NEAR ALGEBRAS

BY
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1. Introduction

A near algebra is an algebraic system with two binary operations satisfying all of the axioms for a ring, except possibly one distributive law, and admitting a field as a left operator domain. The purpose of this paper is to study the structure of certain classes of near algebras.

In Section 2 of this paper the basic concepts are defined and some examples of near algebras are given. The structure of the multiplicative semigroup of a near algebra is investigated and necessary and sufficient conditions for the admissibility of division are determined in Section 3. The concepts of a distributor

$$[a, b, c] = (a + b)c - ac - bc$$

and of a distributor chain are next introduced. These concepts are investigated in Section 4. Semi-simple near algebras are considered in Section 5. In Section 6 near algebras whose linear structures form Banach spaces (topological near algebras) are investigated; and the main result—any semi-simple topological near algebra such that right multiplication is differentiable at the origin is a semi-simple algebra—is proved.

Similar algebraic systems, near rings, were first considered by Dickson [6]. Zassenhaus [11] and Kalscheuer [9] investigated division near rings (Fastkörper). More recently near rings have been studied extensively by Beidleman [1], Blackett [3], Wielandt [10] and others. A comprehensive study of the known results on near rings is contained in [1].

The investigation of near algebras is motivated in part by a possible application to physics, for quantum mechanical models have been considered in which the operators form only a near algebra.

2. Basic concepts

A (left) *near algebra* over a field F is a linear space N over F on which a multiplication is defined such that

(i) N forms a semigroup under multiplication.
(ii) Multiplication is left distributive over addition.
(iii) $a(\delta b) = \delta(ab)$ for $a, b \in N$ and $\delta \in F$.

We will denote the additive identity of N by **0** and the vector space structure of N by N^+, and, if N has a multiplicative identity, we will denote it by **1**.

A natural example of a near algebra over a field F which is not an algebra

Received January 17, 1967.

is the system of all the transformations of a non-zero linear space over F into itself.

In analogy to the n^3 structural constants of an n-dimensional algebra, the multiplicative structure of an n-dimensional near algebra over a field F is completely determined relative to a basis $\{w_1, \cdots, w_n\}$ by n^2 structural functions from F^n into F, say $\{\pi_{ij} : i, j = 1, \cdots, n\}$, which are defined as follows:

$$xw_j = \sum_i \pi_{ij}(\mathbf{x})w_i,$$

where $\mathbf{x}$ denotes the column vector corresponding to $x \in N$. Then, for any $a, b \in N$,

$$(2.1) \qquad (\mathbf{ab}) = \| \pi_{ij}(\mathbf{a}) \| \mathbf{b}.$$

The associativity of multiplication implies that

$$(2.2) \qquad \| \pi_{ij}(\| \pi_{ij}(\mathbf{a}) \| \mathbf{b}) \| = \| \pi_{ij}(\mathbf{a}) \| \; \| \pi_{ij}(\mathbf{b}) \|.$$

If N is an algebra, $\| \pi_{ij}(\mathbf{a}) \|$ is the matrix of the left regular representation of a.

Conversely, if $\{\pi_{ij}\}$ is any set of n^2 functions from F^n into F satisfying (2.2), an n-dimensional vector space over F becomes a near algebra if, relative to a fixed basis, multiplication is defined by (2.1).

Let $\{w_i\}$ and $\{v_i\}$ be two bases for an n-dimensional near algebra N, $\{\pi_{ij}\}$ and $\{\delta_{ij}\}$ the respective n^2 structural functions, and T the change of basis matrix relative to $\{w_i\}$. Since multiplication in N is independent of basis,

$$(\mathbf{ab}) = T^{-1} \| \delta_{ij}(T\mathbf{a}) \| T\mathbf{b}$$

for any $a, b \in N$. Thus

$$\| \pi_{ij}(\mathbf{x}) \| = T^{-1} \| \delta_{ij}(T\mathbf{x}) \| T.$$

As an example of these concepts we will determine the 2-dimensional near algebras N with identity satisfying $\mathbf{0}N \neq \mathbf{0}$ over a field F.

Let $\{\mathbf{1}, w\}$ be a basis for N. For $x \in N$, say $x = \alpha_1 \mathbf{1} + \alpha_2 w$, the correspondence $\pi : x \to \| \pi_{ij}(\mathbf{x}) \|$ is a faithful representation of the multiplicative semigroup of N by 2×2 matrices. Moreover, $\pi_{11}(\mathbf{x}) = \alpha_1$ and $\pi_{21}(\mathbf{x}) = \alpha_2$. Since π is a representation,

$$(2.3) \qquad \pi_{12}(\mathbf{xy}) = \alpha_1\pi_{12}(\mathbf{y}) + \pi_{12}(\mathbf{x})\pi_{22}(\mathbf{y})$$

and

$$(2.4) \qquad \pi_{22}(\mathbf{xy}) = \alpha_2\pi_{12}(\mathbf{y}) + \pi_{22}(\mathbf{x})\pi_{22}(\mathbf{y}).$$

By (2.4), $\pi_{22}(\mathbf{0}) = (\pi_{22}(\mathbf{0}))^2$, i.e. $\pi_{22}(\mathbf{0}) = 0$ or 1. If $\pi_{22}(\mathbf{0}) = 0$, we have $\pi_{12}(\mathbf{0}) = 0$, and $\mathbf{0}N = \mathbf{0}$. Thus $\pi_{22}(\mathbf{0}) = 1$. Let $\pi_{12}(\mathbf{0}) = k$. By letting $y = \mathbf{0}$ in (2.3), and (2.4) we get

$$\pi_{12}(\mathbf{x}) = k - k\alpha_1; \qquad \pi_{22}(\mathbf{x}) = 1 - k\alpha_2.$$

Hence the multiplication on N relative to the basis $\{\mathbf{1}, w\}$ is given by the functional matrix

$$\left\| \begin{matrix} \alpha_1 & k - k\alpha_1 \\ \alpha_2 & 1 - k\alpha_2 \end{matrix} \right\|. \tag{2.5}$$

Although as a direct consequence of the definition of a near algebra N we have

$$x\mathbf{0} = \mathbf{0}; \qquad -(xy) = x(-y), \tag{2.6}$$

for $x, y \in N$, the above example shows that we need not have $\mathbf{0}x = \mathbf{0}$ or $-(xy) = (-x)y$. We also note that in this example $F\mathbf{1}$ does not lie in the center of N.

A linear space V over F is called a *right N-space* if V admits N as a right operator domain such that for $v \in V$, $x, y \in N$ and $\delta \in F$

(i) $v(x + y) = vx + vy$.
(ii) $v(xy) = (vx)y$.
(iii) $v(\delta x) = \delta(vx)$.
(iv) $v\mathbf{1} = v$ if N has an identity $\mathbf{1}$.

If V is also a subspace of N, V is called a *right module*.

Let V and V' be two right N-spaces. A mapping π of V into V' is called an *N-homomorphism* if π is a right N-operator linear transformation on V. If π is also one-to-one, π is called an *N-isomorphism.*

The *right ideals* of N are defined as the kernels of N-homomorphisms on N, and the *left ideals* are defined as the subspaces of N closed under left multiplication by N.

Let N and N' be near algebras over F. A mapping π from N into N' is called a *near algebra homomorphism* if π is a multiplication-preserving linear transformation on N. If π is also one-to-one, π is called a *near algebra isomorphism.* For example, the linear transformation given by

$$\left\| \begin{matrix} 1 & -k \\ 0 & 1 \end{matrix} \right\|$$

is a near algebra isomorphism which carries the 2-dimensional near algebra with functional matrix (2.5) onto the near algebra with functional matrix

$$\left\| \begin{matrix} \alpha_1 & 0 \\ \alpha_2 & 1 \end{matrix} \right\| \tag{2.7}$$

relative to a basis $\{\mathbf{1}, w\}$.

The *ideals* of N are defined as the kernels of near algebra homomorphisms on N.

As for near rings (see e.g. Blackett [3]) we have:

(2.8) R is a right ideal of a near algebra N iff R is a subspace of N^+ and

$(x + r)y - xy \in R$ for $x, y \in N$ and $r \in R$;

(2.9) I is an ideal of a near algebra N iff I is both a right ideal and a left ideal of N.

We note that in general a right ideal is not a right module. For example, in the 2-dimensional near algebra with functional matrix (2.7), the subspace $\{\delta(1 + w) : \delta \in F\}$ is a right ideal which is not a right module. If however $\mathbf{0}N = \mathbf{0}$, then $(\mathbf{0} + r)x - \mathbf{0}x = rx$, and any right ideal is also a right module. Also, in general a right module is not a right ideal. For example, in the near algebra of all mappings of a non-zero linear space into itself, the constant mappings form a right module which is not a right ideal.

It is immediate that the intersection or the sum of two right ideals or ideals is again a right ideal or ideal.

For a near algebra N over F, let

$$A = \{\mathbf{0}N\} \quad \text{and} \quad B = \{x \in N : \mathbf{0}x = \mathbf{0}\}.$$

A direct computation gives that A is a left ideal and a right module and that B is a sub near algebra of N. Moreover, $N = A \oplus B$ as a linear space.

In the remainder of this paper we will consider only near algebras of the type of B, i.e. we will assume the following axiom:

(2.10) Axiom. $\mathbf{0}N = \mathbf{0}$.

We note that with this axiom all of the standard isomorphism theorems for rings, the Zassenhaus lemma, and, where applicable, the Jordan, Hölder, Schreier theorem all hold for near algebras. The proofs carry over almost verbatim from ordinary ring theory.

For a non-empty subset S of a near algebra N, we denote by $A(S)$ the set $\{x \in N : Sx = \mathbf{0}\}$. Using (2.8) and (2.9), a direct computation gives:

(2.11) $A(S)$ is a right ideal, and, if S is a right module, $A(S)$ is an ideal.

3. The multiplicative semigroup

A. Clifford [4], [5] and others have studied extensively the theory of semigroups. Here we apply some of these notions to the multiplicative semigroup of a near algebra N over a field F.

In this section we will assume N has a multiplicative identity $\mathbf{1}$.

A non-empty subset I of N is called a *right* (*left*) *s-ideal* if $IN \subseteq I$ ($NI \subseteq I$).

(3.1) Theorem. *If N is finite-dimensional over F, N contains minimal (non-zero) right s-ideals. Every minimal right s-ideal is a minimal right module, and conversely. Moreover any minimal right s-ideal I is of the form aN for any $a \in I$ such that $aN \neq \mathbf{0}$.*

Proof. Choose $a \in N$ such that $\dim(aN)$ is minimal and non-zero. aN is clearly a minimal right s-ideal. If I is a minimal right s-ideal, then for any

$a \in I$ such that $aN \neq 0$, $aN = I$. By the definition of a near algebra, aN is a right module. Since a right module is also a right s-ideal, I must be minimal as a right module. The converse is now clear.

Since the map $x \to ax$ is an N-homomorphism of N for $a \in N$, we have:

(3.2) THEOREM. *Let N have a minimal right module R. For any $a \in N$ such that $aR \neq 0$, aR is also a minimal right module; and any minimal right module R' such that $R'R \neq 0$ is of the form aR for some $a \in R'$.*

From (3.2) and (2.11) it follows that:

(3.3) COROLLARY. *If N has a minimal right module R and no proper ideals, all minimal right modules are of the form aR and are N-isomorphic.*

(3.4) THEOREM. *A minimal right module R such that $aR \neq 0$ for any $0 \neq a \in R$ when considered as a near algebra has no proper right s-ideals. Moreover, if R has a right identity, R is a division near algebra.*

Proof. Let I be a non-zero right s-ideal of R. Then,

$$0 \neq IR \subseteq I \subseteq R, \quad \text{and} \quad 0 \neq (IR)N \subseteq R.$$

Since R is minimal, $IR = I = R$. If R has a right identity, $aR = R$ for any $0 \neq a \in R$.

(3.5) COROLLARY. *A non-zero near algebra N with a right identity is a division near algebra iff N has no proper right s-ideals.*

(3.6) THEOREM. *If N has a minimal right s-ideal I and no proper ideals, then $NI = \bigcup nI$ $(n \in N)$ is the unique minimal two sided s-ideal of N.*

Proof. By (3.1) and (3.3), NI is the union of all the minimal right s-ideals of N. Let S be a non-zero two sided s-ideal of N. By (2.11), $A(nI) = 0$ if $nI \neq 0$. Thus $(nI)S$ is a non-zero right s-ideal for any minimal right s-ideal nI, and $nI = (nI)S \subseteq S$. Hence $NI \subseteq S$.

(3.7) COROLLARY. *If N has a minimal right module R and no proper ideals, then N has no proper two sided s-ideals iff each principal right s-ideal (or equivalently, by (3.1), principal right module) $nN \neq 0$ is minimal.*

Proof. If every non-zero principal right s-ideal is minimal, by (3.6) $N = \bigcup nN$ $(n \in N)$ is the only non-zero two sided s-ideal in N. Conversely, if N has no proper two sided s-ideals, $N = \bigcup nR$ $(n \in N)$. Thus for any $0 \neq a \in N$, a is in some minimal right s-ideal, say A; and $0 \neq aN \subseteq A$. Hence $aN = A$ is a minimal right s-ideal.

Since for a near algebra N the subset $\{a : \dim(aN) < k\}$ is a two sided s-ideal for any integer $k > 0$, it follows that:

(3.8) THEOREM. *A finite-dimensional near algebra N is a division near algebra iff N has no proper two sided s-ideals.*

(3.9) COROLLARY. *Let N be a finite-dimensional near algebra with no proper ideals. N is a division near algebra iff each principal right module is minimal.*

Proof. By (3.1), N contains a minimal right module. Hence N satisfies the hypothesis of (3.7) and has no proper two sided s-ideals.

4. Distributors

Let V be a right N-space. For $a,\ b \in V$ and $n \in N$, we call the element $(a + b)n - an - bn$ of V the *distributor* of a and b with respect to n and denote it by $[a, b, n]$. For a non-void subset A of V and a sub near algebra B of N, we denote by $D_B(A)$ the subspace of V generated by

$$\{[a,\ b,\ n] : a,\ b \in A,\ n \in B\}.$$

We say that A is *right distributive* if $D_N(A) = \mathbf{0}$. For a sub near algebra A of N, we will denote $D_A(A)$ more simply by $D(A)$.

Fröhlich [8] has investigated similar concepts for distributively generated near rings.

(4.1) THEOREM. *For a near algebra N, $D(N)$ is an ideal of N.*

Proof. For $a, b, c \in N$ and $\delta \in F$, $\delta[a, b, c] = [a, b, \delta c]$. Hence

$$D(N) = \{\sum [a_i, b_i, c_i] : a_i, b_i, c_i \in N\}.$$

Also, a direct computation gives that for $x, y \in N$

$$x(\sum [a_i, b_i, c_i]) = \sum [xa_i, xb_i, c_i], \tag{4.2}$$

$$\begin{aligned}(x + [a, b, c])y - xy = [x, [a, b, c], y] - [ac + bc, [a, b, c], y] \\ + [a, b, cy] - [ac, bc, y],\end{aligned} \tag{4.3}$$

and, if $d = \sum_{i=1}^{k} [a_i, b_i, c_i]$ $(k > 1)$ and $d' = \sum_{i=1}^{k-1} [a_i, b_i, c_i]$,

$$\begin{aligned}(x + d)y - xy = \{((x + d') + [a_k, b_k, c_k])y - (x + d')y\} \\ + \{(x + d')y - xy\}.\end{aligned} \tag{4.4}$$

Using (4.3), (4.4) and induction on k, we get that

$$(x + d)y - xy \in D(N)$$

for $d \in D(N)$, $x, y \in N$. By (2.8), $D(N)$ is a right ideal. (4.2) shows that $D(N)$ is also a left ideal, and, by (2.9), $D(N)$ is an ideal of N.

(4.5) THEOREM. *Let N be a finite-dimensional near algebra with an identity and no proper ideals. If $D_N(aN)$ is either a minimal right module or zero for each $a \in N$, then N is a full matrix ring over a division ring or N is a division near algebra.*

Proof. By (4.1), $D(N) = \mathbf{0}$ or $D(N) = N$. In the first case N is a simple finite-dimensional ring which, by the Wedderburn theorem, must be of the first type. In the latter case we have $\mathbf{1} = \sum [a_i, b_i, c_i]$, $a_i, b_i, c_i \in N$. From (4.2) it follows that $D_N(aN) \supseteq aN$. Since aN is a right module, $D_N(aN) \subseteq aN$. Thus $D_N(aN) = aN$, and, by (3.9), N is of the second type.

If A is an ideal (right ideal) of N, we denote by N/A the near algebra (right N-space) of cosets with the operations induced by those in N. (Using (2.8) and (2.9) one can easily verify that these induced operations are well defined.)

(4.6) THEOREM. *For a near algebra N, $N/D(N)$ is right distributive. If N/A is right distributive for an ideal (right ideal) A, then $A \supseteq D(N)$.*

Proof. The first statement is clear since we have factored out exactly the right distributivity relation. The second statement follows from the observation that if π is the natural near algebra homomorphism (N-homomorphism) from N onto N/A, $D(N) \subseteq \ker \pi$.

We define repeated distributors as follows: $D^0(N) = N$, $D^1(N) = D(N)$, $D^k(N) = D(D^{k-1}(N))$. By (4.6) each $D^{k-1}(N)/D^k(N)$ is right distributive. A sub near algebra A of N is called *d-solvable* if for some k, $D^k(A) = \mathbf{0}$.

(4.7) THEOREM. *Every sub near algebra of a d-solvable near algebra is d-solvable. Every homomorphic image of a d-solvable near algebra N is d-solvable. If a near algebra N contains a d-solvable ideal A such that N/A is d-solvable, then N is d-solvable.*

Proof. The proof of the first statement is clear, and from

$$D(N/A) = (D(N) + A)/A$$

the second statement follows. If N/A is d-solvable, $D^k(N) \subseteq A$ for some k, and if A is also d-solvable, $D^{s+k}(N) \subseteq D^s(A) = \mathbf{0}$ for sufficiently large s.

(4.8) COROLLARY. *The sum of a d-solvable sub near algebra A of N and of a d-solvable ideal B of N is d-solvable.*

Proof. By one of the standard isomorphism theorems, $(A + B)/B$ is isomorphic to $A/(A \cap B)$. Since $A/(A \cap B)$ is the homomorphic image of a d-solvable near algebra, it is also d-solvable. Thus $(A + B)/B$ is d-solvable, and, since B is d-solvable, $A + B$ is d-solvable.

For the remainder of this section, we will assume N satisfies the a.c.c. on ideals.

By (4.8), N contains a unique maximal d-solvable ideal which we will call the *d-radical* of N and denote by $dR(N)$. We say that N is *d-semi-simple* if $dR(N) = \mathbf{0}$.

(4.9) THEOREM. *$N/dR(N)$ is d-semi-simple for any near algebra N.*

Proof. If $A/dR(N)$ is a d-solvable ideal in $N/dR(N)$, by (4.7), A is d-solvable. Hence $A \subseteq dR(N)$.

For two sub near algebras A, B of N, we define their product AB as the sub near algebra generated by the set $\{ab : a \in A, b \in B\}$ and a power A^n by $A^n = A(A^{n-1})$. We note that in general, because of the missing distributive law, the multiplication of sub near algebras is non-associative. A sub near algebra A of N is said to be *nilpotent* if $A^k = \mathbf{0}$ for some $k > 0$.

Since for a sub near algebra A we have $D^k(A) \subseteq A^{2^{k+1}}$, it follows that:

(4.10) A nilpotent sub near algebra is d-solvable.

Thus,

(4.11) A d-semi-simple near algebra is semi-simple in the ring theoretic sense.

The converse of (4.11) is false, e.g. any semi-simple, non-zero algebra.

A chain starting at N and terminating at $\mathbf{0}$,

$$N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_k = \mathbf{0},$$

is called a *distributive series* for N if each N_{i+1} is an ideal of N_i and each factor N_i/N_{i+1} is right distributive with respect to N_i. Such a chain is called a *d-composition series* if each term N_{i+1} is maximal in N_i with respect to the property that N_i/N_{i+1} is right distributive and non-zero.

(4.12) THEOREM. *Let N be a finite-dimensional near algebra. Any distributive series for N can be refined into a d-composition series.*

(4.13) THEOREM. *The factors of a given d-composition series for a near algebra N are near algebra isomorphic to the factors in any d-composition series for N in some sequential order.*

The proofs of (4.12) and (4.13) are completely analogous to the proofs of the corresponding statement for composition series in group theory. (Note that with Axiom (2.10) the Zassenhaus lemma holds for near algebras.)

(4.14) THEOREM. *A near algebra N is d-solvable iff N possesses a distributive series.*

Proof. If N is d-solvable, the series

$$N = D^0(N) \supseteq D^1(N) \supseteq \cdots \supseteq D^k(N) = \mathbf{0}$$

is a distributive series for N. Conversely, if

$$N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_k = \mathbf{0}$$

is a distributive series for N, by (4.6), $\mathbf{0} = N_k \supseteq D^k(N)$.

As an example of these concepts we will construct, for a given integer $n > 0$, a near algebra with d-composition series length n.

Let V be an n-dimensional vector space over $\mathcal{R}$ in a coordinate representation relative to a fixed basis. We define a multiplication on V as follows:

$$(4.15) \qquad (\alpha_1, \cdots \alpha_n)\ (\beta_1, \cdots, \beta_n) = (f_1(a)\beta_1, \cdots f_n(a)\beta_n),$$

where $a = (\alpha_1, \cdots, \alpha_n)$ and $f_1(a) = \alpha_1$, $f_i(a) = 0$ if $\alpha_{i-1} = 0$ and $f_i(a) = \alpha_i$ otherwise $(1 < i \leq n)$. A direct computation gives that V with the multiplication defined by (4.15) is a near algebra satisfying $\mathbf{0}V = \mathbf{0}$. Moreover, $D^k(V)$ is the $(n - k)$-dimensional subspace of V consisting of all vectors with first k components zero, and

$$V = D^0(V) \supseteq D^1(V) \supseteq \cdots \supseteq D^n(V) = \mathbf{0}$$

is a d-composition series for V of length n.

5. Semi-simple near algebras

A near algebra N is said to be *semi-simple* if N satisfies the d.c.c. on right modules and has no non-zero nilpotent right modules. N is said to be *simple* if N is a semi-simple, non-zero near algebra with no proper ideals.

We note that the concept of semi-simplicity can also be defined in terms of a radical. Namely, the J-radical of a near algebra N, denoted by $J(N)$, is defined as the intersection of all right annihilator ideals $A(R)$ (see (2.11)) where R ranges over the minimal right modules of N. Betsch [2] has shown that if N satisfies the d.c.c. on right modules, N is semi-simple iff $J(N) = \mathbf{0}$ and that $N/J(N)$ is semi-simple.

(5.1) THEOREM. *Let N be a near algebra such that N satisfies the d.c.c. on right modules and $N/J(N)$ is right distributive. N is right distributive iff $D(N)$ contains no non-zero nilpotent right modules.*

Proof. By (4.1), $D(N)$ is an ideal and therefore contains a minimal right module, say R. By (4.6), $J(N) \supseteq D(N)$. Thus $D(N) \subseteq A(R)$, and we have $R^2 = \mathbf{0}$. By hypothesis we must have $R = D(N) = \mathbf{0}$. The converse is clear.

Blackett [3] has proved that a semi-simple near ring can be decomposed into a ring theoretic direct sum of simple near rings. Blackett's proof carries over directly to near algebras, and we have:

(5.2) THEOREM. *A semi-simple near algebra can be decomposed into a ring theoretic direct sum of simple near algebras.*

6. Topological near algebras

In this section we will assume that N is a near algebra over the real or complex number field such that N^+ is a Banach space with respect to some norm. We will call such a near algebra a *topological near algebra*.

We say that multiplication in a topological near algebra N is continuous in the first (second) factor if the map $x \to xa$ $(x \to ax)$ of N into N is continuous in x for each $a \in N$.

As an example of a topological near algebra we give the "twisted quaternions" of Kalscheuer [9].

Let N^+ be the vector space of quaternions over $\mathcal{R}$ with the usual norm topology. For $a, b \in N^+$, we define their product $a * b$ as follows:

$$a * b = b * a = \mathbf{0}, \qquad a = \mathbf{0},$$

$$a * b = a \cdot \delta_y(a) \cdot b \cdot (\delta_y(a))^{-1}, \quad a \neq \mathbf{0},$$

where

$$\delta_y(a) = \cos(1/y(\log Na)) + i \sin(1/y(\log Na))$$

for $0 \neq y \in \mathcal{R}$. Here $\cdot$ denotes the usual quaternion multiplication, and Na denotes the norm of a.

Kalscheuer proved that the members of this one parameter family of near algebras are the only finite-dimensional topological division near algebras over $\mathcal{R}$ with continuous multiplication which are not algebras.

(6.1) THEOREM. *Let N be a topological near algebra over $\mathcal{R}$ such that multiplication is continuous in the second factor. Then, the near algebra axiom $\delta(ab) = a(\delta b)$ follows from the other axioms.*

Proof. For $a, b \in N$,

$$a((\tfrac{1}{2})b) = a(b - (\tfrac{1}{2})b) = ab - a((\tfrac{1}{2})b),$$

and by an induction argument,

$$a((\tfrac{1}{2})^n b) = (\tfrac{1}{2}^n)(ab), \qquad n > 0.$$

For $1 > \delta > 0$, let $\sum (\delta_i/2^i)$ be the dyadic expansion of δ. For $n > 0$,

$$a((\textstyle\sum_{i=1}^n \delta_i/2^i)b) = (\textstyle\sum_{i=1}^n \delta_i/2^i)\ (ab),$$

and letting $n \to \infty$, we have $a(\delta b) = \delta(ab)$. Since $a(2b) = 2(ab)$ and $a((-1)b) = -(ab)$, the conclusion follows.

Let N be a finite-dimensional near algebra, and $\{w_1, \cdots, w_n\}$ be a basis for N. For $a \in N$, we will denote by $L(a)$ the matrix relative to $\{w_i\}$ corresponding to the linear transformation $x \to ax$ of N into N. We note that the correspondence $x \to L(x)$ is a homomorphism of the multiplicative structure of N.

(6.2) THEOREM. *Let N be an n-dimensional topological near algebra with identity $\mathbf{1}$ such that multiplication is continuous in the first factor. Then, the unit group G of N contains a neighborhood of $\mathbf{1}$ and thus generates N additively. Moreover, the map $x \to x^{-1}$ is a continuous map on G.*

Proof. Choose a basis $\{w_i\}$ for N, and let

$$L(N) = \{L(a) : a \in N\}$$

have the topology induced by norm $\| a_{ij} \| = \max | a_{ij} |$. For $a \in N$ and $\varepsilon > 0$, let $S(\varepsilon)$ be the ε-sphere about $\mathbf{0}$. Since multiplication is continuous in the first factor, we can find neighborhoods S_j of a, $j = 1, \cdots, n$, such that

$x \in S_j$ implies that $xw_j \in \{S(\varepsilon) + aw_j\}$. Let $S = \bigcap S_j$, a neighborhood of a. For $x \in S$, since aw_j and xw_j are the j^{th} columns of $L(a)$ and $L(x)$ respectively, $\text{norm}(L(a) - L(x))$ is less than ε. Thus $a \to L(a)$ is a continuous function of a. Since the map $L(a) \to \det L(a)$ is continuous on $L(N)$, $a \to \det L(a)$ is a continuous map on N. In particular, there is a neighborhood U of $\mathbf{1}$ such that $x \in U$ implies that $\det L(x) - \det L(\mathbf{1}) < 1$. Thus $x \in U$ implies that $L(x)$ is invertible. For $x \in U$ and $y \in N$, $xy = \mathbf{0}$ implies that $L(y) = L^{-1}(x)L(\mathbf{0}) = L(\mathbf{0})$. But $\mathbf{1} \in N$ implies that $L(y) = L(\mathbf{0})$ iff $y = \mathbf{0}$. Since N is finite-dimensional, $xN = N$ for $x \in U$, i.e. $x \in G$. Moreover, $L^{-1}(x) = L(x^{-1})$. Since $L^{-1}(x)$ is a continuous function of the entries of $L(x)$ and $L(x)$ is a continuous function of x, $L(x^{-1})$ is a continuous function of x. Thus $L(x^{-1})\,\mathbf{1} = x^{-1}$ is a continuous function of x.

We note that by (6.2) the unit group of a near algebra N satisfying the hypothesis of (6.2) is a Lie group containing a basis for N. Therefore, one could study the structure of such near algebras by employing Lie theory. This approach, however, has difficulties as is demonstrated in the following two examples of topological near algebras.

Let N be the near algebra over $\mathbb{R}$ generated by the mappings $xf_1 = x$ and $xf_2 = |x|$ from $\mathbb{R}$ into $\mathbb{R}$. N is a 2-dimensional near algebra over $\mathbb{R}$, and, since f_2 is non-linear, N is not an algebra. The component of the identity of the unit group of N is the set

$$\{\alpha f_1 + \beta f_2 : \alpha > 0, \alpha > |\beta|\}.$$

Let N' be the 2-dimensional (commutative) algebra of 2×2 matrices generated by

$$g_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad g_2 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}.$$

The component of the identity of the unit group of N' is

$$\{\alpha g_1 + \beta g_2 : \alpha > 0, \alpha > |\beta|\}.$$

The components of the identity of N and N' are near algebra isomorphic, even on their boundaries, but their global structures are quite different. Thus to apply Lie theory to the study of these near algebras one would face the difficult task of determining all distinct global extensions of the operations in the component of the identity.

(6.3) THEOREM. *Let N be a finite-dimensional topological near algebra with identity such that multiplication is continuous in both factors. Then N is a division near algebra iff for each $a \neq \mathbf{0}$ in N and each sequence $\{n_i\}$ in N such that $\{an_i\}$ converges to $\mathbf{0}$, $\{n_i\}$ converges.*

Proof. Let a^{-1} exist for each $a \neq \mathbf{0}$ in N, and $\{an_i\}$ converge to $\mathbf{0}$. For any open neighborhood S of $\mathbf{0}$, since $x \to a^{-1}x$ is continuous at $\mathbf{0}$,

$$S' = \{x : a^{-1}x \in S\}$$

is also a neighborhood of $\mathbf{0}$. Thus for sufficiently large i, $an_i \in S'$, i.e. $n_i \in S$; and $\{n_i\}$ converges to $\mathbf{0}$. Conversely, assume that $\{an_i\}$ converges to $\mathbf{0}$ implies $\{n_i\}$ converges, $a \neq \mathbf{0}$. Since N is finite-dimensional, if for some $\mathbf{0} \neq a \in N$, a^{-1} does not exist, we must have $ax = \mathbf{0}$ for some $x \neq \mathbf{0}$. Define the sequence $\{n_i\}$ by $n_i = x$ if $i \equiv 0\ (2)$ and $n_i = 2x$ if $i \equiv 1\ (2)$. Now $\{an_i\}$ is identically $\mathbf{0}$, but $\{n_i\}$ does not converge.

Let N be a topological near algebra and M a non-zero right module of N. If π is any zero-preserving map from N into N, we will denote by $D_M \pi$ the derivative, if it exists, of π on M at $\mathbf{0}$; i.e. $D_M \pi$, if it exists, is the unique linear transformation on M satisfying

$$\lim_{x \to 0} \frac{\text{norm } (x\pi - xD_M \pi)}{\text{norm } x} = 0; \qquad x \in M, x \neq \mathbf{0}.$$

(See e.g. Dieudonné [7].)

(6.4) THEOREM. *Let N be a topological near algebra such that for some fixed non-zero right module M the map $x \to xa$ (i.e. right multiplication) is differentiable on M at $\mathbf{0}$ for each $a \in N$. Then, there is a well defined near algebra homomorphism from N onto a right distributive near algebra N'. Moreover, N' is non-zero if at least one of the above derivatives is non-zero.*

Proof. For $a \in N$, let $R(a)$ denote the map $x \to xa$ of N into N. A direct computation gives that the correspondence $a \to R(a)$ is a near algebra homomorphism of N onto $R(N)$ where the operations in $R(N)$ are defined in the usual manner. By hypothesis $D_M R(a)$ exists for each $a \in N$, and by the usual rules for the derivative of a linear combination of functions, the correspondence $R(a) \to D_M R(a)$ is a well defined linear transformation from $R(N)$ onto the vector space $N' = \{D_M R(a) : a \in N\}$. Moreover, since $\mathbf{0}R(a) = \mathbf{0}$ for each $a \in N$, by the chain rule

$$D_M(R(a)R(b)) = (D_M R(a))(D_M R(b)).$$

Thus the correspondence

$$a \to R(a) \to D_M R(a)$$

is a well defined near algebra homomorphism from N onto N'. Since the $D_M R(a)$ are linear transformations, N' is a right distributive near algebra.

(6.5) THEOREM. *Let N be a semi-simple topological near algebra with identity such that the map $x \to xa$ is differentiable on N at $\mathbf{0}$ for each $a \in N$. Then N is a semi-simple algebra.*

Proof. By (5.2), $N = \oplus_{i=1}^{k} N_i$ where each N_i is a simple near algebra. Let

$$\mathbf{1} = e_1 + \cdots + e_k, \qquad e_i \in N_i.$$

Since the multiplication in N can be carried out component-wise, each $R(e_i)$ is the i^{th} projection map. Thus $D_N R(e_i) = R(e_i) \neq 0$. By (6.4) the map

$a \to D_N R(a)$ restricted to N_i is a near algebra homomorphism of N_i onto a non-zero right distributive near algebra N_i, $i = 1, \cdots k$. Since each N_i is simple, these maps must be near algebra isomorphisms. Thus each N_i is a right distributive near algebra, and by a argument symmetric to that in (6.1), $\delta(a_i b_i) = (\delta a_i) b_i$ for $a_i, b_i \in N_i$ and scalar δ. Therefore N is the ring theoretic direct sum of simple algebras and hence is a semi-simple algebra.

We note that the continuity of multiplication is not sufficient to assure that even a simple topological near algebra is an algebra; e.g. Kalscheuer's "twisted quaternions."

Acknowledgments. The author does thank J. Beidleman, H. Wielandt and H. Zassenhaus for their valuable comments.

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