ON REPRESENTABILITY OF CONTRAVARIANT FUNCTORS OVER NON-CONNECTED CW COMPLEXES

BY

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1. Introduction

We assume throughout that each space under consideration has a prescribed base point and that all maps and homotopies preserve this base point. Let W denote the category of spaces admitting the structure of a finite CW complex, the base point being a vertex, and all (continuous) maps. Let W' denote the full subcategory of connected spaces.

Recall that a space B_F is a classifying space for a based-set-valued contravariant function F defined on a category \mathbb{C} of spaces if F is naturally equivalent to the functor $[, B_F]$, the homotopy classes [f] of maps f into B_F . If B_F exists we say that F is representable. In [2], E. H. Brown, Jr., has given a set of conditions on F which will imply that F is representable when $\mathbb{C} = \mathcal{W}'$. In [6] we showed that if F mapped \mathcal{W}' into the category \mathfrak{C} of abelian groups, then we could take B_F to be a weakly homotopy abelian and weakly homotopy associative H-space such that F and $[, B_F]$ are naturally equivalent as functors into \mathfrak{C} . In this note we show how to extend this to representability for functors F defined on the larger category \mathcal{W} .

Since this paper was written the paper [7] of Brown has appeared in which he formalizes the methods of [2] to obtain a very general representability theorem. In particular, his result covers the case in which the domain category of the functor F in question is \mathfrak{W} . However, our main result (1.1) is quite different in that it relates the classifying spaces of F and its restriction to \mathfrak{W}' .

Before giving the precise statement of the main theorem we must recall some definitions. Maps f and g from a space X to a space Y are said to be weakly homotopic if the induced maps f_* and g_* from [K, X] to [K, Y] are equal for every finite CW complex K. Here, $f_*[\varphi] = [f\varphi]$ for $[\varphi] \in [K, X]$. An H-structure map $\mu : B \times B \to B$ on an H-space B is weakly homotopy associative if $\mu(\mu \times 1)$ and $\mu(1 \times \mu)$ are weakly homotopic maps from $B \times B \times B$ to B. Let

$$T: B \times B \to B \times B$$

be defined by $T(x, y) = (y, x), x, y \in B$. Then μ is said to be weakly homotopy abelian if μT and μ are weakly homotopic. Suppose that ν is an *H*-structure for a space *A* and $f: A \to B$ is a map; we say that *f* is a weak homomorphism if $\mu(f \times f)$ and $f\nu$ are weakly homotopic maps.

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Let (X, A) be a pair in \mathbb{W} with A non-empty and having the same base point as X. Let $i: A \subset X$ be the inclusion map and $q: X \to X/A$ the identification map which collapses A to the base point of X. Now it is easy to see that if $F: \mathbb{W} \to \mathfrak{A}$ is a representable contravariant functor, then the sequence

$$F(X/A) \xrightarrow{F(q)} F(X) \xrightarrow{F(i)} F(A)$$

is exact. Dold [3] has called any contravariant functor $F: \mathfrak{W} \to \mathfrak{A}$ satisfying this condition *half exact*.

For the hypothesis of the main theorem (1.1) below we let $F: \mathfrak{W} \to \mathfrak{A}$ be a contravariant functor and let F' be its restriction to \mathfrak{W}' . Suppose B'_F is a connected space with a weakly homotopy associative and weakly homotopy abelian H-structure μ' such that the functors F' and $[, B'_F]$ from \mathfrak{W}' to \mathfrak{A} are naturally equivalent. Regard $F(S^0)$ as a discrete space with base point 0, set $B_F = B'_F \times F(S^0)$, and let $\mu : B_F \times B_F \to B_F$ be the map defined by

$$\mu(a, x, b, y) = (\mu'(a, b), x + y), \qquad a, b \in B'_{F}, x, y \in F(S^{0})$$

Then μ is an *H*-structure on B_F which is weakly homotopy associative and weakly homotopy abelian. Hence the functor $[, B_F]$ maps \mathfrak{W} into \mathfrak{a} and its restriction to \mathfrak{W}' is naturally equivalent of F'.

(1.1) THEOREM. If $F: \mathfrak{W} \to \mathfrak{A}$ is as above and is half exact in the sense of Dold, then there is a natural equivalence $\Phi: [, B_F] \cong F$ of functors from \mathfrak{W} to \mathfrak{A} .

Suppose that $G: \mathfrak{W} \to \mathfrak{A}$ is another such functor and B'_F is a countable CW complex. If $T: F \to G$ is a natural transformation, then there exists a weak homomorphism $f: B'_F \to B'_G$ such that the weak homomorphism

$$f \times T_{S^0} : B_F \to B_G$$

represents T, i.e. $\Phi T = (f \times T_{s^0})_* \Phi$.

Example 1. An Eilenberg-MacLane space of type (G, n), G abelian and $n \geq 1$, is a classifying space for the reduced singular cohomology functor $\tilde{H}^{n}(;G)$ on both \mathfrak{W} and \mathfrak{W}' since $\tilde{H}^{n}(S^{0};G) = 0$.

Example 2. Consider $\tilde{K} : \mathfrak{W} \to \mathfrak{A}$ as defined in [1], for example. Now $\tilde{K} \mid \mathfrak{W}'$ has a countable connected CW complex B_U as classifying space and there exists a weakly homotopy associative and weakly homotopy abelian H-structure on B_U representing the addition on $\tilde{K} \mid \mathfrak{W}'$; see [6] for example. Since \tilde{K} satisfies the half exactness property and $\tilde{K}(S^0) = Z$, the integers, it follows from theorem (1.1) that $Z \times B_U$ is a classifying space for \tilde{K} on \mathfrak{W} , a well-known fact. Similarly for $\tilde{K}O$.

The organization of this note is as follows. In Section 2 we prove two elementary lemmas concerning half exact functors. The next section introduces a technical device, due essentially to Dold [3], which will be the crux of the proof of (1.1) given in Section 5. Section 4 establishes the needed results about representability of natural transformations,

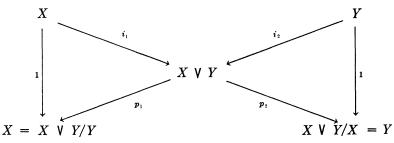
2. Elementary properties of half exact functors

Let $X \vee Y$ denote the disjoint union of the spaces X and Y with their base points identified. Let $i_1: X \to X \vee Y$ and $p_1: X \vee Y \to X$ be the canonical injection and projection maps, and similarly for i_2 and p_2 . The following two results are due to Dold [3].

(2.1) LEMMA. If $F: \mathfrak{W} \to \mathfrak{A}$ is half exact, then

$$F(X \lor Y) = F(p_1)F(X) \oplus F(p_2)F(Y).$$

Proof. Apply F to the commutative diagram

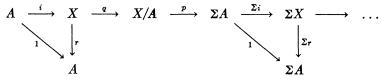


and use [4, Lemma 13.1, p. 32].

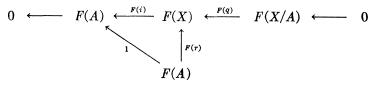
(2.2) LEMMA. Let $r: X \to A$ be a retraction of X onto A in W, and let $q: X \to X/A$ be the identification map. Then

 $F(X) = F(r)F(A) \oplus F(q)F(X/A).$

Proof. The Puppe sequence [5] of the inclusion map $i : A \subset X$ gives rise to the following commutative diagram:



Here Σ denotes the reduced suspension functor. Using [5, Th. 5], one easily shows that the half exact functor F takes the row into an exact sequence of abelian groups. But F(i) and $F(\Sigma i)$ are epimorphisms since F(i)F(r) = F(1). Hence F(p) = 0. We thus obtain the following commutative diagram in which the row is exact:



This splitting establishes the lemma.

3. Special wedges of 0-spheres

Let $X \in W$ have path components X_0, X_1, \dots, X_n where the base point * belongs to X_0 . We write $X = X_0 + \dots + X_n$. Let $S_j^0, j = 1, \dots, n$, be a 0-sphere consisting of the point * and a vertex $x_j \in X_j$. The subcomplex $A = S_1^0 \vee \cdots \vee S_n^0$ of X will be called a *special wedge* of 0-spheres for X. Notice that if we let x_j be the base point of X_j , then

$$X/A = X_0 \lor X_1 \lor \cdots \lor X_r$$

belongs to \mathfrak{W}' . Moreover, the map $r: X \to A$ defined by $r(X_0) = \{*\}$, $r(X_j) = \{x_j\}, 1 \leq j \leq n$, is a retraction.

Concerning naturality properties of special wedges we have the following result.

(3.1) LEMMA. Let A be a special wedge of 0-spheres for the finite CW complex X, let Y be a finite CW complex, and let $f: X \to Y$ be a map. Then there exist a special wedge of 0-spheres C for Y and a map $g: (X, A) \to (Y, C)$ such that g is homotopic to f.

Proof. We write

$$X = X_0 + \cdots + X_n$$
, $Y = Y_0 + \cdots + Y_m$, and $A = \{x_0, x_1, \cdots, x_n\}$

where $x_0 = *$ and x_j is a vertex of X_j . By the cellular approximation theorem we may assume that the map f is cellular; in particular, each $f(x_j)$ is a vertex of Y.

The set C is constructed as follows. Let $f(x_0) \\ \\\epsilon C$. Let $f(x_k) \\\epsilon C$ if and only if $f(x_k)$ does not belong to the path component of $f(x_i)$ for all i less than k. Finally, if $Y_j \\cap f(X)$ is empty, choose any vertex $y_j \\\epsilon Y_j$ and let $y_j \\\epsilon C$. Clearly C is a special wedge of 0-spheres for Y.

It remains to construct the map $g: X \to Y$. Let k be an integer such that $f(x_i) \in C$ for i < k and $f(x_k) \in C$. Then there exists j < k and a path $\gamma: I \to Y$ with $\gamma(0) = f(x_k)$ and $\gamma(1) = f(x_j)$. Define the map

$$F: X_k \times \{0\} \cup \{x_k\} \times I \to Y$$

by F(x, 0) = f(x), $x \in X_k$, and $F(x_k, t) = \gamma(t)$, $t \in I$. By the homotopy extension theorem there exists an extension map $F: X_k \times I \to Y$. Define the map $g_1: X \to Y$ by $g_1(x) = f(x)$ if $x \notin X_k$, $g_1(x) = F(x, 1)$ if $x \notin X_k$. Then $g_1 \simeq f$ and $g_1(x_i) \notin C$ for i < k + 1. Similarly, there exists a map $g_2: X \to Y$ such that $g_2 \simeq g_1$ and $g_2(x_i) \notin C$ for i < k + 2. Continuing in this way we obtain a map $g: X \to Y$ such that $g \simeq f$ and $g(A) \subset C$.

4. Natural transformations

(4.1) PROPOSITION. Let $T : [, B_1] \rightarrow [, B_2]$ be a natural transformation of functors from W' to \mathfrak{A} . Let μ_i be a multiplication on B_i representing the multiplication on the functor $[, B_i]$, i = 1, 2. If B_1 is a countable connected CW complex, then there is a weak homomorphism $f : B_1 \rightarrow B_2$ such that $f_* = T$.

Proof. The existence of f is established in [2, Lemma 2.1]. (The hypothesis that $Y' \in \mathbb{C}_{\omega}$ in [2] is not necessary.) It remains to show that f is a weak homomorphism. For this it suffices to show that if A is a connected finite subcomplex of B_1 containing the base point, then $\mu_2(f' \times f') \simeq f\mu'_1$ as maps

from $A \times A$ to B_2 where f' = f | A and $\mu'_1 = \mu_1 | A \times A$. Let p_1 and p_2 denote the canonical projections of $A \times A$ onto A followed by inclusion into B_1 and let $\Delta : A \to A \times A$ be the diagonal map defined by $\Delta(a) = (a, a)$, $a \in A$. Then

$$\begin{split} [f\mu_1'] &= f_*[\mu_1'] = f_*([p_1] + [p_2]) = f_*[p_1] + f_*[p_2] \\ &= [\mu_2(fp_1 \times fp_2)\Delta] = [\mu_2(f' \times f')] \end{split}$$

as we were to show.

5. Proof of (1.1)

We assume the notation and hypothesis of (1.1) except that the subscripts on B_F , etc., will be suppressed when confusion won't arise.

(5.1) LEMMA. If A is a wedge product of 0-spheres, then there is an isomorphism $\Phi: [A, B] \cong F(A)$ of abelian groups. Moreover, if C is also a wedge product of 0-spheres and $f: A \to C$ is a map, then the diagram

$$[C, B] \xrightarrow{\Phi} F(C)$$

$$\downarrow f^* \qquad \qquad \downarrow F(f)$$

$$[A, B] \xrightarrow{\Phi} F(A)$$

is commutative.

Proof. First, suppose
$$A = S^0$$
 and $x \in S^0$ is not the base point. Let $p_2: B = B' \times F(S^0) \to F(S^0)$

be the projection map and define the function Φ by

$$\Phi[f] = p_2 f(x) \epsilon F(S^0), \qquad [f] \epsilon [S^0, B].$$

Since B' is connected and $F(S^0)$ is discrete it follows that Φ is a well-defined bijection. To show that Φ is an isomorphism we simply note that if [f] and [g] are in $[S^0, B]$ then

$$\Phi([f] + [g]) = \Phi[\mu(f \times g)\Delta] = p_2 \mu(f(x), g(x))$$

= $(p_2 f(x)) + (p_2 g(x)) = \Phi[f] + \Phi[g].$

More generally, let $A = S_1^0 \vee \cdots \vee S_n^0$ where S_j^0 is a 0-sphere. If $p_j : A \to S_j^0$ denotes the projection map, $1 \leq j \leq n$, then $p_j^*[S_j^0, B]$ and $F(p_j)F(S_j^0)$ are canonically isomorphic groups since p_j^* and $F(p_j)$ are monomorphisms. From (2.1) we conclude that

$$[A, B] = p_1^*[S_1^0, B] \oplus \cdots \oplus p_n^*[S_n^0, B]$$
$$\cong F(p_1)F(S_1^0) \oplus \cdots \oplus F(p_n)F(S_n^0)$$
$$= F(A).$$

Let Φ denote this isomorphism.

It remains to prove naturality. Let $C = S_1^0 \vee \cdots \vee S_m^0$ and let $f : A \to C$ be a map. If $A = S^0$ then either f = 0 or f is a homeomorphism onto some $S_j^0 \subset C$. Easy computations of f^* and F(f) show that $F(f)\Phi = \Phi f^*$. More generally, let $A = S_1^0 \vee \cdots \vee \vee S_n^0$. Now the homomorphism

$$F(f): F(C) \to F(A) = F(p_1)F(S_1^0) \oplus \cdots \oplus F(p_n)F(S_n^0)$$

is given by

$$F(f)(\alpha) = (F(f \mid S_1^0)\alpha, \cdots, F(f \mid S_n^0)\alpha), \qquad \alpha \in F(C)$$

Similarly for $f^* : [C, B] \to [A, B]$. Therefore if $\alpha \in [C, B]$ then

$$\Phi f^*(\alpha) = (\Phi(f \mid S_1^0)^* \alpha, \cdots, \Phi(f \mid S_n^0)^* \alpha)$$
$$= (F(f \mid S_1^0) \Phi \alpha, \cdots, F(f \mid S_n^0) \Phi \alpha)$$
$$= F(f) \Phi(\alpha),$$

which completes the proof of (5.1).

Proof of (1.1). Let $X \in W$ and let A be a special wedge of 0-spheres for X. Let $Y \in W$ and let $f: X \to Y$ be a map. By Lemma (3.1), there exist a special wedge of 0-spheres C for Y for a map $g: (X, A) \to (Y, C)$ homotopic to f. Let $g': X/A \to Y/C$ be the induced map. By hypothesis there is a natural equivalence between [, B] | W' and F'. From (2.2) and (5.1) we have the following commutative diagram:

$$[Y, B] \cong [C, B] \oplus [Y/C, B] \cong F(C) \oplus F(Y/C) \cong F(Y)$$
$$\downarrow^{\mathfrak{g}^*} \qquad \downarrow^{(\mathfrak{g}|A)^* \oplus \mathfrak{g}'^*} \qquad \downarrow^{F(\mathfrak{g}|A) \oplus F(\mathfrak{g}')} \qquad \downarrow^{F(\mathfrak{g})}$$
$$[X, B] \cong [A, B] \oplus [X/A, B] \cong F(A) \oplus F(X/A) \cong F(X).$$
Since $f^* = g^*$ and $F(f) = F(g)$, the first part is proved. Let
 $\Phi : [\ , B] \cong F()$

denote the above composition.

Next, let $G: \mathfrak{W} \to \mathfrak{A}$ and $T: F \to G$. By (4.1) there exists a weak homomorphism $f: B'_F \to B'_G$ representing the restriction $T': F' \to G'$ of T. Let $h = f \times T_{S^0}: B_F \to B_G$. Now if $X \in \mathfrak{W}$ and A is a special wedge of 0-spheres for X, then the diagram

$$[X, B_F] \cong [A, F(S^0)] \oplus [X/A, B'_F] \cong F(A) \oplus F'(X/A) \cong F(X)$$
$$\downarrow^{h_*} \qquad \downarrow^{(TS^0)_* \oplus f_*} \qquad \downarrow^{T_A \oplus T'_{X/A}} \qquad \downarrow^{T_X}$$
$$[X, B_G] \cong [A, G(S^0)] \oplus [X/A, B'_G] \cong G(A) \oplus G'(X/A) \cong G(X)$$

is commutative and the horizontal composite maps are both Φ . This completes the proof.

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