THE COHOMOLOGY OF CERTAIN ELLIPTIC CURVES OVER LOCAL AND QUASI-LOCAL FIELDS

BY

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Dedicated with deepest admiration and respect to Richard Brauer on the occasion of his 65th birthday

Introduction

Using the abstract theory of theta functions developed by Tate (see §1), we prove a duality theorem in the cohomology theory of certain elliptic curves over local fields (Theorem 1). For these curves, our theorem extends the known results of Tate [13] to the equi-characteristic case.

When the ground field is quasi-local (§3), we can prove only that the cohomology groups, $H^r(k, A)$, of these curves A vanish for $r \geqq 2$. This gives a partial answer to a question raised by Serre [10, p. II-29].

1. Abstract theta functions (Tate)

If k is a field complete with respect to a rank one valuation $|\ \ |$, then Tate has constructed, for each $q \in k^*$ with $|q| < 1$, a canonical elliptic curve $A(q)$ over k whose j invariant satisfies

$$j(A) = 1/q + \xi, \qquad |\xi| \leqq 1.$$

These curves are constructed in a universal manner, and satisfy the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\rho} k^* \to A(q)_k \to 0$$

where the map ρ is given by $n \to q^n$. Without giving any details, we wish to recall here the definition and construction of these curves over k.[2]

One starts with the power series

$$x(w) = \textstyle\sum_{m=-\infty}^{\infty} q^m w/(1 - q^m w)^2 - 2\sum_{m=1}^{\infty} q^m/(1 - q^m)^2$$

which is absolutely convergent and defined on $k^*/\rho(\mathbf{Z})$ in virtue of the obvious "loxodromic" periodicity: $x(w) = x(qw)$. The function x plays the role of a Weirstrass $\wp$-function in uniformizing the elliptic curve $A(q)$ to be constructed—in fact, in the case $k = \mathbf{C}$, the transformation $u \to w = e^u$ sends $x(w)$ into $\wp(u) + \frac{1}{12}$ for suitable periods $\omega_1 = 2\pi i$ and $\omega_2 = \log q$.

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[2] I understand that Tate has recast his theory in a much more elegant form than the already elegant form presented here.

One defines the function $y(w)$ via the differential equation

$$x'(w) = x(w) + 2y(w)$$

and verifies that

$$y(w) = \sum_{m=-\infty}^{\infty} (q^m w)^2/(1 - q^m w)^3 + \sum_{m=1}^{\infty} q^m/(1 - q^m)^2.$$

Moreover, it turns out that x and y satisfy the functional equation

$$(*) \qquad y^2 + xy = x^3 - b_2 x - b_3$$

where b_2, b_3 are power series in q with *rational integer* coefficients. Tate then proves the

THEOREM (Tate). *The map $w \to \varphi(w) = \langle x(w), y(w) \rangle$ is a surjection $k^* \to A(q)_k$ with kernel $\rho(\mathbf{Z})$, where $A(q)$ is the cubic curve given by $(*)$. The discriminant Δ_q of this cubic is never zero; hence, $A(q)$ is always an elliptic curve. The j invariant, $j(A)$, of $A(q)$ is given by*

$$j(A) = 1/q + 744 + 196884q + \cdots ;$$

hence, in the non-archimedean case, $j(A)$ is not an integer of k. Moreover,

$$q = 1/j - 744(1/j^2) + \cdots$$

so that each such j is obtainable from a unique q with $|q| < 1$.

2. Cohomology of certain elliptic curves over local fields

By a local field, we mean a field complete with respect to a non-archimedean, discrete, rank one valuation, with finite residue class field. In view of the extensive results of Tate in the case of characteristic zero [13], [14], the main interest in the theorem below is in the equi-characteristic case.

THEOREM 1. *Let k be a local field and let A be an elliptic curve over k whose j invariant is not an integer of k. Then the pairing described by Tate in* [14] *establishes a duality of topological groups*

$$H^0(k, A) \times H^1(k, A) \to H^2(k, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z},$$

and $H^r(k, A)$ vanishes for $r \geqq 2$.

Proof. For convenience, we recall the Tate pairing: Let $\alpha \in H^1(k, A)$ and $b \in H^0(k, A)$ be chosen. Let $\{a_\sigma\}$ be a 1-cocycle representing α and let $\{\mathfrak{a}_\sigma\}$ be a 1-cochain of $\mathfrak{G}_k$ with coefficients in the divisors of degree zero on A which is a pre-image of $\{a_\sigma\}$. One checks immediately that $\delta\mathfrak{a}_{\sigma,\tau} = \sigma\mathfrak{a}_\tau - \mathfrak{a}_{\sigma\tau} + \mathfrak{a}_\sigma$ is linearly equivalent to zero for each σ, τ; hence, there is a function $c_{\sigma,\tau}$ on A whose divisor is $\delta\mathfrak{a}_{\sigma,\tau}$. Choose a divisor of degree zero, $\mathfrak{b}$, to represent b; then $c_{\sigma,\tau}(\mathfrak{b})$ is defined (i.e., the choices may be so made), and is an element of k_s^*. The invariant of the two cohomology class of $\{c_{\sigma,\tau}(\mathfrak{b})\}$ is independent of all the above choices and is denoted (α, b). This is Tate's pairing (in the case of curves).

The first step in the proof is the case when A is a canonical Tate curve $A(q)$ for some q with $|q| < 1$. In this case, we have the exact sequence

$$\text{(T)} \qquad 0 \to \mathbf{Z} \xrightarrow[\rho]{} k_s^* \xrightarrow[\varphi]{} A(q) \to 0;$$

hence, the cohomology sequence

$$0 \to H^1(k, A) \to H^2(k, \mathbf{Z}) \xrightarrow{\rho^*} H^2(k, k_s^*) \xrightarrow{\varphi^*} H^2(k, A) \to 0.$$

(Here, A stands for $A(q)$, and $H^1(k, A)$ denotes $H^1(\mathfrak{G}_k, A)$, etc.) If λ denotes the reciprocity law isomorphism of local class field theory

$$\lambda : \hat{k}^* = \text{proj lim}_K\, k^*/N_{K/k}\, K^* \to \mathfrak{G}_k^{ab},$$

and if one takes note of the isomorphisms

$$\mathrm{Hom}_c(\mathfrak{G}_k^{ab}, \mathbf{Q}/\mathbf{Z}) \to H^2(k, \mathbf{Z}),$$

$$\mathrm{Hom}_c(\mathfrak{G}_k^{ab}, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}_c(\hat{k}^*, \mathbf{Q}/\mathbf{Z}),$$

$$H^2(k, k_s^*) \xrightarrow{\text{inv}} \mathbf{Q}/\mathbf{Z},$$

then one obtains the exact sequence

$$0 \to H^1(k, A) \xrightarrow{\eta} \mathrm{Hom}_c(\hat{k}^*, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\theta} \mathbf{Q}/\mathbf{Z} \to H^2(k, A) \to 0.$$

The well-known relation between inv and λ shows that the map θ is given by $\theta(f) = f(q)$. The Pontrjagin dual of the last exact sequence is the sequence

$$0 \to H^2(k, A)^D \to \hat{\mathbf{Z}} \xrightarrow{\theta^D} \hat{k}^* \xrightarrow{\eta^D} H^1(k, A)^D \to 0.$$

Using the description of θ given above, one checks trivially that $\theta^D(n) = q^n$.

The definition of the topology in $\hat{k}^*$ shows that: If $q^{m_n} \to 1$ in $\hat{k}^*$, then for any integers $d_1, \cdots, d_n$, there exist unramified extensions $K_1, \cdots, K_r$ of large enough degrees $s_1, \cdots, s_r$ over k and an integer $N > 0$, such that for every $n > N$, the assertion $q^{m_n} \in \bigcap_j N_{K_j/k} K_j^*$ implies the assertion $m_n \in \bigcap_j d_j \mathbf{Z}$. This immediately implies θ^D is injective, so that $H^2(k, A)$ vanishes; and η^D establishes an isomorphism

$$\hat{k}^*/\hat{\mathbf{Z}} \approx H^1(k, A)^D.$$

However, the natural isomorphism (of *compact* groups) $k^*/\mathbf{Z} \to \hat{k}^*/\hat{\mathbf{Z}}$ together with the map φ yields the isomorphism

$$\text{(a)} \qquad H^0(k, A) \approx H^1(k, A)^D.$$

On the other hand, by tracing through all these isomorphisms, one finds that the isomorphism of (a) is induced by the following pairing: Given $\alpha \in H^1(k, A)$ and $b \in A_k$, let $\{z_\sigma\}$ be a 1-cochain in k_s^* which is a pre-image under φ of the 1-cocycle $\{a_\sigma\}$, and let $w \in k^*$ be a pre-image of b. The exact sequence (T) shows that $\delta z_{\sigma,\tau}$ has the form $q^{n(\sigma,\tau)}$, where $n(\sigma, \tau)$ is a two cocycle in $\mathbf{Z}$. The invariant of the cohomology class of $\{w^{n(\sigma,\tau)}\}$ is independent of all the choices made, and is the required pairing.

Now if l is any integer, the Artinian group scheme A_l of "points of order l" on A is self-dual under a non-degenerate pairing e_l. This is rendered explicit by considering the sequence

$$0 \to A_l \to A \xrightarrow{l} A \to 0$$

and its "dual sequence"

$$0 \to \mathbf{Hom}(A_l, \mathbf{G}_m) \to \mathbf{Ext}(A, \mathbf{G}_m) \xrightarrow{l} \mathbf{Ext}(A, \mathbf{G}_m) \to 0.$$

The pairing e_l becomes the canonical evaluation pairing

$$A_l \times \mathbf{Hom}(A_l, \mathbf{G}_m) \to \mathbf{G}_m$$

under the isomorphism $\mathbf{Ext}(A, \mathbf{G}_m) \xrightarrow{\sim} \mathbf{Pic}\ A$, see [4], [8]. (Here, $\mathbf{Pic}\ A$ is the Picard variety of A = Jacobian of $A = A$.) This being said, one finds that the pairing inducing (a) commutes with e_l in the sense that the diagram

$$(**)\qquad \begin{array}{ccccccc} 0 \to A_k/lA_k & \to & H^1(k, A_l) & \to & H^1(k, A)_l \to 0 \\ \big\downarrow{\scriptstyle (a)_l} & & \big\downarrow{\scriptstyle e_l^{(1)}} & & \big\downarrow{\scriptstyle (a)_l} \\ 0 \to (H^1(k, A)_l)^D & \to & H^1(k, A_l)^D & \to & (A_k/lA_k)^D \to 0 \end{array}$$

commutes for each l. However, it is known [13], [14] that Tate's pairing also commutes in diagram (**) above. Since $e_l^{(1)}$ is an isomorphism for every l [11], this shows that Tate's pairing and the pairing inducing (a) agree on A_k/lA_k. As $A_k = \text{proj lim}\, A_k/lA_k$, we deduce that (a) is also induced by Tate's pairing.

In the general case, one knows that there is a finite separable extension K/k of degree 2, such that over K, the given curve A becomes isomorphic to the canonical curve $A(q)$ having the same j. Moreover, because $j \neq 0$, $\neq 1728$, one knows that the k-rational points of A and $A(q)$ are related by:

$$A_k = \{x \in A(q)_K = A_K \mid \sigma x + x = 0\}$$

where σ is the generator of $\mathfrak{G}(K/k)$. It follows from this that

$$(\mathrm{b})\qquad \begin{aligned} \tilde{H}^{-1}(K/k, A) &\simeq \tilde{H}^0(K/k, A(q)) \\ \tilde{H}^0(K/k, A) &\simeq \tilde{H}^{-1}(K/k, A(q)). \end{aligned}$$

(Here, $\tilde{H}^n(K/k, \text{-})$ is the n^{th} reduced cohomology group of $\mathfrak{G}(K/k)$, [3].)

Now consider the commutative diagram

$$\begin{array}{ccccccc} H^1(K, A(q))^D_{\mathfrak{G}} & \xrightarrow{\mathrm{res}^D} & H^1(k, A(q))^D & \to & H^1(K/k, A(q))^D & \to & 0 \\ \uparrow{\scriptstyle (\mathrm{a})_K} & & \uparrow{\scriptstyle (\mathrm{a})_k} & & \uparrow{\scriptstyle \psi} & & \\ H^0(K, A(q))_{\mathfrak{G}} & \xrightarrow{\mathrm{tr}} & H^0(k, A(q)) & \to & \tilde{H}^0(K/k, A(q)) & \to & 0, \end{array}$$

where tr denotes the transfer map and $\mathfrak{G} = \mathfrak{G}(K/k)$. It shows that the Tate pairing induces an isomorphism, ψ, in the right-hand vertical column. This isomorphism, together with those of (b), yields the duality

$$\text{(c)} \qquad H^1(K/k, A) \times \tilde{H}^0(K/k, A) \to \mathbf{Q}/\mathbf{Z}.$$

If t_k denotes the map $H^0(k, A) \to H^1(k, A)^D$ induced by the Tate pairing (so that $t_K = (\mathrm{a})_K$), then the duality (c) and the commutative diagram

$$\begin{array}{ccccccc} H^1(K, A)^D_{\mathfrak{G}} & \to & H^1(k, A)^D & \to & H^1(K/k, A)^D & \to & 0 \\ \uparrow{\scriptstyle t_K = (\mathrm{a})_K} & & \uparrow{\scriptstyle t_k} & & \uparrow{\scriptstyle (\mathrm{c})} & & \\ H^0(K, A)_{\mathfrak{G}} & \to & H^0(k, A) & \to & \tilde{H}^0(K/k, A) & \to & 0 \end{array}$$

show that t_k is an isomorphism, as required. (Observe that diagram (**) shows that t_k is injective.)

There remains the proof that $H^2(k, A) = (0)$. (Since k is local, we know that $H^r(k, A) = (0)$ for $r \geqq 3$, [13].) For this, there are two methods available. One is presented in §3 below because it works equally well in the quasi-local case, the second is the following very elegant method of Tate (together with the main result of [11]): For any integer l, consider the exact sequence

$$0 \to A_l \to A \xrightarrow{l} A \to 0.$$

The cohomology sequence yields (with obvious notations)

$$H^1(A) \xrightarrow{l} H^1(A) \to H^2(A_l) \to H^2(A) \xrightarrow{l} H^2(A).$$

By the duality established above, the cokernel of the map l on $H^1(A)$ is dual to the group $H^0(k, A_l)$; in particular they have the same order. However, by the main result of [11], this order is precisely the order of $H^2(A_l)$. It follows that the map l on $H^2(A)$ is injective, which implies $H^2(A) = (0)$, Q.E.D.

Remarks. 1. The proof that $H^2(k, A) = (0)$ given above also shows that this result holds when A is an elliptic curve with Hasse invariant zero.

2. By the same methods, using Morikawa's theta functions [7], one can prove a duality theorem for the class of abelian varieties parametrized by these theta functions. Since our method does not give full generality, we have restricted attention to the simpler case of dimension one.

3. The case of quasi-local fields

By a quasi-local field (also called a general local field in [2]) we mean a field complete with respect to a non-archimedian, discrete, rank one valuation with quasi-finite [9] residue class field. To investigate the cohomology of elliptic curves defined over quasi-local fields, we need the following cohomological lemma. It is stated in a more general form than the one in which we will use it.

LEMMA. *Let $\mathfrak{G}$ be a profinite group, let $\mathfrak{N}$ be a closed normal subgroup of $\mathfrak{G}$, and let A be a $\mathfrak{G}$-module. Assume either*

(1) s.c.d. $\mathfrak{G} = r > 2$ (c.d. $\mathfrak{G} = r > 2$ *if A is a torsion module*)
(2) $H^1(\mathfrak{N}, A) = \text{—} = H^{r-2}(\mathfrak{N}, A) = (0)$
(3) $\mathfrak{G}/\mathfrak{N}$ *is a finite cyclic group*

or

(1′) s.c.d. $\mathfrak{G} = 2$ (*resp.* c.d. $\mathfrak{G} = 2$)
(2′) $\mathfrak{G}/\mathfrak{N}$ *is a finite solvable group.*

Then, whenever $H^r(\mathfrak{N}, A) = (0)$, we have $H^r(\mathfrak{G}, A) = (0)$ and the sequence

$$0 \to H^{r-1}(\mathfrak{G}/\mathfrak{N}, A^{\mathfrak{N}}) \to H^{r-1}(\mathfrak{G}, A) \to H^{r-1}(\mathfrak{N}, A)^{\mathfrak{G}/\mathfrak{N}} \to H^r(\mathfrak{G}/\mathfrak{N}, A^{\mathfrak{N}}) \to 0$$

is exact.

(For the notations c.d., s.c.d. see [12], [13].)

Proof. Under hypothesis (1′), one may use induction on the length of a composition series for $\mathfrak{G}/\mathfrak{N}$ to reduce (2′) to (3). Hence, we may assume that $\mathfrak{G}/\mathfrak{N}$ is always finite cyclic.

Form the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{G}/\mathfrak{N}, H^q(\mathfrak{N}, A)) \Rightarrow H^*(\mathfrak{G}, A)$$

and observe that our hypotheses yield $E_2^{p,q} = (0)$ for $q \neq 0, \neq r-1$. It follows from Theorem 5.11 of [3, Chapter XV], that the sequence

$$\cdots \to E_2^{n,0} \to H^n \to E_2^{n-r+1,r-1} \to E_2^{n+1,0} \to H^{n+1} \to E_2^{n-r+2,r-1} \to \cdots$$

is exact. Let $n = r-1$ in the above sequence, then

$$\cdots \to H^{r-1} \to E_2^{0,r-1} \xrightarrow[\beta]{} E_2^{r,0} \to H^r \to E_2^{1,r-1} \xrightarrow[\alpha]{} E_2^{r+1,0} \to H^{r+1} \to E_2^{2,r-1} \to E_2^{r+2,0} \to H^{r+2} \to \cdots$$

is exact. We deduce that

(i) $E_2^{v,r-1} \to E_2^{v+r,0}$ is bijective for $v \geqq 2$
(ii) α is surjective.

Our aim is to show that β is surjective and α is injective. The lemma follows immediately from these assertions.

Consider the diagrams

$$\begin{tikzcd}
H^3(\mathfrak{G}/\mathfrak{N}, H^{r-1}(\mathfrak{N}, A)) \arrow[r, "d_{2,r}^{3,r-1}"] & H^{r+3}(\mathfrak{G}/\mathfrak{N}, H^0(\mathfrak{N}, A)) \\
H^1(\mathfrak{G}/\mathfrak{N}, H^{r-1}(\mathfrak{N}, A)) \arrow[u, "\theta"] \arrow[r, "d_{2,r}^{1,r-1}", "\alpha"'] & H^{r+1}(\mathfrak{G}/\mathfrak{N}, H^0(\mathfrak{N}, A)) \arrow[u, "\theta"] \\
H^0(\mathfrak{G}/\mathfrak{N}, H^{r-1}(\mathfrak{N}, A)) \arrow[d, "\pi"] \arrow[r, "d_{2,r}^{0,r-1}", "\beta"'] & H^r(\mathfrak{G}/\mathfrak{N}, H^0(\mathfrak{N}, A)) \arrow[dd, "\theta"] \\
\hat{H}^0(\mathfrak{G}/\mathfrak{N}, H^{r-1}(\mathfrak{N}, A)) \arrow[d, "\theta"] & \\
H^2(\mathfrak{G}/\mathfrak{N}, H^{r-1}(\mathfrak{N}, A)) \arrow[r, "d_{2,r}^{2,r-1}"] & H^{r+2}(\mathfrak{G}/\mathfrak{N}, H^0(\mathfrak{N}, A)).
\end{tikzcd}$$

Here, θ is the cyclic cohomology isomorphism obtained by cup product with the fundamental class in $H^2(\mathfrak{G}/\mathfrak{N}, \mathbf{Z})$, the maps $d_{2,r}^{p,q}$ are those defined in Chapter XV of [3]—they are essentially the differentials $d_r^{p,q}$, and π is the projection from H^0 to $\hat{H}^0$. When these diagrams are proved commutative or anti-commutative, our assertions concerning α, β will follow from (i).

Now[3] the spectral sequence for the $\mathfrak{G}$-module A is a "module" over the corresponding spectral sequence for $\mathbf{Z}$:

$$H^p(\mathfrak{G}/\mathfrak{N}, H^q(\mathfrak{N}, \mathbf{Z})) \Rightarrow H^*(\mathfrak{G}, \mathbf{Z}).$$

Since the element θ can be identified as an element of $H^2(\mathfrak{G}/\mathfrak{N}, H^0(\mathfrak{N}, \mathbf{Z}))$, and since θ under this identification becomes a universal cocycle in the spectral sequence for $\mathbf{Z}$, we see that for every $r \geqq 2$,

$$d_r(\theta \cup \xi) = d_r \theta \cup \xi \pm \theta \cup d_r \xi = \pm \theta \cup d_r \xi.$$

This proves that our diagrams either commute or anticommute, as required, Q.E.D.

COROLLARY 1. *Let k be a quasi-local field and let $\mathfrak{G}_k$ denote the Galois group of the separable closure, k_s, of k over k with Krull topology. Let A be a $\mathfrak{G}_k$-*

[3] I owe the following argument to the referee who suggested it as a simplification of my original argument. In the latter argument, I used an oral communication of L. Charlap in which he informed me that Theorem 4 of [6] as generalized by Charlap-Vasquez [5] and Andrée [1], could be further generalized so as to yield: For arbitrary $\mathfrak{G}$, $\mathfrak{N}$, A as above, if $d_2^{p,q} = \cdots = d_{r-1}^{p,q} = 0$, then $d_r^{p,r-1}$ is given by cup product with a certain "characteristic class". Of course the commutativity (up to ± 1) of the diagrams follows from this result.

module. If K is a finite, normal, separable, extension of k, then $H^2(\mathfrak{G}_K, A) = (0)$ *implies* $H^2(\mathfrak{G}_k, A) = (0)$.

COROLLARY 2. *Let k be a field, and assume* c.d. $\mathfrak{G}_k = 2$. *If K/k is a finite solvable extension, then $Br(K) = (0)$ implies $Br(k) = (0)$. (Here, $Br(\text{-})$ is the Brauer group of -.)*

THEOREM 2. *Let k be a quasi-local field, and let A be an elliptic curve over k whose j-invariant is not an integer of k. Then $H^r(k, A) = (0)$ for $r \geqq 2$. If A is a canonical Tate curve, then A_k is dense in $H^1(k, A)^D$.*

Proof. If A is a canonical Tate curve, the same argument as used in Theorem 1 shows that $H^2(k, A) = (0)$. (That argument used only local class field theory and the existence theorem—both of which are valid in the case of quasi-local fields.) In the general case, A becomes isomorphic to $A(q)$ over a quadratic, separable extension K/k. Hence, $H^2(K, A) = (0)$. Now, Corollary 1 shows that $H^2(k, A) = (0)$ as well. When $A = A(q)$, the argument of Theorem 1 shows that $H^1(k, A)^D$ is isomorphic to $\hat{k}^*/\hat{\mathbf{Z}}$. However, A_k is isomorphic to $k^*/\mathbf{Z}$ which is dense in $\hat{k}^*/\hat{\mathbf{Z}}$, Q.E.D.

Remark. Theorem 2 is germane to a question raised by Serre [10, p. II-29].

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