$ (direct sum of vector spaces), and thus $C^1(M, L/M) = \mathrm{Hom}(M, L/M)$ is canonically isomorphic to $\mathrm{Hom}(M, W)$; we shall identify $C^1(M, L/M)$ with $\mathrm{Hom}(M, W)$. With respect to this identification, the action of $\rho_x$ on $\mathrm{Hom}(M, W)$ can be described as follows: if $f \in \mathrm{Hom}(M, W)$, then

$$\rho_x \cdot f = \mathrm{ad}(x) \circ f - f \circ \mathrm{ad}(x).$$

(Here we have used the fact that $M$ and $W$ are both stable under $\mathrm{ad}(x)$; we have also identified $\mathrm{Hom}(M, W)$ with a subspace of $\mathrm{Hom}(L, L)$, as described in 0.)

The direct sum decompositions $M = \oplus_{\lambda \in \Delta'} L(x, \lambda)$ and $W = \oplus_{\lambda \in \Delta''} L(x, \lambda)$ determine a canonical isomorphism of $\mathrm{Hom}(M, W)$ (and, hence of $C^1(M, L/M)$ with the vector space

$$H = \oplus_{(\lambda, \mu) \in \Delta' \times \Delta''} \mathrm{Hom}(L, (x, \lambda), L(x, \mu)).$$

We shall identify $\mathrm{Hom}(M, W)$ with $H$ by means of this isomorphism. Each direct summand in the above decomposition of $\mathrm{Hom}(M, W)$ is stable under $\rho_x$.

There exists an integer $n$ such that $(\mathrm{ad}(x) - \lambda I)^n$ annihilates $L(x, \lambda)$ and $(\mathrm{ad}(x) - \mu I)^n$ annihilates $L(x, \mu)$. It follows from the formula

$$\rho_x \cdot f = \mathrm{ad}(x) \circ f - f \circ \mathrm{ad}(x)$$

that

$$(\rho_x - (\mu - \lambda)I) \cdot f = (\mathrm{ad}(x) - \mu I) \circ f - f \circ (\mathrm{ad}(x) - \lambda I).$$

An easy argument now shows that $(\rho_x - (\mu - \lambda)I)^{2n}$ annihilates $\mathrm{Hom}(L(x, \lambda), L(x, \mu))$. Since $\lambda \neq \mu$, this implies that the restriction of $\rho_x$ to $\mathrm{Hom}(L(x, \lambda), L(x, \mu))$ is non-singular. Hence $\rho_x$ itself is non-singular. The conclusion of 12.5 now follows from 12.4.

COROLLARY 12.6. *Let $L$ be* (i) *a real or complex Lie algebra or* (ii) *the Lie algebra of a linear algebraic group over an algebraically closed field. Let $x \in L$ and let $M$ be as in* 12.5. *Then $M$ is rigid with respect to the group $G$ of inner automorphisms of $L$.*

Let $L$ and $M$ be as in 12.6. Let $N$ be a subalgebra of $L$ such that $M + N = L$ (not necessarily a direct sum). If $x \in L$, then it follows easily that there exists $y \in N$ such that $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ induce the same crossed homomorphism of $M$ into $L/M$. In particular, if $\mathrm{ad}(N)$ is the Lie algebra of an algebraic subgroup (or Lie subgroup) $G$ of $\mathrm{Aut}(L)$, then $M$ is rigid with respect to $G$.

Let the base field of $L$ be algebraically closed and of characteristic 0. Then it follows from a result of Chevalley [3, Théorème 15. p. 177] that the Lie algebra

$$\mathrm{ad}([L, L]) = [\mathrm{ad}(L), \mathrm{ad}(L)]$$

is the Lie algebra of an algebraic group, which we denote by $\exp(\mathrm{ad}([L, L]))$; the group $\exp(\mathrm{ad}([L, L]))$ is an algebraic subgroup of $\mathrm{Aut}(L)$. Corollary 12.6 can be strengthened as follows:

COROLLARY 12.7. *Let $L$ be a Lie algebra over an algebraically closed field of characteristic* 0, *let $x \in L$, and let $\Delta$ denote the set of eigenvalues of* $\mathrm{ad}(x)$. *Let $M$ be a subalgebra of $L$ of the form $\oplus_{\lambda\in\Delta'} L(x, \lambda)$, where $\Delta'$ is a subset of $\Delta$ and $0 \in \Delta'$. Then $M$ is rigid with respect to the group* $\exp(\mathrm{ad}([L, L]))$.

*Proof.* Let $\Delta'' = \Delta - \Delta'$ and set $W = \oplus_{\lambda\in\Delta''} L(x, \lambda)$. Then it is obvious that $W \subset [L, L]$. Thus $M + [L, L] = L$. The conclusion now follows from the remarks above.

An element $x$ of a Lie algebra $L$ is *regular* if $\dim L(x, 0) = \mathrm{Min}_{y\in L}(\dim L(y, 0))$. If $x$ is a regular element of $L$, then the subalgebra $L(x, 0)$ is called a *regular subalgebra* of $L$. If the base field of $L$ is of characteristic 0, then the regular subalgebras of $L$ are precisely the Cartan subalgebras of $L$ [4, Proposition 9, p. 207 and Proposition 16, p. 216].

COROLLARY 12.8. *Let $L$ be a Lie algebra over an algebraically closed field of characteristic* 0 (*resp. the Lie algebra of a linear algebraic group over an algebraically closed field*) *and let $M$ be a regular subalgebra of $L$. Then $M$ is rigid with respect to* $\exp(\mathrm{ad}([L, L]))$ (*resp. with respect to the group of inner automorphisms of $L$*).

COROLLARY 12.9. *Let $S$ be a semi-simple Lie algebra over an algebraically closed field of characteristic* 0, *let $G$ denote the group of inner automorphisms of $S$, and let $H$ be a Cartan subalgebra of $S$. Let $M$ be a subalgebra of $S$ which contains $H$. Then $M$ is rigid with respect to $G$.*

*Proof.* Since $[S, S] = S$, it follows that $S$ admits a natural (algebraic) group of inner automorphisms. Let $H = S(x, 0)$. Since $M$ is a subalgebra, it is stable under $\mathrm{ad}(x)$. Furthermore, if $\lambda \neq 0$ is an eigenvalue of $\mathrm{ad}(x)$, then it is known that $S(x, \lambda)$ is 1-dimensional. It follows that $M$ is of the form described in 12.5. The conclusion follows from 12.5.

(c) **Ideals which are rigid subalgebras.** Let $M$ be an ideal of a Lie algebra $L$. Then it follows from the definition of an ideal that $L/M$ is a trivial $M$-module. An elementary computation shows that a necessary and sufficient condition that $H^1(M, L/M) = 0$ is that $M = [M, M]$. In fact, this condition implies that $Z^1(M, L/M) = 0$. Thus we obtain:

PROPOSITION 12.10. *Let $L$ be a Lie algebra over an algebraically closed field and let $\mathfrak{A} \subset \Gamma_n(L)$ denote the set of n-dimensional subalgebras of $L$. Let $M$ be* *an $n$-dimensional ideal in $L$ such that $[M, M] = M$ and let $m$ be the point of $\Gamma_n(L)$ corresponding to $M$. Then $m$ is an isolated point of $\mathfrak{A}$. Thus, there exists only a finite number of ideals $M'$ of $L$ such that $[M', M'] = M'$.*

We remark that the last conclusion of 12.10 is valid for Lie algebras over an arbitrary field, as follows immediately.

## 13. The case of associative algebras

The results of the previous sections can be carried over with only minor changes to obtain rigidity theorems for subalgebras of an associative algebra. (We do not require that an associative algebra have an identity element.) We sketch the details.

If $V$ and $W$ are vector spaces, we define $\mathrm{Lin}^n(V, W)$ to be the vector space of all multilinear maps of the $n$-fold Cartesian product $V \times \cdots \times V$ into $W$; we set

$$\mathrm{Lin}(V, W) = \oplus_{n \geq 0} \mathrm{Lin}^n(V, W).$$

If $A$ is an associative algebra and if $S, T \in \mathrm{Lin}^1(V, A)$, we let $[S, T] \in \mathrm{Lin}^2(V, A)$ be the map

$$(x, y) \to (Sx)(Ty) + (Tx)(Sy).$$

Let $B$ be an $n$-dimensional subalgebra of an associative algebra $A$, let $W$ be a subspace of $A$ of codimension $n$ which is transversal to $B$, let $P$ be the projection operator with kernel $B$ and image $W$, and let $Q$ be the complementary projection. Then, if $T \in \mathrm{Lin}^1(B, W)$, a necessary and sufficient condition that the image of $Q + T$ be a subalgebra of $A$ is that $(P - T) \circ [Q + T, Q + T] = 0$. The graded vector space $\mathrm{Lin}(B, A/B)$ is the underlying vector space of the complex $C(B, A/B)$ defined by Hochschild to compute the cohomology space of the associative algebra $B$ with coefficients in the $B$-module $A/B$ [10].

The definition of a rigid subalgebra of an associative algebra is essentially the same as that given for Lie algebras in 9.1. All the computations involved in the proof of Theorem 9.3 carry over for the case of associative algebras except that, in this case, $Z^1(B, A/B)$ is the space of 1-cocycles in the Hochschild complex. Thus, we obtain:

THEOREM 13.1. *Let $A$ be an associative algebra over an algebraically closed field, let $G$ be a closed subgroup of* $\mathrm{Aut}(A)$ *and let $\mathfrak{g}$ be the Lie algebra of $G$. Let $B$ be a subalgebra of $A$ such that every crossed homomorphism of $B$ into $A/B$ is induced by a derivation $D \in \mathfrak{g}$. Then $B$ is a rigid subalgebra of $A$ with respect to $G$.*

In contrast to the case of Lie algebras, an associative algebra $A$ admits a natural group $G$ of inner automorphisms. If $A$ has an identity element, then $G$ can be described as follows. Let $H$ denote the group of units of $A$. If $h \in H$, let $\rho(h)$ be the automorphism

$$x \to hxh^{-1}$$

of $A$. Then $\rho : H \to \mathrm{Aut}(A)$ is a representation and the image $\rho(H) = G$ is the group of inner automorphisms of $A$. If $A$ does not have an identity, then the description of $G$ is slightly more complicated and will be omitted. If the base field is algebraically closed, then the group $G$ of inner automorphisms of $A$ is an algebraic subgroup of $\mathrm{Aut}(A)$; the Lie algebra is the Lie algebra of all inner derivations of $A$. Thus we obtain the following corollary of 13.1:

COROLLARY 13.2. *Let $A$ be an associative algebra over an algebraically closed field and let $G$ be the algebraic group of inner automorphisms of $A$.*

(a) *Let $B$ be a subalgebra of $A$ such that $H^1(B, A/B) = 0$. Then $B$ is a rigid subalgebra of $A$ with respect to $G$.*

(b) *There exists only a finite number of conjugacy classes (under $G$) of subalgebras $B'$ of $A$ such that $H^1(B', A/B') = 0$.*

The cohomology spaces which occur in the statement of 13.2 are the Hochschild cohomology spaces.

As an application of 13.2, a semi-simple subalgebra of an associative algebra $A$ over an algebraically closed field is rigid with respect to the group of inner automorphisms of $A$.

We can obtain similar results for subalgebras of associative algebras with identity. In this case a subalgebra, by definition, contains the identity element. Moreover, the Hochschild complex is replaced by the "normalized standard complex" of Cartan-Eilenberg [1, p. 176]. Otherwise the results are identical with those of 13.1 and 13.2. We omit the details.

## 14. Subalgebras of arbitrary algebras

The reader may have observed that in our proof of the rigidity theorems, we have not used either the Jacobi identity for Lie algebras or the associativity condition for associative algebras. Thus Theorems 9.3 and 11.4 can be carried over without change to subalgebras of any finite-dimensional algebra over an algebraically closed field. (We refer the reader to [5, Chapter IV] for the appropriate definitions.) We note that there is no natural concept of inner automorphism for an algebra. Thus we do not have any natural analogues of 9.4 or 11.6 in this case.

## 15. A rigidity theorem for ideals and submodules

Let $L$ be a Lie algebra over an algebraically closed field and let $\mathcal{I}$ denote the subset of $\Gamma_n(L)$ consisting of all $n$-dimensional ideals. Let $G$ be a closed subgroup of the algebraic group $\mathrm{Aut}(L)$, let $M$ be an $n$-dimensional ideal of $L$, and let $m$ be the point of $\Gamma_n(L)$ corresponding to $M$. Then the ideal $M$ is *rigid* if the orbit $G(m)$ is an open subset of $I$.

Let $L_1$ and $L_2$ be supplementary subspaces of $L$, let $P_1$ be the projection operator on $L$ with image $L_1$ and kernel $L_2$, and let $P_2$ be the complementary projection operator. A necessary and sufficient condition that $L_1$ be an ideal of $L$ is that $P_2 \circ \mathrm{ad}(x) \circ P_1 = 0$ for every $x \in L$.

In particular, let $M$ be an $n$-dimensional ideal of $L$ and let $W$ be a subspace of $L$ supplementary to $M$. Let $P$ be the projection operator on $L$ with image $W$ and kernel $M$ and let $Q$ be the complementary projection. If $T \in \mathrm{Hom}(M, W)$, then $Q + T$ and $P - T$ are complementary projection operators. A necessary and sufficient condition that the image of $Q + T$ be an ideal is that

$$(P - T) \circ \mathrm{ad}(x) \circ (Q + T) = 0$$

for every $x \in L$. For each $x \in X$, let $\psi(x)$ denote the polynomial mapping

$$T \to (P - T) \circ \mathrm{ad}(x) \circ (Q - T)$$

of $\mathrm{Hom}(M, W)$ into $\mathrm{Hom}(M, W)$. Let $Z_x$ denote the kernel of the differential

$$d\psi(x)_{(0)} : \mathrm{Hom}(M, W) \to \mathrm{Hom}(M, W)$$

and let $Z = \bigcap_{x \in L} Z_x$. Let $Z^1$ denote the image of $Z$ under the isomorphism

$$\theta : \mathrm{Hom}(M, W) \to \mathrm{Hom}(M, L/M).$$

A trivial calculation shows that $Z^1$ is precisely the set of all $L$-module homomorphisms of $M$ into $L/M$. If $D$ is a derivation of $L$ and if $\pi : L \to L/M$ denotes the canonical mapping, then the map $x \to \pi(Dx)$ of $M$ into $L/M$ is a homomorphism of $L$-modules. Thus, each derivation of $L$ induces a homomorphism of the $L$-module $M$ into the $L$-module $L/M$. An argument similar to that used in the proof of Theorem 9.3 gives the following theorem:

THEOREM 15.1. *Let $L$ be a Lie algebra over an algebraically closed field and let $G$ be a closed subgroup of* $\mathrm{Aut}(L)$ *with Lie algebra* $\mathfrak{g}$. *Let $M$ be an ideal of $L$ such that every homomorphism of the $L$-module $M$ into the $L$-module $L/M$ is induced by a derivation $D \in \mathfrak{g}$. Then $M$ is a rigid ideal of $L$ with respect to $G$.*

We note that every inner derivation of $L$ induces the 0-homomorphism of $M \to L/M$. Thus if $\mathfrak{g}$ is included in the Lie algebra of inner derivations of $L$, then the hypothesis of Theorem 12.1 implies that $m$ (the point of $\Gamma_n(L)$ corresponding to $M$) is an isolated point of $I$.

Rigidity for (two-sided) ideals of an associative algebra is defined similarly. In this case the analogue of Theorem 15.1 is the following:

PROPOSITION 15.2. *Let $A$ be an associative algebra over an algebraically closed field and let $G$ be an algebraic subgroup of* $\mathrm{Aut}(A)$ *with Lie algebra* $\mathfrak{g}$. *Let $I$ be an ideal in $A$ such that every homomorphism of the* A*-bimodule $I$ into the $A$-bimodule $A/I$ is induced by a derivation $D \in \mathfrak{g}$ of $A$. Then $I$ is a rigid ideal of $A$.*

Let $A$ be an associative algebra over an algebraically closed field and let $V$ be a (finite-dimensional) $A$-module. Then one can define in a manner similar to 9.1 the rigidity of submodules of $V$. We let $\mathrm{Aut}_A(V)$ denote the algebraic group of all automorphisms of the $A$-module $V$. The Lie algebra of

$\mathrm{Aut}_A(V)$ is just the vector space $\mathrm{Hom}_A(V, V)$ of all homomorphisms of the $A$-module $V$, with its natural structure of Lie algebra. In this case we obtain the following analogue of Theorem 15.1.

PROPOSITION 15.3. *Let $A$ be an associative algebra over an algebraically closed field, let $V$ be an $A$-module, and let $G$ be a closed subgroup of $\mathrm{Aut}_A(V)$ with Lie algebra $\mathfrak{g}$. Let $M$ be a submodule of the $A$-module $V$ such that every homomorphism of the $A$-module $M$ into the $A$-module $V/M$ is induced by an element of $\mathfrak{g}$. Then $M$ is a rigid submodule of $V$.*

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