A NOTE ON FOURIER-LAGRANGE INTERPOLATION

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1. Introduction

Marcinkiewicz has given an example of a continuous function f whose Fourier series $\{S_n(x, f)\}$ converges uniformly (on $[0, 2\pi]$) but whose sequence of Fourier-Lagrange interpolation polynomials $\{I_n(x, f)\}$ diverges almost everywhere (see [2, page 40]). In this note we give a continuous function $\phi(x)$ with the property that $\{I_n(x, \phi)\}$ converges uniformly to $\phi(x)$ but $\{S_n(x, \phi)\}$ diverges at a point. Using a standard construction, ϕ can be modified to give an example with $\{S_n(x, \phi)\}$ diverging on an everywhere dense set in $[0, 2\pi]$. The details of this latter construction are not carried out.

We are indebted to Professor G. Alexits for suggesting the problem treated here and for helpful discussions during its preparation. Also we remark that several classical results will be used without specific reference. All of these may be found in Zygmund [1], [2].

2. A preliminary construction

We first define a set of functions which will be basic in the construction of the example. To describe these functions it is convenient to introduce certain sets of integers and certain subsets of $[0, 2\pi]$.

D1. $p_1, p_2, \cdots, p_k$ or simply $\{p_i\}_k$ will denote the first k odd primes, indexed in order.

D2. p^* will denote a certain member of $\{p_i\}_k$. $A(p^*)$ will denote the set $\bigcup_{\nu=0}^{[p^*/4]} [4\nu\pi/p^*, 2(2\nu + 1)\pi/p^*]$.

D3. e^* is a positive integer, subject only to the restriction that if m is an integer with $1 \leqq (2m + 1) \leqq p_k$, $(2m + 1) = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where $0 \leqq e_i \leqq e^*$.

D4. For each number $y = (2\pi\mu)/(p_1^{e^*} \cdots p_k^{e^*})$ (with μ an integer), $y \in A(p^*)$, let $\lambda = \lambda(y)$ be a positive number less than $2/(p_1 \cdots p_k)^{2e^*}$. Denote the totality of such y by $B(k, p^*, e^*)$ and let $C(k, p^*, e^*, \lambda)$ denote the set $A(p^*)\backslash \bigcup_y (y - \lambda(y), y + \lambda(y))$ where the union is taken over all $y \in B(k, p^*, e^*)$.

For some applications, the members of $B(k, p^*, e^*)$ will be subscripted in order from left to right (i.e., $y_1 < y_2 < \cdots < y_n$).

DEFINITION 1. For a given choice of k, p^*, e^* and λ as defined in D1–D4, let $\phi(k, p^*, e^*, \lambda; x)$ be the continuous function on $[0, 2\pi]$ defined as follows:
1° $\phi(k, p^*, e^*, \lambda; x) = 0$ for $x \in B(k, p^*, e^*) \cup ([0, 2\pi]\backslash A(p^*))$;

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2° $\phi(k, p^*, e^*, \lambda; x) = 1$ for $x \in C(k, p^*, e^*, \lambda)$;

3° $\phi(k, p^*, e^*, \lambda; x)$ is extended to the rest of $[0, 2\pi]$ so as to be continuous on $[0, 2\pi]$ and linear on each subinterval for which it has not been defined in 1° and 2°.

For example if $y \in B(k, p^*, e^*)$ but is not an endpoint of one of the intervals in $A(p^*)$, then

$$\phi(k, p^*, e^*, \lambda; x) = (1/\lambda(y))(x - y) \qquad \text{for} \quad 0 \leqq (x - y) \leqq \lambda(y)$$

and

$$\phi(k, p^*, e^*, \lambda; x) = 1 - (1/\lambda(y))(x - y + \lambda(y)) \quad \text{for} \quad -\lambda(y) \leqq (x - y) \leqq 0.$$

In Definition 1, the functions $\phi(k, p^*, e^*, \lambda; x)$ are 0 in $[\pi, 2\pi]$ so that in certain formulas involving the Dirichlet kernel $D_n(u)$, the substitution $1/u$ for $1/(2 \sin u/2)$ can be made with impunity.

Notice that $\phi(k, p^*, e^*, \lambda; x)$ is very nearly the characteristic function of $A(p^*)$. However, it is continuous and also is zero at a certain critical set of points. For simplicity $\phi(k, p^*, e^*, \lambda; x)$ will sometimes be denoted by $\phi(-; x)$. In what follows certain properties of $\{I_m(x, \phi(-; x))\}$ will be discussed and these will be denoted by P followed by a suitable integer. Later on properties of $\{S_m(x, \phi(-; x))\}$ will be developed and these will be denoted by Q followed by a suitable integer.

P1. $I_m(x, \phi(-; x)) \equiv 0$ for $1 \leqq (2m + 1) \leqq p_k$ or if $2m + 1$ divides $(p_1 \cdots p_k)^{e*}$.

For let $x_\mu^{(m)} = 2\pi\mu/(2m + 1)$. Then

$$I_m(x, \phi(-; x)) = 2/(2m + 1) \textstyle\sum_{\mu=0}^{2m} \phi(-; x_\mu^{(m)}) D_m(x - x_\mu^{(m)}) = 0$$

since $\phi(-; x_\mu^{(m)}) = 0$ for each $x_\mu^{(m)}$ (see D3 and 1° of Definition 1).

P2. If $2m + 1$ and $(p_1 p_2 \cdots p_k)^{e^*}$ are relatively prime and if all points $x_\mu^{(m)}$ which are in $A(p^*)$ are also in $C(k, p^*, e^*, \lambda)$ then

$$I_m(x, \phi(-; x)) - O(1) + O((\log p^*)/q)$$

where $q = (2m + 1)/p^*$.

One has

$$\begin{aligned} I_m(x, \phi(-; x)) &= 2/(2m + 1) \textstyle\sum_{\mu=0}^{2m} \phi(-; x_\mu^{(m)}) D_m(x - x_\mu^{(m)}) \\ &= 2/(2m + 1) \textstyle\sum_{r=0}^{[p^*/4]} \sum_{\mu=[2rq]+1}^{[(2r+1)q]} D_m(x - x_\mu^{(m)}) \end{aligned}$$

(since $\phi(-; x_\mu^{(m)}) = 1$ in the terms left in the sum). Let

$$x = (2\mu_0 \pi)/(2m + 1) + (2\alpha)/(2m + 1)$$

where $0 \leqq \alpha < \pi$. Then

$$D_m(x - x_\mu^{(m)}) = \{\sin((m + \tfrac{1}{2})(\mu_0 \pi - \mu\pi + \alpha))\cdot 2/(2m + 1)\}$$
$$/\{2 \sin(((\mu_0 - \mu)\pi + \alpha)/(2m + 1))\}$$
$$= \{(\sin(\mu_0 - \mu)\pi) \cos\alpha + (\cos(\mu_0 - \mu)\pi) \sin\alpha\}$$
$$/\{2 \sin((\mu_0 - \mu)\pi + \alpha)/(2m + 1)\}.$$

Therefore

$$(*) \quad I_m(x, \phi(-;x)) = \{\textstyle\sum_{r=0}^{[p^*/4]} \sum'^{[(2r+1)q]}_{\mu=[2rq]+1} (-1)^{\mu_0-\mu}/(\mu_0 - \mu)\pi\} \sin\alpha + O(1).$$

In $(*)$ the symbol $\sum'$ means that if one of the ranges $([2rq] + 1, [(2r + 1)q])$ includes μ_0 then this term is not included in the sum.

Consider a block of terms $\sum_{\mu=[2rq]+1}^{[(2r+1)q]} (-1)^{\mu_0-\mu}/(\mu_0 - \mu)\pi$ and suppose that $\mu \neq \mu_0$ in this range. Suppose for example that $\mu_0 < [2rq] + 1$. If $[(2r + 1)q] - [2rq]$ is even, then successive terms of the block can be paired to give some terms of a series which is absolutely convergent (i.e., $\sum_{n=1}^{\infty} \pm 1/(\pi(n)(n + 1))$). If $[(2r + 1)q] - [2rq]$ is odd, this pairing leaves over one term, $(-1)^{\mu_0-[(2r+1)q]}/((\mu_0 - [(2r + 1)q])\pi)$. The set of all paired terms of $(*)$ is dominated by the series $\sum_{n=1}^{\infty} 1/(\pi(n)(n + 1))$ (or more properly by twice this series since in general we have terms to the left and right of x).

The worst possible case for the unpaired terms is to start with $0 < \mu_0 < [q]$ and to have each integer $[(2r + 1)q] - [2rq]$ odd. But then the sum of the unpaired terms is dominated by $1 + \sum_{n=1}^{[p^*/4]} 1/qn = O((\log p^*)/q) + 1$. The result P2 follows.

P3. Suppose $(2m + 1)$ is not relatively prime to $(p_1 \cdots p_k)^{e^*}$ but does not divide $(p_1 \cdots p_k)^{e^*}$. Suppose further that each point of $\{(2\mu\pi)/(2m + 1)\}$ which is in $A(p^*)$ but not in $B(k, p^*, e^*)$ is in $C(k, p^*, e^*, \lambda)$. Then $I_m(x, \phi(-;x)) = O(1)$.

Let $2m + 1 = p_1^{f_1} \cdots p_k^{f_k} p_{k+1}^{f_{k+1}} \cdots p_s^{f_s}$ where either $s > k$ and $f_s \geqq 1$ or $f_{i_1} > e^*, \cdots, f_{i_r} > e^*$ with $r \geqq 1$ $(1 \leqq i_1 < i_2 < \cdots < i_r \leqq k)$. If

$$q = p_{i_1}^{f_{i_1}-e^*} \cdots p_{i_r}^{f_{i_r}-e^*} p_{k+1}^{f_{k+1}} \cdots p_s^{f_s},$$

the numbers $\{(2\mu\pi)/(2m + 1)\}$ which are in $B(k, p^*, e^*)$ are just those of the form $\{(2\nu q\pi)/(2m + 1)\}$. Since q is odd, $q - 1$ is even and the numbers

$$(\{(2\mu\pi)/(2m + 1)\}\backslash\{(2\nu q\pi)/(2m + 1)\}) \cap A(p^*)$$

occur in blocks of $q - 1$ consecutive integers where $\phi(-;x_i^{(m)}) = 1$ while at the remaining points of $\{x_i^{(m)}\}$, $\phi(-;x_i^{(m)}) = 0$. Therefore,

$$I_m(x, \phi) = 2/(2m + 1) \textstyle\sum_\nu \sum_{i=\nu q+1}^{(\nu+1)q-1} D_m(x - x_i^{(m)})$$

and with a suitable pairing of consecutive terms in this latter sum it is clear that

$$|I_m(x, \phi(-; x))| < \textstyle\sum_1 1/(\pi(n(n+1))) + 1.$$

P1–P3 show that if m is not too large (relative to p_k) and if p_k/p^* is of the same order of magnitude as $\log p^*$, $|I_m(x, \phi(-; x))|$ is bounded (the bound is uniform if $(p^* \log p^*)/p_k$ is uniformly bounded and if the other hypotheses of P1–P3 are met uniformly). We now put some further restrictions on the numbers $\{\lambda(y_i)\}$ which will make $|I_m(x, \phi(-; x))|$ bounded for all m and x. Let

$$\lambda(y_i) = (p_1 p_2 \cdots p_k)^{-e^*(1+i)}$$

(recall the definition of y_i from D4). With this choice for $\lambda(y_i)$ and with $(p^* \log p^*)/p_k < M$ (say) we have

LEMMA 1. *Suppose we have a class of functions*

$$\{\phi(k, p^*, e^*, \lambda; x)\}$$

with $|p^* \log p^*/p_k| < M$ *where the* ϕ*'s of the class are constructed in accordance with Definition* 1 *and the* $\lambda(y)$ *are chosen as above. Then* $\{I_m(x, \phi(-; x))\}$ *is uniformly bounded in* m, x *and the class* $\{\phi(-; x)\}$.

The proof of Lemma 1 consists of examining $|I_m(x, \phi(-; x))|$ for m in several ranges (the ranges adding up to all of the positive integers) and showing boundedness in each of these ranges. We emphasize that for a single $\phi(-; x)$ the boundedness is trivial since each $\phi(-; x)$ is Lipschitz (and hence $I_m(x, \phi(-; x)) \to \phi(-; x)$ uniformly. However in the construction of the example in Theorem 1 below we need to consider a sequence of functions of the type $\phi(-; x)$ and to have $|I_m(x, \phi(-; x))|$ uniformly bounded even though the set of Lipschitz constants of the sequence of functions is not bounded.

If $2m + 1 \leqq p_k$, $|I_m(x, \phi(-; x))| = 0$ by P1, so Lemma 1 is valid for this range of m. If $p_k \leqq (2m + 1) \leqq (p_1 p_2 \cdots p_k)^{e^*}$, and $2m + 1$ and $(p_1 p_2 \cdots p_k)^{e^*}$ are relatively prime, Lemma 1 follows from P2 since $p^*(\log p^*) /(2m + 1) < m$ and all points of the form $(2\pi\mu)/(2m + 1)$ in $A(p^*)$ are also in $C(k, p^*, e^*, \lambda)$ (since the minimum distance from points $\{2\pi\mu/(2m + 1)\}$ to points $\{2\pi\nu/(p_1 \cdots p_k)^{e^*}\}$ is greater than $2\pi/(p_1 \cdots p_k)^{2e^*}$). A similar argument shows that if $p_k \leqq (2m + 1) \leqq (p_1 \cdots p_k)^{e^*}$ and $2m + 1$ are not relatively prime then the hypotheses of P3 are satisfied and $\{|I_m(x, \phi(-; x))|\}$ is uniformly bounded in this case.

Finally we treat the case $(2m + 1) > (p_1 \cdots p_k)^{e^*}$. This will be handled by some sublemmas which will be prefixed by P (continuing from the previous set P1 – P3).

P4.

$$I_m(x, \phi(-; x)) = (1/\pi)\Big(\textstyle\sum_{i=0}^{\prime 2m} \phi(-; x_i^{(m)})(-1)^{i_0 - i}/(i_0 - i)\Big) \sin\alpha + O(1)$$

where $x = 2\pi i_0/(2m + 1) + 2\alpha/(2m + 1)$ $(0 \leqq \alpha < \pi)$ and $\sum'$ means the term $i = i_0$ is deleted. $O(1)$ is uniform for the class of functions $\{\phi(-;x)\}$.

For $I_m(x; \phi(-;x))$

$$= (2/(2m+1)) \textstyle\sum_{i=0}^{2m} \phi(-;x_i^{(m)}) D_m(x - x_i^{(m)})$$
$$= (2/(2m+1)) \textstyle\sum_{i=0}^{\prime 2m} \phi(-;x_i^{(m)})$$
$$\cdot(\sin(\pi(i_0 - i) + \alpha))/2\pi(i_0 - i)/(2m+1) + O(1)$$
$$= (1/\pi)(\textstyle\sum_{i=0}^{\prime 2m} \phi(-;x_i^{(m)})(-1)^{i_0-i}/(i_0 - i)) \sin\alpha + O(1).$$

The $O(1)$ term in this last formula is uniform for the class of functions $\{\phi(-;x)\}$ and P4 is established.

P5. Suppose $\phi(-;x)$ is linear in an interval $[a, b]$ (and either increases from 0 to 1 or decreases from 1 to 0 there). Suppose further that

$$|\phi(-;x_i^{(m)}) - \phi(-;x_{i+1}^{(m)})| = \eta \qquad \text{for} \quad x_i^{(m)} \in [a, b].$$

Then

$$\textstyle\sum_{i=N_1}^{\prime N_2} \phi(-;x_i^{(m)})(-1)^{i_0-i}/(i_0 - i)$$
$$= O(1) \textstyle\sum_{i=[N_1/2]}^{\prime [N_2/2]} 1/(i_0 - 2i)(i_0 - (2i+1))$$
$$+ O(1)\eta \textstyle\sum_{i=[N_1/2]}^{\prime [N_2/2]} 1/(i_0 - 2i)$$

where N_1 and N_2 are (respectively) the smallest and largest indexes of i with $x_i^{(m)} \in [a, b]$.

Assume first that $N_2 - N_1 + 1$ is even and that $\phi(-;x_i^{(m)})$ decreases from 1 to 0 in $[a, b]$. Then

$$\textstyle\sum_{i=N_1}^{\prime N_2} \phi(-;x_i^{(m)})(-1)^{i_0-i}/(i_0 - i)$$
$$= (-1)^{i_0-N_1}\{(\phi(-;x_{N_1}^{(m)})/(i_0 - N_1) - \phi(-;x_{N_1+1}^{(m)})/(i_0 - N_1 - 1)$$
$$+ \cdots + \phi(-;x_{N_2-1}^{(m)})/(i_0 - N_2 + 1) - \phi(-;x_{N_2}^{(m)})/(i_0 - N_2)\}$$
$$= (-1)^{i_0-N_1}\{\phi(-;x_{N_1}^{(m)})(1/(i_0 - N_1) - 1/(i_0 - N_1 - 1))$$
$$+ \eta 1/(i_0 - N_1 - 1) + \cdots + \phi(-;x_{N_2-1}^{(m)})(1/(i_0 - N_2 + 1)$$
$$- 1/(i_0 - N_2)) + \eta 1/(i_0 - N_2)\}$$
$$= O(1) \textstyle\sum_{i=[N_1/2]}^{\prime [N_2/2]} 1/((i_0 - 2i)(i_0 - (2i+1)))$$
$$+ O(1)\eta \textstyle\sum_{i=[N_1/2]}^{\prime [N_2/2]} 1/(i_0 - 2i).$$

A similar argument holds when $N_2 - N_1 + 1$ is odd. In this case there is an unpaired term (say $\phi(-;x_{N_2}^{(m)})$) and this gives rise to a term $O(1)\eta 1/(i_0 - N_2)$. The case where $\phi(-;x)$ increases from 0 to 1 is treated by taking $\phi(-;x_{N_1}^{(m)})$ as the unpaired term. (The factors $O(1)$ are universally bounded-independent of $[a, b]$ and N_1 and N_2.)

The case $(2m + 1) > (p_1 \cdots p_k)^{e^*}$ is somewhat different than when

$2m + 1 \leqq (p_1 \cdots p_k)^{e^*}$. In P6 the range

$$(p_1 \cdots p_k)^{e^*} < 2m + 1 < (p_1 \cdots p_k)^{Ne^*}$$

is treated where N is the number of points in $B(k, p^*, e^*)$. The case $2m + 1 > (p_1 \cdots p_k)^{Ne^*}$ requires only a slight modification and will not be treated explicitly. We emphasize that $O(1)$ is used to mean bounded for the class of functions considered in Lemma 1.

P6. If m is such that

$$(p_i \cdots p_k)^{ne^*} \leqq (2m + 1) < (p_1 \cdots p_k)^{(n+1)e^*}$$

then $|I_m(x; \phi(-; x))| = O(1)$.

First note that $\phi(-; \mathrm{x})$ is either 0 or 1 except in "small" neighborhoods o^f the points $\{y_i\}$ where it consists of one or two linear parts (depending on i). The fundamental points $\{x_i^{(m)}\}$ where $\phi(-; x_i^{(m)}) \neq 0, 1$ are in the neighborhoods of points $\{y_i\}$ where $1 \leqq i \leqq (n - 1)$.

Let $2m + 1 = \theta(p_1 \cdots p_k)^{ne^*}$ where $1 \leqq \theta < (p_1 \cdots p_k)^{e^*}$. The points $\{x_i^{(m)}\}$ where $\phi(-; x_i^{(m)}) \neq 0, 1$ are of the form $2\pi\mu/(2m + 1)$ where

$$\begin{aligned} N_1(i) &= \nu\theta(p_1 \cdots p_k)^{(n-1)e^*} - \theta(p_1 \cdots p_k)^{(n-i)e} \\ &\leqq \mu \leqq \nu\theta(p_1 \cdots p_k)^{(n-1)e^*} + \theta(p_1 \cdots p_k)^{(n-i)e^*} \\ &= N_2(i) \end{aligned}$$

(ν is determined by the index of the point y_i). From P4 and P5 it follows that

$$\begin{aligned} &2/(2m + 1)\left|\textstyle\sum'^{N_2(i)}_{j=N_1(i)} \phi(-; x_j^{(m)}) D_m(x - x_j^{(m)})\right| \\ &\quad = \Big|\{O(1) \textstyle\sum'^{[N_2(i)/2]}_{j=[N_1(i)/2]} 1/(2i_0 - j)(2(i_0 - j) + 1) \\ &\qquad\qquad + O(1)\eta_i \textstyle\sum'^{[N_2(i)/2]}_{j=[N_1(i)/2]} 1/(i_0 - 2j)\} \sin\alpha\Big| \end{aligned}$$

where

$$\eta_i = 1/\theta(p_i \cdots p_k)^{(n-i)e^*}$$

and

$$x = 2\pi i_0/(2m + 1) + 2\alpha/(2m + 1) \qquad (0 \leqq \alpha < \pi).$$

Now

$$\begin{aligned} &\left|\sum'^{[N_2(i)/2]}_{j=[N_1(i)/2]} 1/(i_0 - 2j)\right| \\ &\quad = O(1) \log\left|\frac{i_0 - \nu\theta(p_1 \cdots p_k)^{(n-1)e^*} + \theta(p_1 \cdots p_k)^{(n-i)e^*}}{i_0 - \nu\theta(p_1 \cdots p_k)^{(n-1)e^*} - \theta(p_1 \cdots p_k)^{(n-i)e^*}}\right| \\ &\quad = O(1)\left\{\frac{\theta(p_1 \cdots p_k)^{(n-i)e^*}}{(i_0 - \nu\theta(p_i \cdots p_k))^{(n-1)e^*}}\right\}. \end{aligned}$$

Let $i_0 = \nu_0 \theta(p_1 \cdots p_k)^{(n-1)e^*}$. One has

$$\sum_{i=1}^{n-1} \eta_j \left| \sum_{j=[N_1(i)/2]}^{\prime [N_2(i)/2]} 1/(i_0 - 2j) \right|$$
$$= O(1) \sum_{i=1}^{n-1} (1/p_1 \cdots p_k)^{(n-i)e^*} \theta(p_1 \cdots p_k)^{(n-i)e^*}/\theta(\nu_0 - \nu(i))$$
$$\cdot (p_1 \cdots p_k)^{(n-1)e^*} + O(1)$$
$$= O(1)(\sum_{i=1}^{\prime n-1} 1/(\nu_0 - \nu(i)))1/(p_1 \cdots p_k)^{(n-1)e^*} + O(1) = O(1)$$

since $n \leqq (p_1 \cdots p_k)^{e^*}$. Clearly

$$\sum_{i=1}^{n-1} \sum_{j=[N_1(i)/2]}^{\prime [N_2(i)/2]} 1/2(i_0 - j)(2(i_0 - j) + 1) = O(1).$$

Therefore

$$\text{(A)} \qquad \sum_{i=1}^{n=1} \sum_{j=N_1(i)}^{\prime N_2(i)} \phi(-; x_j^{(m)}) D_m(x - x_j^{(m)}) = O(1)$$

where the $O(1)$ in (A) is uniform over the class of $\phi(-; x)$'s we are considering.

In the expression

$$\text{(B)} \quad I_m(x; \phi(-; x)) = 2/(2m + 1) \sum_{n=0}^{2m} \phi(-; x_i^{(m)})) D_m(x - x_i^{(m)})$$

the set of terms where $\phi(-; x_i^{(m)}) = 1$ (i.e., the set of terms corresponding to $x_i^{(m)} \in C(k, p^*, e^*, \lambda)$) can be divided into blocks in $C_j(k, p^*, e^*, \lambda)$ where $C_j(k, p^*, e^*, \lambda)$ is the arc-component of $C(k, p^*, e^*, \lambda)$ between y_j and y_{j+1}. If there is an even number of terms of (B) in a given $C_j(k, p^*, e^*, \lambda)$ we leave this block unchanged. If there is an odd number of terms, we lump the "last" term (one farthest to the right) with the contiguous block in (A). With this modification, the estimate in (A) is still valid (see P5) and the remaining terms in (B) occur in blocks of even numbers of consecutive terms, i.e.,

$$\text{(C)} \qquad \sum_i \sum_{x_j^{(m)} \in C_i(k,p^*,e^*,\lambda)} D_m(x - x_j^{(m)})$$

That (C) is bounded follows as in P3. Therefore (B) is uniformly bounded, over the class $\{\phi(-; x)\}$ we are considering in Lemma 1, and Lemma 1 is proved.

3. Properties of $S_m(0, \phi(-; x))$

In this section we discuss certain properties of $S_m(0, \phi(-; x))$. The properties will be labelled by Q followed by an integer.

Q1. Let $n = [p^*/2]$. Then $S_n(0, \phi(-; x)) \geqq (1/20) \log n$ for n sufficiently large.

Recall that

$$(1/\pi) \int_0^\pi | D_n(x) | \, dx = (2/\pi^2) \log n + O(1) \text{ where } O(1) < 2.$$

From this it is clear that if $\chi_n(x)$ is the characteristic function of $A(p^*)$ then

$$S_n(0, \chi_n(x)) = (1/\pi) \int_0^{2\pi} \chi_n(x) D_n(x)\, dx$$

$$\geqq \frac{1}{2} (1/\pi) \int_0^{\pi} | D_n(x)|\, dx = (1/\pi^2) \log n + O(1) \geqq (1/10) \log n.$$

Now $\phi(k, p^*, e^*, \lambda; x)$ is an approximation to $\chi_n(x)$ and it is clear that if p_k is large enough relative to p^* that

$$S_n(0, \phi(-; x)) > (1/2) S_n(0, \chi_n(x)) > (1/20) \log n.$$

Q2. If $m/p^* = \delta$, then $S_m(0, \phi(-; x)) = O(1)\delta \log m + O(1)$.

Notice that $\phi(-; x + 2\pi/p^*) = \phi(-; x)$ for $x, x + 2\pi/p^* \in C([0, \pi])$. Let $d_1(\nu)$, $d_2(\nu)$, $d_3(\nu)$ and $d_4(\nu)$ be defined by the conditions

1° $d_j(\nu) = 2\pi\nu_j/p^*$ $(j = 1, 2, 3, 4)$ (ν_j an integer $\leqq [p^*/2]$);

2° $0 \leqq d_1(\nu) - 2\nu\pi/m < 2\pi\delta/m$, $0 \leqq (2\nu + 1)\pi/m - d_2(\nu) < 2\pi\delta/m$

$0 \leqq d_3(\nu) - (2\nu + 1)\pi/m < 2\pi\delta/m$,

$0 < 2(\nu + 1)\pi/m - d_4(\nu) < 2\pi\delta/m$;

3° $d_2(\nu) - d_1(\nu) = d_4(\nu) - d_3(\nu)$.

Then
$$S_m(0, \phi(-; x)) = (1/\pi) \int_0^{\pi} \phi(-; x)(\sin mx)/x\, dx + O(1)$$

$$= (1/\pi) \sum_{\nu=1}^{[m/2]} \int_{2\nu\pi/m}^{2(\nu+1)\pi/m} \phi(-; x) \sin mx/x\, dx + O(1)$$

$$= (1/\pi) \left\{ \sum_{\nu=1}^{[m/2]} \int_{2\nu\pi/m}^{d_1(\nu)} + \int_{d_1(\nu)}^{d_2(\nu)} + \int_{d_2(\nu)}^{(2\nu+1)\pi/m} + \int_{(2\nu+1)\pi/m}^{d_3(\nu)} \right.$$

$$\left. + \int_{d_3(\nu)}^{d_4(\nu)} + \int_{d_4(\nu)}^{2(\nu+1)\pi/m} \phi(-; x) \sin mx/x\, dx \right\} + O(1)$$

$$= O(1) \left\{ \sum_{\nu=1}^{[m/2]} 1/\nu + \sum_{\nu=1}^{[m/2]} \int_{d_1(\nu)}^{d_2(\nu)} \phi(-; x) \left(\sin mx/x \right.\right.$$

$$\left.\left. + \frac{\sin m(x + d_3 - d_1)}{x + d_3 - d_1} \right) dx \right\} + O(1)$$

$$= O(1)\delta \ln m + O(1)$$

since $|(d_3 - d_1) - \pi/m| < 4\pi\delta/m$. This proves Q2.

Q3. If $m \gg (p_1 \cdots p_k)^{e^*}$ then $|S_m(0, \phi(-; x))| \leqq 2$. For $\phi(-; x)$ is a Lipschitz function and its Fourier series converges to it.

4. An example

We state our main result as

THEOREM 1. *There exists a continuous function* $\phi(x)$ *with* $\{S_n(0, \phi(x))\}$ *diverging but with* $\{I_n(x, \phi(x))\}$ *converging uniformly to* $\phi(x)$ *(on* $[0, 2\pi]$*).*

The function $\phi(x)$ is of the form

$$(\dagger) \qquad \phi(x) = \sum_{i=1}^{\infty} (1/i^2)\phi(k_i, p_i^*, e_i^*, \lambda_i; x)$$

where the parameters $\{k_i\}$, $\{p_i^*\}$, $\{e_i^*\}$ and $\{\lambda_i\}$ are chosen so that $\phi(x)$ satisfies the conditions in Theorem 1.

The sequences of parameters are generated inductively. For the first set of parameters choose $p_1^* = 3$, $k_1 = 4$ (so that $p_{k_1} = 11$) $e_1^* = 3$ and the set λ_1 by the formulas in the paragraph preceding Lemma 1. Now suppose for $i < n$ parameters have been chosen so that

1° $p_i^* > 2^{2^i}$, $(p_{i-1}^* \log p_{i-1}^*)/p_i^* < 1$, and p_i^* satisfies the conditions on m in the hypotheses of Q3;
2° k_i satisfies the conditions $(p_i^* \log p_i^*)/p_{k_i} < 1$, and $p_{k_i} > (p_i^*)^2$;
3° e_i^* satisfies the conditions of D3;
4° for the given choice of k_i, p_i^*, e_i^* the set λ_i is chosen in accordance with the formula $\lambda_i(y_j^{(i)}) = (p_1 \cdots p_{k_i})^{(1+j)e_i*}$ (see the paragraph preceding Lemma 1).

One now proceeds to generate the parameters for the index n. First one chooses p_n^* so that the conditions of 1° are met for $i = n$. One has only to choose p_n^* large enough. Then k_n is chosen so that the conditions of 2° are met (for the given p_n^*). Clearly this will be possible for k_n sufficiently large. Finally e_n^* and λ_n are chosen so as to satisfy 3° and 4°.

Let ϕ be the function defined by the formula ($\dagger$) with the set of parameters satisfying 1°–4°. We show that $\{S_m(0, \phi(x))\}$ diverges. Given p_j^*, let $n_j = [p_j^*/2]$. Then

$$\begin{aligned} S_{n_j}(0, \phi(x)) &= S_{n_j}(0, \textstyle\sum_{i=1}^{j-1} (1/i^2)\phi(k_i, p_i^*, e_i^*, \lambda_i; x)) \\ &\quad + S_{n_j}(0, (1/j^2)\phi(k_j, p_j^*, e_j^*, \lambda_j; x)) \\ &\quad + S_{n_j}(0, \textstyle\sum_{i=j+1}^{\infty} (1/i^2)\phi(k_i, p_i^*, e_i^*, \lambda_i; x)). \end{aligned}$$

From 1° and Q1,

$$S_{n_j}(0, (1/j^2)\phi(k_j, p_j^*, e_j^*, \lambda_j; x)) \geqq (1/20j^2) \log p_j^*/2 > (2^j \log 2)/(20j^2)$$

while

$$S_{n_j}(0, \textstyle\sum_{i=1}^{j-1} (1/i^2)\phi(-; x)) + S_{n_j}(0, \sum_{i=j+1}^{\infty}(1/j^2)\phi(-; x)) = O(1)$$

from Q2 and Q3. Hence the subsequence $\{S_{n_j}(0, \phi(x))\}$ is unbounded so that $\{S_m(0, \phi(x))\}$ diverges.

To prove that $\{I_m(x, \phi(x))\}$ converges uniformly to $\phi(x)$, we remark first that from 2°–4° it follows that Lemma 1 holds for the set $\{\phi(k_i, p_i^*, e_i^*, \lambda_i; x)\}$ used in the definition of ϕ. Given $\varepsilon > 0$, choose n_0 so that for $n \geqq n_0$

$$|\phi(x) - \sum_{i=1}^{n} (1/j^2)\phi(-; x)| < \varepsilon/3.$$

If M is a bound for $\{I_m(x, \phi(k_i, p_i^*, e_i^*, \lambda_i; x))\}$ choose n_1 so that $M/n_1 < \varepsilon/3$. Finally choose m_0 so that for $m \geqq m_0$ and $n_2 = \max(n_0, n_1)$,

$$|I_m(x, \sum_{i=1}^{n_2} (1/i^2)\phi(-; x)) - \sum_{i=1}^{n_2} (1/i^2)\phi(-; x)| < \varepsilon/3$$

(this latter is possible since $\sum_{i=1}^{n_2} (1/i^2)\phi(-; x)$ is Lipschitz). Then for $m \geqq m_0$

$$\begin{aligned} |I_m(x, \phi(x)) - \phi(x)| \leqq\ & |\phi(x) - \sum_{i=1}^{n_2} (1/i^2)\phi(-; x)| \\ & + |\sum_{i=1}^{n_2} (1/i^2)\phi(-; x) - \sum_{i=1}^{n_2} (1/i^2)I_m(x, \phi(-; x))| \\ & + \sum_{n_2+1}^{\infty}(1/i^2)|I_m(x, \phi(-; x))| \\ & < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore $\{I_m(x, \phi(x))\}$ converges uniformly to $\phi(x)$ and Theorem 1 is established.

5. Concluding remarks

The example in Theorem 1 is such that $\{S_m(x, \phi(x))\}$ diverges at 0 and π while converging at other points of $[0, 2\pi]$. First, $\phi(x)$ can be modified so that $\{S_m(x, \phi)\}$ diverges only at a single point (say x_0) and $\{I_m(x, \phi)\}$ converges uniformly to $\phi(x)$. Secondly, given an arbitrary sequence of points $\{x_i\} \subset [0, 2\pi]$, we can construct a set of functions $\{\phi_i(x)\}$, with $\phi_i(x)$ having the behavior at x_i that $\phi(x)$ has at 0 and with $\{I_m(x, \phi_i(x)\}$ converging uniformly to $\phi_i(x)$. Using a standard construction, the $\{\phi_i(x)\}$ can be used to construct a function whose Fourier series diverges at least at each of the points $\{x_i\}$ but whose interpolation series converges uniformly.

Finally we remark that it is relatively simple to construct a function $\psi(x)$ with $\{S_m(0, \psi(x))\}$ divergent and with $\{I_m(x, \psi(x))\}$ convergent to $\psi(x)$ but not uniformly convergent. The major complications in our proof are forced by wishing to make the convergence of $\{I_m(x, \phi)\}$ uniform.

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