MINIMAL SETS AND ERGODIC MEASURES FOR $\beta N \backslash N$

BY

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0. Introduction

Let N be the additive semigroup of positive integers with the discrete topology, $m(N)$ the space of bounded real-valued functions on N with the usual sup norm, and $m(N)^*$ the conjugate Banach space of $m(N)$. $\varphi' \in m(N)^*$ is a mean if $\| \varphi' \| = 1$, and $(\varphi', f) \geq 0$ whenever $f \geq 0$. A mean is said to be invariant if $(\varphi', f) = (\varphi', \tau f)$ for each $f \in m(N)$, where $\tau f \in m(N)$ is defined by $(\tau f)(n) = f(n+1), n \in N$. The set of all invariant means on N is denoted by M'. It is well known that M' is non-empty, convex and weak* compact (Day [2]).

The structure of M' as a set is already known. There are various descriptions of how to get the set of invariant means on $m(N)$, e.g., Jerison [5], and Raimi [8]. In §3 we study a geometrical property of the set M'. Namely, we show that extreme points of the convex compact set M' are insufficient in the sense that there exists $\varphi' \in M'$ such that φ' is not a countably convex combination of extreme points of M'. This answers a question asked by R. G. Douglas.

As is well known [10], N may be embedded densely into βN, its Stone-Cech compactification, and $m(N)$ and $C(\beta N)$, the space of real-valued continuous functions of βN, are isomorphic as Banach spaces. Any $\varphi' \in M'$, as a functional on $C(\beta N)$, is represented, by the Riesz Representation Theorem, as a measure. In §1 and §2 these measures are studied, continuing work begun in [9]. In particular, since extreme points of M' are represented by ergodic measures on βN (relative to the homeomorphism of βN into βN induced by the translation on N), it is quite natural to ask whether the support of an ergodic measure is a minimal set (definition 1.1). The answer turns out to be negative; this is the main result of §2.

1. Invariant means and invariant measures

The continuous mapping $n \to n + 1$, $n \in N$, of N into the compact space βN has a unique continuous extension to βN. We will denote this extended mapping by τ. If $\varphi' \in m(N)^*$, then φ' corresponds to a Borel measure φ on βN. The correspondence is characterized by $(\varphi', f) = \int_{\beta N} \bar{f}\, d\varphi$, where $f \in m(N)$ and $\bar{f}$ denotes its continuous extension to βN. In the sequel, for $\varphi' \in m(N)^*$, φ will always denote the corresponding measure. If $\varphi' \in M'$,

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then φ is a probability measure, i.e., φ is positive and $\varphi(\beta N) = 1$, and φ is invariant in the sense that $\varphi(\tau\beta) = \varphi(B)$ for each Borel set $B \subset \beta N$. We denote by M the set of invariant probability measures corresponding to M'. It is easy to see that if $\varphi \in M$ then supp $\varphi \subset \beta N \setminus N = \hat{N}$. (For a measure ψ, the support of ψ is denoted by supp ψ.)

DEFINITION 1.1. A subset X of $\hat{N}$ is said to be invariant if $\tau X \subset X$. A set $K \subset \hat{N}$ is said to be minimal if K is closed, non-empty, invariant, and minimal with respect to these three properties.

It is an easy consequence of Zorn's Lemma that each non-empty, closed invariant set in $\hat{N}$ contains a minimal set.

DEFINITION 1.2. The orbit of a point $\omega \in \hat{N}$ is the set

$$o(\omega) = \{\tau^i\omega : i = 0, 1, 2, \cdots\}.$$

The orbit closure of $\omega \in \hat{N}$ is the closure of $o(\omega)$ in $\hat{N}$ and we denote it by $\bar{o}(\omega)$.

Rudin [11] showed that for each $\omega \in \hat{N}$, $\tau^i\omega \neq \tau^j\omega$ if $i \neq j$. Thus $o(\omega)$ is an infinite set and hence is not closed, since a closed infinite subset of $\hat{N}$ contains 2^c points [10]. (Here c denotes the cardinality of the continuum.) If K is a minimal set then $K = \bar{o}(\omega)$ for each $\omega \in K$.

DEFINITION 1.3. (Raimi [8]) For each subset $A \subset N$ we define the upper and lower densities as follows:

$$\bar{d}(A) = \sup \{(\varphi', \chi_A) : \varphi' \in M'\};$$

$$\underline{d}(A) = \inf \{(\varphi', \chi_A) : \varphi' \in M'\}.$$

Here χ_A denotes the characteristic function on N of the set A.

The following lemma has appeared in several different forms. For our convenience, we quote the one in [8].

LEMMA 1.1. *Given* $A \subset N$,

$$\bar{d}(A) = \lim_n \sup \sup_{m \in N} (1/n) \text{ Card } \{A \cap \{m, m+1, \cdots, m+n-1\}\};$$

$$\underline{d}(A) = \lim_n \inf \inf_{m \in N} (1/n) \text{ Card } \{A \cap \{m, m+1, \cdots, m+n-1\}\}.$$

The above lemma is, indeed, a simple consequence of the Krein-Milman Theorem.

A set $A \subset N$ is said to be *relatively dense* if there exists a $p \in N$ such that for each $n \in N$,

$$\{n, n+1, \cdots, n+p-1\} \cap A \neq \emptyset.$$

DEFINITION 1.4. A point $\omega \in \hat{N}$ is said to be almost periodic if for each neighborhood U of ω the set $\{n \in N : \tau^n\omega \in U\}$ is relatively dense (c.f. Gottschalk-Hedlund [4]). The set of all almost periodic points is denoted by A^τ.

A point $\omega \in \hat{N}$ is said to be τ-discrete if $o(\omega)$ is a discrete space in its subspace topology. The set of τ-discrete points is designated by D^τ.

A point $\omega \in \hat{N}$ is said to be strongly τ-discrete if there exists a neighborhood U of ω such that $\tau^n U \cap \tau^m U = \emptyset$ for $m \neq n$. The set of strongly τ-discrete points is denoted by SD^τ. Clearly, $SD^\tau \subset D^\tau$.

A subset $A \subset N$ is said to be thin if $A \cap \tau^n A$ is a finite set for each positive integer n.

As usual, if $A \subset N$, we denote $\hat{A} = \mathrm{cl}_{\beta N} A \cap \hat{N}$. $\hat{A}$ is closed and open in $\hat{N}$ and sets of form $\hat{A}$ form a topological basis for $\hat{N}$. It is also true that sets of form $\hat{A}$, $A \subset N$, are the only open and closed subsets of $\hat{N}$. $\hat{A}$ is empty if and only if A is a finite set. (All of these facts are in [10].)

PROPOSITION 1.2. (1) *If A is a thin set, then $\bar{d}(A) = 0$.*

(2) $SD^\tau = \bigcup \{\hat{A} : A$ *a thin subset of* $N\}$.

(3) SD^τ *is dense in* $\hat{N}$.

Proof. (1) Note that a set $A \subset N$ is thin if and only if $\hat{A} \cap \tau^n \hat{A} = \emptyset$ for each $n \in N$. Let A be a thin set and let $\varphi \in M$ be given.

$$1 = \varphi(\hat{N}) \geq \varphi(\hat{A} \cup \tau\hat{A} \cup \cdots) = \varphi(\hat{A}) + \varphi(\tau\hat{A}) + \cdots$$

but $\varphi(\hat{A}) = \varphi(\tau\hat{A}) = \cdots$. Thus $\varphi(\hat{A}) = 0$, but φ is an arbitrary element in M, we conclude that $\bar{d}(A) = 0$.

(2) Obvious.

(3) Let B be an arbitrary infinite subset of N. By induction, we construct an infinite subset $A = \{a_1, a_2, \cdots\}$ of B such that $a_{n+1} - a_n > n + 1$. But, then the set A is thin, indeed, Card $(A \cap \tau^n A) \leq n$. This says that for each non-empty clopen subset $\hat{B}$ of $\hat{N}$ there exists a non-empty clopen set $\hat{A}$ such that $\hat{A} \subset \hat{B}$ and such that A is a thin set. Combining this with (2) and the fact that clopen sets form a topological basis of $\hat{N}$, we conclude that SD^τ is dense in $\hat{N}$.

As in Raimi [9, p. 3], we set $K^\tau = \mathrm{cl}\, [\bigcup \{\mathrm{supp}\, \varphi : \varphi \in M\}]$. We have the following characterization of the set K^τ:

LEMMA 1.3. *For $\omega \in \hat{N}$, $\omega \in K^\tau$ if and only if whenever $\omega \in \hat{A}$ we always have $\bar{d}(A) > 0$.*

Proof. Let $\omega \in K^\tau$ and suppose $\omega \in \hat{A}$. Since $\hat{A}$ is open, by the definition of K^τ, there exists $\varphi \in M$ such that $\hat{A} \cap \mathrm{supp}\, \varphi \neq \emptyset$. By the definition of supp φ, we conclude that $\varphi(\hat{A}) > 0$, and hence $\bar{d}(A) > 0$.

Conversely, if for each clopen neighborhood $\hat{A}$ of ω, $\bar{d}(A) > 0$, then there exists a $\varphi \in M$ such that $\varphi(\hat{A}) > 0$. This implies that $\hat{A} \cap \mathrm{supp}\, \varphi \neq \emptyset$. Thus $\omega \in K^\tau$.

COROLLARY 1.4. $K^\tau \cap SD^\tau = \emptyset$.

Proof. Apply Lemma 1.3 and Proposition 1.2.

The following corollary is Theorem 2.7 of Raimi [9]. Our proof is easier and completely different from his proof.

COROLLARY 1.5. *K^τ is nowhere dense in $\hat{N}$.*

Proof. Follows from Corollary 1.4 and Proposition 1.2.

A point $\omega \in \hat{N}$ is called a P-point of $\hat{N}$ if it has the property that whenever $\{U_n\}$ is a sequence of neighborhoods of ω then $\bigcap_{n=1}^{\infty} U_n$ is again a neighborhood of ω. We denote by P the set of all P-points of $\hat{N}$.

PROPOSITION 1.6. $P \subset SD^\tau$.

Proof. Let $\omega \in P$. For each n, choose a neighborhood U_n of ω such that $\tau^n\omega \notin U_n$. (We can do this since $\tau^n\omega \neq \omega$.) Then $V_0 = \bigcap_{n=1}^{\infty} U_n$ is a neighborhood of ω and $\tau^n\omega \notin V_0$ for $n \in N$. Since $\tau^n\omega$, $n = 1, 2, \cdots$, are also P-points, by induction, we can find V_i, $i = 0, 1, 2, \cdots$, such that

(1) V_i is a neighborhood of $\tau^i\omega$, $i = 0, 1, 2, \cdots$.
(2) $V_i \cap V_j = \emptyset$ if $i \neq j$.
(3) For each i, $\tau^n\omega \notin V_i$ if $n \neq i$.

Let $V = V_0 \cap \tau^{-1}V_1 \cap \cdots \cap \tau^{-n}V_n \cap \cdots$. Then V is a neighborhood of ω and it can be directly checked that if $n \neq m$, then $\tau^n V \cap \tau^m V = \emptyset$. Thus $\omega \in SD^\tau$.

COROLLARY 1.7. ([9, Theorem 2.11]) $P \cap K^\tau = \emptyset$.

Proof. Follows from Corollary 1.4 and Proposition 1.6.

Although the set K^τ is invariant and nowhere dense, it is very scattered as the following proposition shows. In particular, we give a negative answer to the following natural question: Does there exist any $\varphi \in M$ such that $\operatorname{supp} \varphi = K^\tau$? We need the following characterization of sets with upper density 1 which is contained in Mitchell [6, p. 258].

LEMMA 1.8. *Let $A \subset N$. The following are equivalent.*

(1) $\bar{d}(A) = 1$.
(2) *There exists $\varphi \in M$ such that* $\operatorname{supp} \varphi \subset \hat{A}$.
(3) *For each $n \in N$, there exists $m \in N$ such that*

$$\{m, m+1, \cdots, m+n-1\} \subset A.$$

Mitchell obtained this as a corollary of a theorem concerning general amenable semigroups. However, it is worth mentioning that we can prove this lemma by applying Lemma 1.1.

PROPOSITION 1.9. *For each $A \subset N$ with $\bar{d}(A) = 1$ there exist c many disjoint clopen subsets $\hat{B} \subset \hat{A}$ such that $\bar{d}(B) = 1$, $d(B) = 0$, and $\bar{d}(B \,\Delta\, \tau^k B) = 0$ for each $k \in N$.*

Proof. Assume that $\bar{d}(A) = 1$. Choose $k_1 \in A$ and set $A_1 = \{k_1\}$. Assume that we have constructed $A_1, \cdots, A_n$ such that

$$A_i = \{k_i, k_i + 1, \cdots, k_i + i - 1\} \tag{1}$$

$$\text{for some } k_i \in N \text{ and } A_i \subset A \ (i = 1, 2, \cdots, n);$$

$$k_i > k_{i-1} + 2i - 1, \qquad i = 2, \cdots, n. \tag{2}$$

Now $\bar{d}(A) = 1$ implies that $\bar{d}(A \setminus \{1, 2, \cdots, k_n + 2n\}) = 1$. Applying Lemma 1.8 to the set $A \setminus \{1, 2, \cdots, k_n + 2n\}$, we find a $k_{n+1} \in N$ such that

$$\{k_{n+1}, k_{n+1} + 1, \cdots, k_{n+1} + n\} \subset A \setminus \{1, 2, \cdots, k_n + 2n\}.$$

It is clear that $A_1, \cdots, A_n, A_{n+1}$ also satisfy (1) and (2) for $n + 1$. By induction we construct a sequence of subsets A_i of A such that (1) and (2) are true for all $n \in N$. If I is an infinite subset of N and $B = \bigcup_{i \in I} A_i$, then it is not hard to see that $\bar{d}(B) = 1$, $\underline{d}(B) = 0$, and $\bar{d}(B \,\Delta\, \tau^k B) = 0$, by applying Lemma 1.1.

Now let θ be a one-one mapping of N onto the set of rational numbers. For each irrational number r, choose a fixed sequence of increasing rationals, $s_n(r)$, converging to r. Note that if r_1 and r_2 are two different irrational numbers then $s_n(r_1) \neq s_n(r_2)$ for all large n. Set

$$I(r) = \{\theta^{-1}(s_n(r)) : n = 1, 2, \cdots\}$$

for each irrational r, $I(r)$ is an infinite subset of N and if $r_1 \neq r_2$ then $I(r_1) \cap I(r_2)$ is finite. Let E be the set of irrationals. For $r \in E$, set $A(r) = \bigcup_{i \in I(r)} A_i$. Then $A(r_1)^\wedge \cap A(r_2)^\wedge = \emptyset$, if $r_1 \neq r_2$. Since each of the set $A(r)^\wedge$ is of the form B mentioned earlier, $\{A(r)^\wedge : r \in E\}$ is the family of clopen sets we wanted. Note that the last step of our proof is similar to [3, Problem 6Q].

COROLLARY 1.10. *If $\bar{d}(A) = 1$ and φ_n, $n = 1, 2, \cdots$, is a sequence of probability measures, then*

$$\mathrm{cl}\,[U_{n=1}^{\infty} \operatorname{supp} \varphi_n] \not\supset K^\tau \cap \hat{A}.$$

Proof. Let $A(r)^\wedge$, $r \in E$, be a family of mutually disjoint clopen sets constructed as in Proposition 1.9, for the set A. For each n, $\operatorname{supp} \varphi_n$ intersects at most countably many $A(r)^\wedge$. Thus there is an $A(r_0)$ such that

$$A(r_0)^\wedge \cap \operatorname{supp} \varphi_n = \emptyset \qquad \text{for all } n.$$

Now $A(r_0)^\wedge$ is open and

$$A(r_0)^\wedge \cap K^\tau \neq \phi \quad \text{and} \quad A(r_0)^\wedge \cap \mathrm{cl}\,[U_{n=1}^{\infty} \operatorname{supp} \varphi_n] = \emptyset.$$

The results follows.

The above corollary says that, in some sense, the set K^τ is very scattered.

2. The existence of an ergodic measure with non-minimal support

As usual, a measure $\varphi \in M$ is said to be *ergodic* if whenever a Borel set $B \subset \hat{N}$ satisfying $\varphi(B \Delta \tau^n B) = 0$ for all $n \in N$, then $\varphi(B) = 1$ or $= 0$. It is well known [4, Theorem 2] that $\varphi \in M$ is an ergodic measure if and only if φ is an extreme point of M. The support of an invariant measure φ is closed and invariant. If, in addition φ is ergodic, then supp φ cannot be divided into two measure theoretically non-trivial sets. On the other hand, it follows easily from the Markov Fixed Point Theorem and the Krein-Milman Theorem, that for each minimal set K, there exists ergodic φ such that supp $\varphi = K$. With these facts, one is quite natural to conjecture that the support of each ergodic measure is minimal. The main purpose of this section is to show that the conjecture is not true. We will also show that $\mathrm{cl}\, A^\tau \subsetneq K^\tau$, i.e., that the almost periodic points of $\hat{N}$ are not sufficient to generate all support sets for members of M.

Notation. If $\omega \in \beta N$, ω' will denote the element in $m(N)^*$ defined by $(\omega', f) = \bar{f}(\omega)$, where $f \in m(N)$, and $\bar{f}$ is the continuous extension of f to βN.

If K is a subset of a vector space, ex K denotes the set of all extreme points of K.

Suppose $B \subset N$ and $k \geq 1$. B is called a k-chain if whenever p and q are two consecutive numbers of B, $|q - p| \leq k$. (Thus a 1-chain is a set of consecutive numbers of N.) A 1-chain will be called connected. Clearly, B is a k-chain, alternatively, if and only if $\bigcup_{i=0}^{k-1} \tau^i B$ is connected. We know (Lemma 1.8) $\bar{d}(A) = 1$ if and only if A contains connected subsets of unbounded length. An easy consequence of this result is the following

LEMMA 2.1. *Let A be a subset of N and k a positive integer. Then*

$$\bar{d}(\textstyle\bigcup_{i=0}^{k-1} \tau^i A) = 1$$

if and only if A contains k-chains of unbounded length.

Note that every k-chain in a set $A \subset N$ is contained in a maximal k-chain, and so A is partitioned into k-components $A_1, A_2, \cdots$, which are disjoint maximal k-chains.

If A is a subset of N and $\bar{d}(A) > 0$, one might think of extending Mitchell's result to say that there exists some $k > 0$ such that

$$\bar{d}(A \cup \tau A \cup \cdots \cup \tau^{k-1} A) = 1.$$

But this is not true:

PROPOSITION 2.2. *There exists a set $A_0 \subset N$ such that*

(p1) $\bar{d}(A_0 \cup \tau A_0 \cup \cdots \cup \tau^{k-1} A_0) < 1$, *for* $k \in N$,
(p2) $\bar{d}(A_0) > 0$,
are satisfied.

Proof. We shall construct such a set A_0 by induction. Let $A_1 = \{1, 2, 3\}$. Assume that we have constructed $A_1, A_2, \cdots, A_n$. Set

$$a_n = \text{Card } A_n, \qquad b_n = \text{Card } (\{1, 2, \cdots, \sup A_n\} \backslash A_n).$$

Then define,

$$A_{n+1} = A_n \cup [a_n + b_n + n + A_n] \cup [2a_n + 2b_n + 2n + A_n].$$

And, finally, set

$$A_0 = A_1 \cup A_2 \cup \cdots .$$

A_0 satisfies (p1). Let $k \in N$ and let $B_1, B_2, \cdots$ be the k-components of A_0. By the construction of the set A_0, it is obvious that Card $(B_i) = a_k$, $i = 1, 2, \cdots$. Thus, by Lemma 2.1,

$$\bar{d}(A_0 \cup \tau A_0 \cup \cdots \cup \tau^{k-1} A_0) < 1.$$

A_0 satisfies (p2). It is easy to see that the a_n's satisfy the following equation:

$$a_{n+1} = 3 \cdot a_n, \qquad n = 1, 2, \cdots .$$

But $a_1 = 3$, hence $a_n = 3^n$. The b_n's satisfy the following equation:

$$b_{n+1} = 3b_n + 2n, \qquad b_1 = 0.$$

Thus $b_{n+1} = \frac{1}{2}(3^{n+1} - 3) - n$. Let φ' be a weak* limiting point of the sequence

$$(*) \qquad \frac{1' + 2' + \cdots + (3^n + \frac{1}{2}(3^n - 3) - (n-1))'}{3^n + \frac{1}{2}(3^n - 3) - (n-1)}, \qquad n = 1, 2, \cdots .$$

It is directly calculated that $\varphi' \in M'$. Now note that the numerator in (*) is

$$1' + 2' + \cdots + (a_n + b_n)'$$

and that exactly a_n of these numbers lie in A_0. Thus

$$(\varphi', \chi_{A_0}) = \lim_{n \to \infty} \frac{3^n}{3^n + \frac{1}{2}(3^n - 3) - (n-1)} = \frac{2}{3}$$

Thus $\bar{d}(A_0) \geq \frac{2}{3} > 0$.

Recall that A^τ is the set of almost periodic points of $\hat{N}$ (Definition 1.4) and K^τ is the set of $\omega \in \hat{N}$ such that if $\omega \in \hat{A}$ then $\bar{d}(A) > 0$ (Lemma 1.3).

Proposition 2.3. cl $A^\tau \subset_{\neq} K^\tau$.

Proof. If $\omega \in A^\tau$ and $A \subset N$ such that $\omega \in \hat{A}$, then $\{n : \tau^n \omega \in \hat{A}\}$ is relatively dense. Let φ' be a weak* limiting point of $\{\omega', (\omega' + (\tau\omega)')/2, \cdots\}$; then $\bar{d}(A) \geq (\varphi', \chi_A) > 0$. Thus $A^\tau \subset K^\tau$. K^τ is closed, hence cl $A^\tau \subset K^\tau$. To prove they are not equal, let $A_0 \subset N$ satisfying (p1) and (p2) of Proposition 2.2. We assert

(1) $A^\tau \cap \hat{A}_0 = \emptyset$,
(2) $K^\tau \cap \hat{A}_0 \neq \emptyset$.

Since $\hat{A}_0$ is open, (1) implies $\mathrm{cl}\, A^\tau \cap \hat{A}_0 = \emptyset$. Thus if (1) and (2) are true then $\mathrm{cl}\, A^\tau \subsetneq K^\tau$.

Assume that (1) is false and let $\omega \in A^\tau \cap \hat{A}_0$; then by the definition of almost periodic point, $\{n : \tau^n \omega \in \hat{A}_0\}$ is relatively dense in N. This equivalent to the existence of $k \in N$ such that the orbit

$$o(\omega) \subset \hat{A}_0 \cup \tau\hat{A}_0 \cup \cdots \cup \tau^{k-1}\hat{A}_0 .$$

But $\hat{A}_0 \cup \tau\hat{A}_0 \cup \cdots \cup \tau^{k-1}\hat{A}_0$ is closed, hence

$$\bar{o}(\omega) \subset \hat{A}_0 \cup \tau\hat{A}_0 \cup \cdots \cup \tau^{k-1}\hat{A}_0 = (A_0 \cup \tau A_0 \cup \cdots \cup \tau^{k-1}A_0)^\wedge.$$

Thus $(A_0 \cup \tau A_0 \cup \cdots \cup \tau^{k-1}A_0)^\wedge$ contains a closed invariant set; hence

$$\bar{d}(A_0 \cup \cdots \cup \tau^{k-1}A_0) = 1,$$

denying (p1). Thus $\hat{A}_0 \cap A^\tau = \emptyset$.

(2) is true. By (p2), $\bar{d}(A_0) > 0$, hence $\hat{A}_0 \cap K^\tau \neq \emptyset$. Indeed, if A is any subset of N such that $\bar{d}(A) > 0$, then there exists $\varphi' \in M'$ such that $(\varphi', \chi_A) > 0$, i.e., $\varphi(\hat{A}) > 0$. Because $\hat{A}$ is open some point $\omega \in \hat{A}$ is in the support of φ. Hence $\omega \in K^\tau$.

To prove the next proposition, we need an essentially known result.

LEMMA 2.4. $A^\tau = \{\omega \in \hat{N} :$ *ω is contained is some minimal set*$\}$.

Proof. $\omega \in A^\tau$ implies that ω is contained in some minimal set is proved implicitly in Rudin [11, Theorem 3]. The converse statement is proved by Gottschalk and Hedlund [4, p. 32].

PROPOSITION 2.5. *There exists an ergodic measure with non-minimal support.*

Proof. Let A_0 be a set satisfying (p1) and (p2). Since $\bar{d}(A_0) > 0$, there exists $\varphi \in M$ such that $\varphi(\hat{A}_0) > 0$; hence there exists an ergodic measure ψ such that $\psi(\hat{A}_0) > 0$. Then $\hat{A}_0 \cap \mathrm{supp}\, \psi \neq \emptyset$. Thus by Lemma 2.4, $\mathrm{supp}\, \psi$ contains a point which is contained in no minimal set and is therefore not minimal.

By a similar argument, we have

PROPOSITION 2.6. *The closed convex hull of invariant measures with minimal support is properly contained in M.*

3. Countably convex combinations of ergodic measures

Let $\{c_n\}_{n=1}^\infty$ be a sequence of non-negative real numbers such that $\sum_{n=1}^\infty c_n = 1$ and let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of probability measures on $\hat{N}$. It is easy to see that $\sum_{n=1}^k c_n \varphi_n$, $k = 1, 2, \cdots$, is a Cauchy sequence with respect to the total variation norm. Thus $\sum_{n=1}^k c_n \varphi_n$ converges in norm to a Borel measure on $\hat{N}$. We will denote this measure by $\sum_{n=1}^\infty c_n \varphi_n$ and call it a countably convex combination of $\{\varphi_n\}$ (with coefficients $\{c_n\}$). Note that the summation is absolute in the sense that for each permutation π of N, we have

$$\sum_{n=1}^\infty c_{\pi(n)} \varphi_{\pi(n)} = \sum_{n=1}^\infty c_n \varphi_n .$$

If all the φ_n's are in M, then since M is weak* closed and convex, we see that $\sum_{n=1}^{\infty} c_n \varphi_n \in M$.

We will denote the set of all countably convex combinations of a set X by cco (X), whenever it is defined.

Since M is convex and weak* compact, by a combination of the Krein-Milman Theorem and the Riesz Representation Theorem, each $\varphi \in M$ is represented by a probability measure μ defined on the weak* compact set cl (ex M) in the weak* sense, that is to say that for each $\hat{f} \in C(\hat{N})$, we have

$$\varphi(\hat{f}) = \int_{\mathrm{cl}(\mathrm{ex} M)} \psi(\hat{f})\, d\mu(\psi)$$

(Phelps [7, Proposition 1.2]).

It is then quite natural to ask whether we can improve the above result to show that each $\varphi \in M$ is represented by a discrete measure on cl (ex M). (A measure is said to be discrete if it is concentrated on a countable set.) R. G. Douglas asked whether each φ can be represented by a discrete measure on ex M itself. The answer to our question (and hence to Douglas' weaker question) turns out to be negative. Note that $\varphi \in M$ can be represented by a discrete measure on cl (ex M) if and only if $\varphi \in$ cco (cl (ex M)).

DEFINITION 3.1. A subset A of N is said to be nearly invariant if $\bar{d}(A \,\Delta\, \tau^n A) = 0$ for each $n \in N$.

LEMMA 3.1. *If $\varphi \in$ cl (ex M) and A is nearly invariant then* supp $\varphi \subset \hat{A}$ *or* supp $\varphi \subset (N \backslash A)^\wedge$.

Proof. Let A be nearly invariant. Then for each $\varphi \in M$, $\varphi(\hat{A} \,\Delta\, \tau^n \hat{A}) = 0$ for all $n \in N$. If, in addition, $\varphi \in$ ex M, then φ is an ergodic measure, hence $\varphi(\hat{A}) = 1$ or $\varphi((N \backslash A)^\wedge) = 1$. But $\hat{A}$ and $(N \backslash A)^\wedge$ are closed sets. Thus supp $\varphi \subset \hat{A}$ or supp $\varphi \subset (N \backslash A)^\wedge$.

Now let φ be an arbitrary element in cl (ex M). Choose a net φ_α in ex M such that $\varphi_\alpha \to \varphi$ in weak* topology. Thus $\varphi_\alpha(\hat{A})$ converges to $\varphi(\hat{A})$, since $\chi_{A^\wedge}$ is continuous. But $\varphi_\alpha(\hat{A})$ either equals 1 or equals 0 for each α. Thus $\varphi(\hat{A}) = 1$ or 0.

Notation. For each $\omega \in \beta N$, the set of cluster points of the net,

$$(1/n)(\omega' + (\tau\omega)' + \cdots + (\tau^{n-1}\omega)'), \qquad n = 1, 2, \cdots,$$

will be denoted by Q'_ω. It is known and easily calculated, that $Q'_\omega \neq \emptyset$ and $Q'_\omega \subset M'$. As usual, the set of measures corresponding to Q'_ω will be denoted by Q_ω.

PROPOSITION 3.2. $Q_1 \cap$ cco (cl (ex M)) $= \emptyset$

Proof. Given $m \in N$, set

$$N_i = \{i\} \cup \{m + 2i - 1, m + 2i\} \cup \cdots \cup \{(m/2)(n-1)n + (i-1)n + 1, \cdots, (m/2)(n-1)n + in\} \cup \cdots, \qquad i = 1, 2, \cdots, m.$$

Note that

(1) $\bigcup_{i=1}^m N_i = N$, and hence $\bigcup_{i=1}^m \hat{N}_i = \hat{N}$.

It is easily calculated that the Cesáro mean of the sequence χ_{N_i} equals $1/m$ for each $1 \leq i \leq m$. This implies that (by the definition of the set Q_1')

(2) for each $\varphi' \in Q_1'$, and for each $i \in N$, $i \leq m$, we have $(\varphi', \chi_{N_i}) = 1/m$.

Now note that for each $j \in N$ and each $i \leq m$, $N_i \backslash \tau^j N_i$ is of the form

$$F \cup A_1 \cup A_2 \cup \cdots$$

where F is a finite set and for $n \in N$,

$$A_n = \{k_n, k_n + 1, \cdots, k_n + j - 1\}$$

with $k_{n+1} > k_n + j - 1$ and $k_{n+1} - k_n \to \infty$ as $n \to \infty$. Using Lemma 1.1, we conclude that $\bar{d}(N_i \backslash \tau^j N_i) = 0$ for $1 \leq i \leq m$ and $j \in N$. Similarly, $\bar{d}(\tau^j N_i \backslash N_i) = 0$ for $1 \leq i \leq m$ and $j \in N$. Thus

(3) for each $1 \leq i \leq m$, N_i is nearly invariant.

Now to prove the proposition, suppose that there were

$$\varphi \in Q_1 \cap \operatorname{cco}(\operatorname{cl}(\operatorname{ex} M)),$$

say, $\varphi = \sum_{n=1}^\infty c_n \varphi_n$, with $\sum_{n=1}^\infty c_n = 1$, $c_n \geq 0$ and $\varphi_n \in \operatorname{cl}(\operatorname{ex} M)$. Fix $n \in N$. By (1), $\operatorname{supp} \varphi_n \cap \hat{N}_i \neq \emptyset$ for some $i \leq m$. By Lemma 3.1 and (3) above, $\operatorname{supp} \varphi_n \subset \hat{N}_j$. Hence by (2), $1/m = \varphi(\hat{N}_i) \geq c_n \varphi_n(\hat{N}_i) = c_n$. But m is a pre-fixed constant, which is independent of n. Thus we have $c_n = 0$. Again, n is arbitrary, i.e., $c_n = 0$ for all $n \in N$, in other words, $\varphi = 0$. This is a contradiction.

A modification of the proof of Proposition 3.2 will lead us to the following,

PROPOSITION 3.4. *If $\omega \in D^\tau$ (Definition 1.4), then*

$$Q_\omega \cap \operatorname{cco}(\operatorname{cl}(\operatorname{ex} M)) = \emptyset.$$

Proof. Let $\omega \in D^\tau$ and let $m \in N$ be given. Set

$$\Omega_i = \operatorname{cl}\{\tau^n \omega : n \in N_i\}, \qquad i = 1, 2, \cdots, m,$$

where N_i is as in the proof of Proposition 3.2. Since $o(\omega)$ is discrete in the subspace topology, we can construct by induction a disjoint sequence of subsets A_n, $n = 1, 2, \cdots$, of N such that $\tau^n \omega \in \hat{A}_n$. Define $f \in m(N)$ as follows:

$$f(l) = i \quad \text{if} \quad l \in \bigcup \{A_n : n \in N_i\}, \qquad i = 1, 2, \cdots, m;$$

$$f(l) = 0 \quad \text{if} \quad l \notin \bigcup \{A_n : n \in N\}.$$

Denote by $\bar{f}$, the continuous extension of f to βN. Then if $n \in N_i$, we have $\bar{f}(\tau^n \omega) = i$. This implies

(1) $\Omega_i \cap \Omega_k = \emptyset$ if $i \neq k$.

Since the sets Ω_i are closed and mutually disjoint, we can find m mutually disjoint clopen sets $\hat{V}_i$ such that $\Omega_i \subset \hat{V}_i$ for all i. Now, since the Cesáro mean of χ_{N_i} has value $1/m$, and since $\chi_{V_i}{}^\wedge$ is continuous on $\hat{N}$, if $\varphi \in Q_\omega$, we have $\varphi(\hat{V}_i) = 1/m$. Now note that support of $\varphi \subset \bar{o}(\omega)$. It follows easily that

(2) if $\varphi \in Q_\omega$, then $\varphi(\Omega_i) = 1/m$, $i = 1, 2, \cdots, m$.

Again since $o(\omega)$ is discrete, an elementary argument tells us that

$$\begin{aligned} \Omega_i \Delta \tau^n \Omega_i &= \text{cl}\{\tau^j \omega : j \in N_i \Delta \tau^n N_i\} \\ &= \text{cl}\{\tau^j \omega : j \in N_i \backslash \tau^n N_i\} \cup \text{cl}\{\tau^j \omega : j \in \tau^n N_i \backslash N_i\}. \end{aligned}$$

Thus

(3) $\varphi(\Omega_i \Delta \tau^n \Omega_i) = 0$, for each $\varphi \in M$, $n \in N$, and $i \leq m$.

But since the collection $\{\Omega_i\}$ can be separated by the clopen set $\hat{V}_i$, by an argument similar to Lemma 3.1, we see that if $\varphi \in \text{cl}(\text{ex}\, M)$, and $\text{supp}\, \varphi \subset \bar{o}(\omega)$, then $\varphi(\Omega_i) = 1$ or 0.

Now assume that

$$\varphi = \textstyle\sum_{n=1}^{\infty} c_n \varphi_n \in Q_\omega \cap \text{cco}(\text{cl}(\text{ex}\, M)).$$

We may assume that $\text{supp}\, \varphi_n \subset \bar{o}(\omega)$. Then using the facts we developed above, it follows as in Proposition 3.2 that $\varphi = 0$, a contradiction.

Remarks. (1) Let $Q = \bigcup \{Q_\omega : \omega \in \hat{N}\}$. Jerison [5] showed that $M = \overline{\text{co}}\, Q$ ($=$ the closed convex hull of Q). A look at his proof shows that it is not hard to see that we can extend it to say that

$$M = \overline{\text{co}}\, (\bigcup \{Q_\omega : \omega \in K^\tau\}).$$

But by Proposition 2.6, $M \neq \overline{\text{co}}\, (\bigcup \{Q_\omega : \omega \in A^\tau\})$.

Jerison asked whether the set Q is closed. By the Krein-Milman Theorem, if the set Q is closed then all the extreme points of M are contained in Q. Thus to show that Q is not closed it would be more than sufficient to show that Q doesn't contain any ergodic measure. We have already shown that if $\omega \in D^\tau$ then Q_ω contains no ergodic measure (Proposition 3.3). But for the general case it seems very hard to prove, perhaps not true. However, by Proposition 2.5, to show that Q is not closed we need only to show that if $\omega \notin A^\tau$ then Q_ω contains no ergodic measures. It appears to us that we could refine Theorem 6 of Rudin [11] to get this result.

(2) Nearly invariant sets are insufficient in the following sense. There exists a set A such that $\bar{d}(A) = 1$, $\underline{d}(A) = 0$ but A cannot be written either as

(i) $A = A_1 \cup A_2$, A_1 nearly invariant and $\bar{d}(A_2) < 1$, or as

(ii) $A = A_1 \backslash A_2$, A_1 nearly invariant and $\bar{d}(A_2) < 1$.

Indeed, let $\varphi \in \operatorname{ex} M$ such that supp φ is not minimal. Choose $\omega \in \operatorname{supp} \varphi \cap A^{\tau}$. (Such an ω exists since the closed invariant set supp φ contains a minimal set and points in minimal sets are almost periodic.) Since supp φ is not minimal there exists $\omega_1 \in \operatorname{supp} \varphi \backslash \bar{o}(\omega)$. Now we can find a subset $A \subset N$ such that $\bar{o}(\omega) \subset \hat{A}$, $\omega_1 \notin \hat{A}$ and $\hat{A}$ avoids $\bar{o}(\omega_2)$, where ω_2 is an almost periodic point not in $\bar{o}(\omega)$. (It follows from Proposition 1.9 that there exist at least c minimal sets.) Since $\hat{A}$ contains the closed invariant set $\bar{o}(\omega)$, $\bar{d}(A) = 1$. On the other hand, since $\hat{A} \cap \bar{o}(\omega_2) = \emptyset$, $\underline{d}(A) = 0$. We claim the set A cannot have the form (i) or (ii). If we can write $A = A_1 \cup A_2$ as in (i), then since $\bar{d}(A_2) < 1$, some $\tau^i\omega \notin \hat{A}_2$, for if $o(\omega) \subset \hat{A}_2$ then $\bar{d}(A_2) = 1$. Thus some $\tau^i\omega \in \hat{A}_1$. Now A_1 is nearly invariant and $\hat{A}_1 \cap \operatorname{supp} \varphi \neq \emptyset$ we must have supp $\varphi \subset \hat{A}_1$ (Lemma 3.1). This is a contradiction, since $\omega_1 \notin \hat{A}_1$, but $\omega_1 \in \operatorname{supp} \varphi$. Similarly, we can show that (ii) is impossible.

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