ON LOCAL MAXIMALITY FOR THE COEFFICIENT a_8

BY

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1. In our previous paper [5] we proved the local maximality of $\Re a_6$, that is, Bombieri's result [1] at the Koebe function by using Golusin's inequality [3]. By making use of the same idea we shall prove the following

THEOREM. *Let $f(z)$ be a normalized regular univalent function $f(z)$ in the unit disc $|z| < 1$*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then there exist two positive constants ε and A, which are independent of $f(z)$, such that

$$\Re a_8 \leqq 8 - A(2 - \Re a_2)$$

for $0 \leqq 2 - \Re a_2 < \varepsilon$. If $\Re a_8 = 8$ in $0 \leqq 2 - \Re a_2 < \varepsilon$, then $f(z)$ reduces to the Koebe function

$$\frac{z}{(1 - z)^2}.$$

It should be remarked that our local maximality is different from that of Garabedian, Ross and Schiffer [2] in definition.

2. Let $G_\mu(w)$ be the μ^{th} Faber polynomial which is defined by

$$g_\mu(z) = G_\mu(g(z)) = z^\mu + \sum_{\nu=1}^{\infty} b_{\mu\nu}/z^\nu,$$

$$g(z) = f(1/z^2)^{-1/2}.$$

Then it is known that $\nu b_{\mu\nu} = \mu b_{\nu\mu}$. Let

$$Q_m(g(z)) = \sum_{\mu=1}^{m} x_\mu g_\mu(z);$$

then Golusin's inequality has the form

$$\sum_{\nu=1}^{\infty} \nu \left| \sum_{\mu=1}^{m} x_\mu b_{\mu\nu} \right|^2 \leqq \sum_{\nu=1}^{m} \nu |x_\nu|^2,$$

and Grunsky's inequality has the form

$$\left| \sum_{\mu,\nu=1}^{m} \nu b_{\mu\nu} x_\mu x_\nu \right| \leqq \sum_{\nu=1}^{m} \nu |x_\nu|^2.$$

One of the authors [4] pointed out that Grunsky's inequality is a direct consequence of Golusin's.

Received September 8, 1966.

[1] The first author was supported in part by a National Science Foundation grant and the second author was supported by a National Science Foundation Foreign Scientist Fellowship during 1965–66.

By a simple calculation we have

$$b_{1,1} = -\tfrac{1}{2}a_2 ,$$
$$b_{1,3} = -\tfrac{1}{2}(a_3 - (3/4)a_2^2),$$
$$b_{1,5} = -\tfrac{1}{2}(a_4 - (3/2)a_2 a_3 + (5/8)a_2^3),$$
$$b_{1,7} = -\tfrac{1}{2}(a_5 - (3/2)a_2 a_4 - (3/4)a_3^2 + (15/8)a_2^2 a_3 - (35/64)a_2^4),$$
$$b_{1,9} = -\tfrac{1}{2}(a_6 - (3/2)a_2 a_5 - (3/2)a_3 a_4 + (15/8)a_2^2 a_4 + (15/8)a_2 a_3^2 - (35/16)a_2^3 a_3 + (63/128)a_2^5),$$
$$b_{1,11} = -\tfrac{1}{2}(a_7 - (3/2)a_2 a_6 - (3/2)a_3 a_5 + (15/8)a_2^2 a_5 + (30/8)a_2 a_3 a_4 + (5/8)a_3^3 - (35/16)a_2^3 a_4 - (105/32)a_2^2 a_3^2 + (315/128)a_2^4 a_3 - (231/512)a_2^6 - (3/4)a_4^2),$$
$$b_{1,13} = -\tfrac{1}{2}(a_8 - (3/2)a_2 a_7 - (3/2)a_3 a_6 - (3/2)a_4 a_5 + (15/8)a_2^2 a_6 + (15/4)a_2 a_3 a_5 + (15/8)a_2 a_4^2 + (15/8)a_3^2 a_4 - (35/16)a_2^3 a_5 - (105/16)a_2^2 a_3 a_4 - (35/16)a_2 a_3^3 + (315/128)a_2^4 a_4 + (315/64)a_2^3 a_3^2 - (693/256)a_2^5 a_3 + (429/1024)a_2^7).$$

For simplicity's sake we shall denote $b_m = b_{1,m}$, $m = 1, 3, \cdots, 13$. Then we have

$$b_{3,1} = 3b_3 ,$$
$$b_{3,3} = 3b_5 + 3b_1 b_3 + b_1^3 ,$$
$$b_{3,5} = 3(b_7 + b_1 b_5 + b_3^2 + b_1^2 b_3) = (3/5)b_{5,3} ,$$
$$b_{3,7} = 3(b_9 + b_1 b_7 + 2b_3 b_5 + b_1^2 b_5 + b_1 b_3^2) = (3/7)b_{7,3} ,$$
$$b_{5,1} = 5b_5 ,$$
$$b_{5,5} = 5(b_9 + b_1 b_7 + 3b_3 b_5 + b_1^2 b_5 + 3b_1 b_3^2 + b_1^3 b_3 + (1/5)b_1^5),$$
$$b_{5,7} = 5(b_{11} + b_1 b_9 + 3b_3 b_7 + b_1^2 b_7 + 6b_1 b_3 b_5 + b_1^3 b_5 + 2b_5^2 + 3b_1^2 b_3^2 + 2b_3^3 + b_1^4 b_3) = (5/7)b_{7,5} ,$$
$$b_{7,1} = 7b_7 ,$$
$$b_{7,7} = 7b_{13} + 7b_1 b_{11} + 21b_3 b_9 + 35b_5 b_7 + 7b_1^2 b_9 + 63b_3^2 b_5 + 35b_1 b_5^2 + 7b_1^3 b_7 + 49b_1 b_3^3 + 7b_1^4 b_5 + 42b_1^3 b_3^2 + 7b_1^5 b_3 + 63b_1^2 b_3 b_5 + 42b_1 b_3 b_7 + b_1^7 .$$

Further we have

$$-2b_{13} = a_8 - 6b_1 b_{11} - 6b_3 b_9 - 6b_5 b_7 + 12b_1^2 b_9 + 24b_1 b_3 b_7 - 20b_1^3 b_7$$

$$+ 12b_1 b_5^2 + 12b_3^2 b_5 - 60b_1^2 b_3 b_5 + 30b_1^4 b_5 - 20b_1 b_3^3 + 60b_1^3 b_3^2$$
$$- 42b_1^5 b_3 + 8b_1^7,$$
$$b_{7,7} = -(7/2)(a_8 - 8b_1 b_{11} - 12b_3 b_9 - 16b_5 b_7 + 10b_1^2 b_9 + 12b_1 b_3 b_7$$
$$- 22b_1^3 b_7 + 2b_1 b_5^2 - 78b_1^2 b_3 b_5 + 28b_1^4 b_5 - 34b_1 b_3^3 + 48b_1^3 b_3^2$$
$$- 44b_1^5 b_3 + (54/7)b_1^7).$$

From now on we shall use the following notations:

$$a_2 = -2b_1 = 2 - x + ix',$$
$$-2b_3 = y + iy',$$
$$-2b_5 = \eta + i\eta',$$
$$-2b_7 = \xi + i\xi',$$
$$-2b_9 = \varphi + i\varphi',$$
$$-2b_{11} = \tau + i\tau'.$$

3. LEMMA 1.

$$11(\tau^2 + \tau'^2) + 9(\varphi^2 + \varphi'^2) + 7(\xi^2 + \xi'^2) + 5(\eta^2 + \eta'^2) + 3(y^2 + y'^2) + x'^2 \leqq 4x - x^2.$$

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

LEMMA 2. *$y = O(x), \eta = O(x), \xi = O(x)$ as $x \to 0$.*

This was already proved in our previous paper [5]. It is very possible to conjecture that $\tau = O(x)$ and $\varphi = O(x)$ as $x \to 0$.

LEMMA 3.

$$4\eta + (4\gamma - 4)y \leqq (4 + \gamma^2)x - 2x'y' + 2x'^2 - 3\{-\eta' + (1 - (\gamma/2))y' - x'\}^2 - (y' + (\gamma/2)x')^2 - 5\{-\xi' + (1 - (\gamma/2))\eta' - y'\}^2 - 7\{-\varphi' + (1 - (\gamma/2))\xi' - \eta'\}^2.$$

Proof. By Golusin's inequality

$$7 \mid x_3 b_{3,7} + x_1 b_{1,7} \mid^2 + 5 \mid x_3 b_{3,5} + x_1 b_{1,5} \mid^2 + 3 \mid x_3 b_{3,3} + x_1 b_{1,3} \mid^2 + \mid x_3 b_{3,1} + x_1 b_{1,1} \mid^2 \leqq \mid x_1 \mid^2 + 3 \mid x_3 \mid^2,$$

we have, with $x_3 = \frac{2}{3}$ and $x_1 = \gamma$, real,

$$
\begin{aligned}
&7[-\varphi + \xi - \eta - (x'\xi'/2) - y'\eta' + x'\eta' + (y'^2/2) - (\xi/2)\gamma]^2 \\
&\quad + 7[-\varphi' + \xi' - \eta' - (\xi'/2)\gamma]^2 \\
&\quad + 5[-\xi + \eta - y - (\eta/2)\gamma - (x'\eta'/2) + x'y' - (y'^2/2)]^2 \\
&\quad + 5[-\xi' + \eta' - y' - (\eta'/2)\gamma]^2 - 4[-\eta + y + x - \tfrac{1}{2}x'y' + \tfrac{1}{2}x'^2 - (y/2)\gamma] \\
&\quad + 3[-\eta' + y' - x' - (\gamma/2)y']^2 + 2\gamma[y - (\gamma/2)x] + (-y' - (\gamma/2)x')^2 \\
&\leqq 0.
\end{aligned}
$$

Here we omitted higher order terms as $x \to 0$, that is, terms which are $O(x^{3/2})$.

In the sequel we shall omit higher order terms as $x \to 0$, since those are not essential for local maximality.

4. By Golusin's inequality

$$
\begin{aligned}
&7|x_7 b_{7,7} + x_5 b_{5,7} + x_3 b_{3,7} + x_1 b_{1,7}|^2 \\
&\quad + 5|x_7 b_{7,5} + x_5 b_{5,5} + x_3 b_{3,5} + x_1 b_{1,5}|^2 \\
&\quad + 3|x_7 b_{7,3} + x_5 b_{5,3} + x_3 b_{3,3} + x_1 b_{1,3}|^2 \\
&\quad + |x_7 b_{7,1} + x_5 b_{5,1} + x_3 b_{3,1} + x_1 b_{1,1}|^2 \\
&\qquad\qquad \leqq |x_1|^2 + 3|x_3|^2 + 5|x_5|^2 + 7|x_7|^2,
\end{aligned}
$$

we have the following inequality, putting $x_7 = 1$, $x_5 = 14/5$, $x_3 = 35/6$, $x_1 = 21$,

$$
\begin{aligned}
&7\cdot(49/4)(\Re a_8 - (54/7) + \Delta_1 + \Gamma_1)^2 + 5\cdot(49/4)(-(2/7)x_5 + \Delta_2 + \Gamma_2)^2 \\
&\quad + 5\cdot(49/4)\Theta_2^2 + 3\cdot(49/4)(-(2/7)x_3 + \Delta_3 + \Gamma_3)^2 \\
&\quad + 3\cdot(49/4)\Theta_3^2 + (49/4)(-(2/7)x_1 + \Delta_4)^2 + (49/4)\Theta_4^2 \\
&\qquad\qquad \leqq 7 + 5x_5^2 + 3x_3^2 + x_1^2,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_1 &= (-4 + (5/7)x_5)\tau + (-5 - (5/7)x_5 + (3/7)x_3)\varphi \\
&\quad + (-11 + (5/7)x_5 + (3/7)x_3 - (1/7)x_1)\xi \\
&\quad + (-14 - (5/7)x_5 + (3/7)x_3)\eta + (-22 + (5/7)x_5)y + 27x, \\
\Delta_2 &= -\tau + (1 - (5/7)x_5)\varphi - (1 + (3/7)x_3 - (5/7)x_5)\xi \\
&\quad + (1 - (5/7)x_5 + (3/7)x_3 - (1/7)x_1)\eta - (1 - (5/7)x_5 + (3/7)x_3)y \\
&\quad + (5/7)x_5 x, \\
\Delta_3 &= -\varphi + (1 - (5/7)x_5)\xi + (-1 + (5/7)x_5 - (3/7)x_3)\eta \\
&\quad + (-(5/7)x_5 + (3/7)x_3 - (1/7)x_1)y + (3/7)x_3 x, \\
\Delta_4 &= -\xi - (5/7)x_5 \eta - (3/7)x_3 y + (1/7)x_1 x,
\end{aligned}
$$

$$\Gamma_1 = 2x'\tau' + 3y'\varphi' + 4\eta'\xi' + 6x'\varphi' + 6y'\xi' + ((29/2) + (5/4))x'\xi'$$
$$+ (5/2)\eta'^2 + 16y'\eta' + (57/2)x'\eta' + (55/4)y'^2 + 51x'y'$$
$$+ (81/2)x'^2,$$
$$\Gamma_2 = -\tfrac{1}{2}x'\varphi' - (3/2)y'\xi' - (3/4)x'\eta' - \eta'^2 + (1/4)y'^2 + (3/2)x'y' + 2x'^2,$$
$$\Gamma_3 = -\tfrac{1}{2}x'\xi' - y'\eta' - \tfrac{1}{2}y'^2 + (3/4)x'y' + (5/4)x'^2,$$
$$\Theta_2 = -\tau' - \varphi' - (3/2)\xi' - (3/2)\eta' - (3/2)y' - 2x',$$
$$\Theta_3 = -\varphi' - \xi' - (3/2)\eta' - (5/2)y' - (5/2)x',$$
$$\Theta_4 = -\xi' - 2\eta' - (5/2)y' - 3x'.$$

By a simple calculation we have

$$(\Re a_8 - (54/7) + \Delta_1 + \Gamma_1)^2$$
$$\leqq (4/343)(7 + 5x_5^2 + 3x_3^2 + x_1^2) - (5/7)(-(2/7)x_5 + \Delta_2 + \Gamma_2)^2$$
$$- (5/7)\Theta_2^2 - (3/7)(-(2/7)x_3 + \Delta_3 + \Gamma_3)^2 - (3/7)\Theta_3^2$$
$$- (1/7)(-(2/7)x_1 + \Delta_4)^2 - (1/7)\Theta_4^2$$
$$= (4/49) + (5/7)((4/7)x_5\Delta_2 + (4/7)x_5\Gamma_2 - (\Delta_2 + \Gamma_2)^2) - (5/7)\Theta_2^2$$
$$+ (3/7)((4/7)x_3\Delta_3 + (4/7)x_3\Gamma_3 - (\Delta_3 + \Gamma_3)^2) - (3/7)\Theta_3^2$$
$$+ (1/7)((4/7)x_1\Delta_4 - \Delta_4^2) - (1/7)\Theta_4^2 .$$

Here we may omit $(\Delta_2 + \Gamma_2)^2$, $(\Delta_3 + \Gamma_3)^2$ and Δ_4^2. Taking square roots of both hand side terms and expanding the right-hand side in a power series, we have

$$\Re a_8 \leqq 8 - \Delta_1 - \Gamma_1 + (5/7)x_5\Delta_2 + (5/7)x_5\Gamma_2 + (3/7)x_3\Delta_3 + (3/7)x_3\Gamma_3$$
$$+ (1/7)x_1\Delta_4 - (5/4)\Theta_2^2 - (3/4)\Theta_3^2 - (1/4)\Theta_4^2$$
$$- (1/14)(5x_5\Delta_2 + 3x_3\Delta_3 + x_1\Delta_4)^2.$$

Here we may omit the last term. Then this gives just the following inequality:

$$\Re a_8 \leqq 8 + 0.75\eta + 3.25y - 7.75x - \Gamma_1 + 2\Gamma_2 + (5/2)\Gamma_3 - (5/4)\Theta_2^2$$
$$- (3/4)\Theta_3^2 - (1/4)\Theta_4^2 .$$

By Lemma 3 we have

$$0.75\eta + 3.25y \leqq 0.75(1 + (64/9))x - (0.75/2)x'y' + (0.75/2)x'^2$$
$$- (9/16)(\eta' + (7/6)y' + x')^2 - (3/16)(y' + (8/3)x')^2$$
$$- (21/16)(\varphi' + (7/6)\xi' + \eta')^2$$
$$- (15/16)(\xi' + (7/6)\eta' + y')^2$$

with $\gamma = 16/3$. Then we have

$$\Re a_8 \leqq 8 - (5/3)x - (3/8)x'y' + (3/8)x'^2 - \Gamma_1 + 2\Gamma_2 + (5/2)\Gamma_3 - (5/4)\Theta_2^2 - (3/4)\Theta_3^2 - (1/4)\Theta_4^2 - (9/16)(\eta' + (7/6)y' + x')^2 - (3/16)(y' + (8/3)x')^2 - (21/16)(\varphi' + (7/6)\xi' + \eta')^2 - (15/16)(\xi' + (7/6)\eta' + y')^2.$$

Further by Lemma 1 we have

$$\Re a_8 \leqq 8 - L(x', y', \eta', \xi', \varphi', \tau'),$$

$$\begin{aligned} 192L(x', y', \eta', \xi', \varphi', \tau') &= 9072x'^2 + 13332x'y' + 5131y'^2 + 8112x'\eta' + 6144y'\eta' + 2445\eta'^2 \\ &+ 4512x'\xi' + 3948y'\xi' + 2760\eta'\xi' + 1863\xi'^2 + 3024x'\varphi' + 2016y'\varphi' \\ &+ 1176\eta'\varphi' + 1596\xi'\varphi' + 1356\varphi'^2 + 1344x'\tau' + 720y'\tau' + 240\eta'\tau' \\ &+ 720\xi'\tau' + 480\varphi'\tau' + 1120\tau'^2. \end{aligned}$$

Consider

$$\tilde{L} = \begin{bmatrix} 9072 & 6666 & 4056 & 2256 & 1512 & 672 \\ 6666 & 5131 & 3072 & 1974 & 1008 & 360 \\ 4056 & 3072 & 2445 & 1380 & 588 & 120 \\ 2256 & 1974 & 1380 & 1863 & 798 & 360 \\ 1512 & 1008 & 588 & 798 & 1356 & 240 \\ 672 & 360 & 120 & 360 & 240 & 1120 \end{bmatrix}.$$

Take the principal diagonal minor determinants

$$1120, \quad \begin{vmatrix} 1140 & 240 \\ 240 & 1356 \end{vmatrix}, \quad \begin{vmatrix} 1120 & 240 & 360 \\ 240 & 1356 & 798 \\ 360 & 798 & 1863 \end{vmatrix},$$

$$\begin{vmatrix} 1120 & 240 & 360 & 120 \\ 240 & 1356 & 798 & 588 \\ 360 & 798 & 1863 & 1380 \\ 120 & 588 & 1380 & 2445 \end{vmatrix}, \quad \begin{vmatrix} 1120 & 240 & 360 & 120 & 360 \\ 240 & 1356 & 798 & 588 & 1008 \\ 360 & 798 & 1863 & 1380 & 1974 \\ 120 & 588 & 1380 & 2445 & 3072 \\ 360 & 1008 & 1974 & 3072 & 5131 \end{vmatrix}, \quad |\tilde{L}|.$$

Then these are positive. Hence L is positive definite. By continuity we have the desired result.

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