FURTHER GAPS IN THE DIMENSIONS OF TRANSFORMATION GROUPS

BY

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Introduction

It is well known [3] that if a compact Lie group G of homeomorphisms acts effectively on a connected m-manifold M,

$$\dim G \leq m(m+1)/2.$$

In addition, it has been observed previously [5, Chapter IV], [4] that the dimension of G cannot fall into the following two ranges:

$$(m-1)m/2 + 1 < \dim G < m(m+1)/2 \qquad (m \neq 4)$$

$$(m-2)(m-1)/2 + 3 < \dim G < (m-1)m/2 \qquad (m \text{ large}).$$

In [2] we showed that the above two ranges of gaps in dimensions are part of a general pattern. Specifically we established the following result [2, Theorem 2].

THEOREM A. *Let G be a compact Lie group acting effectively on a connected m-manifold M. Then if the dimension of G falls into one of the following ranges*:

$$(m-k)(m-k+1) + k(k+1)/2$$

$$< \dim G < (m-k+1)(m-k+2), \quad k = 1, 2, 3, \cdots$$

we have only three possibilities:

(i) $m = 4$, *G is isomorphic to $SU(3)/Z$ (Z denotes the center of the special unitary group $SU(3)$), M is homeomorphic to the complex projective plane $P^2(C)$ and G acts transitively on M.*

(ii) $m = 6$, *G is isomorphic to the exceptional Lie group G_2, M is homeomorphic to either the sphere S^6 or real projective space $P^6(R)$ and G acts transitively on M.*

(iii) $m = 10$, *G is isomorphic to $SU(6)/Z$, M is homeomorphic to $P^5(C)$ and G acts transitively on M.*

In this paper we show that the pattern of gaps given by Theorem A is but a special case of a still more general pattern of gaps. This, in effect, settles a question which we raised at the end of [2]. Although our present result does not exhaust all possible gaps, we have reason to believe, as will be discussed later, that it produces the most general consistent pattern of gaps.

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2. Preliminaries

The following notation will be helpful. If n is a positive integer,

$$\langle n\rangle = n(n+1)/2.$$

$$\Phi(n) = \text{largest integer } j \text{ such that } \langle n-j\rangle + \langle j\rangle < \langle n-j+1\rangle - 1.$$

In the statement of Theorem A, k runs from 1 to $\Phi(m)$. The following short table of values of $\Phi(n)$ will be of future assistance:

n	$\Phi(n)$
3	1
6	2
10	3
15	4
21	5
28	6
36	7

LEMMA 1. $\Phi(n) = [(\sqrt{\{1+8n\}} - 3)/2]$ *where* $[x]$ *denotes the largest integer* $\leq x$.

LEMMA 2. *If* $n_1 \geq n_2 \geq u \geq 0$,

$$\langle n_1\rangle + \langle n_2\rangle \leq \langle n_1 + u\rangle + \langle n_2 - u\rangle.$$

LEMMA 3. *If* $n_1 \geq n_2 \geq 0$,

(a) $\langle n_1\rangle + \langle n_2\rangle \leq \langle n_1 + n_2\rangle$,

(b) $\langle n_1 - n_2\rangle \leq \langle n_1\rangle - \langle n_2\rangle$.

LEMMA 4. $\langle n+1\rangle - \langle n\rangle = n + 1$.

LEMMA 5. $n - \Phi(n) \geq \langle\Phi(n)\rangle + 1$.

LEMMA 6. $n \leq \langle n - \Phi(n)\rangle$.

LEMMA 7. $\langle n-j-1\rangle + \langle j+1\rangle \leq \langle n-j\rangle + \langle j - \Phi(j)\rangle$ *for* $j \leq \Phi(n)$, $j \geq 1$.

Proof. The result of course follows immediately from the definition of $\Phi(n)$ for $j \leq \Phi(n) - 1$. We let $j = \Phi(n)$. Now

$$\begin{aligned}(1)\quad &\langle n - \Phi(n) - 1\rangle + \langle\Phi(n) + 1\rangle \\ &\qquad = \langle n - \Phi(n)\rangle + \langle\Phi(n)+1\rangle - (n - \Phi(n))\end{aligned}$$

by Lemma 4. Applying Lemma 5,

$$\begin{aligned}&\langle n - \Phi(n) - 1\rangle + \langle\Phi(n) + 1\rangle \\ (2)\quad &\qquad \leq \langle n - \Phi(n)\rangle + \langle\Phi(n)+1\rangle - \langle\Phi(n)\rangle - 1 \\ &\qquad \leq \langle n - \Phi(n)\rangle + \Phi(n) \qquad \text{(Lemma 4)}\end{aligned}$$

Since by Lemma 6,

$$\Phi(n) \leq \langle \Phi(n) - \Phi(\Phi(n)) \rangle \tag{3}$$

the result follows.

We have reduced the next lemma which will be used heavily in the sequel to the following technical form.

LEMMA 8. *Let K, k, u, t_j ($j = 1, 2, \cdots, r$), v, q be non-negative integers satisfying the following conditions:*

(i) $v = 0$ *or* $v \geq 3$, $u \geq 1$, $k \geq 2$,
(ii) $k \leq \Phi(K)$,
(iii) $K - k - u \geq t_j$, *all* j,
(iv) $k - v - q + u \geq 0$,
(v) $\sum_{j=1}^{r} t_j \leq k - v - q + u$.

Then

$$\langle K - k - u \rangle + \sum_{j=1}^{r} \langle t_j \rangle \leq \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2v - q.$$

Proof. It could be checked directly that with the hypothesis above

$$\langle K - k \rangle + \langle k - \Phi(k) \rangle - 2v - q \geq 0.$$

This fact, however, will of course be established indirectly through the course of the proof.

Case I. $v + q \leq k + 1$. By repeated application of Lemma 2,

$$\langle K - k - u \rangle + \sum_{j=1}^{r} \langle t_j \rangle \leq \langle K - k - 1 \rangle + \sum_{j=1}^{r} \langle t_j' \rangle \tag{1}$$

where

$$t_j' \geq 0, \quad \text{all } j, \text{ and } \sum_{j=1}^{r} t_j' \leq k - v - q + 1.$$

Applying Lemma 3(a) and then Lemma 3(b),

$$\begin{aligned} \langle K - k - u \rangle + \sum_{j=1}^{r} \langle t_j \rangle &\leq \langle K - k - 1 \rangle + \langle k - v - q + 1 \rangle \\ &\leq \langle K - k - 1 \rangle + \langle k + 1 \rangle - \langle v + q \rangle \\ &\leq \langle K - k - 1 \rangle + \langle k + 1 \rangle - 2v - q. \end{aligned} \tag{2}$$

The last step follows since $v = 0$ or $v \geq 3$. Finally, applying Lemma 7,

$$\langle K - k - u \rangle + \sum_{j=1}^{r} \langle t_j \rangle \leq \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2v - q. \tag{3}$$

Case II. $v + q \geq k + 2$. Let $\eta = k - v - q + u \geq 0$. Therefore $\eta < u$. Now,

$$\begin{aligned} \langle K - k - u \rangle + \sum_{j=1}^{r} \langle t_j \rangle &\leq \langle K - k - (u - \eta) \rangle && \text{(Lemma 2)} \\ &\leq \langle (K - k - 1) - (u - \eta - 1) \rangle \\ &\leq \langle K - k - 1 \rangle - \langle u - \eta - 1 \rangle \\ & && \text{(Lemma 3}(b)) \\ &\leq \langle K - k - 1 \rangle - \langle v + q - k - 1 \rangle. \end{aligned} \tag{4}$$

Subcase (a). $v + q \geq k + 4$. Now,

$$v + q - k - 1 \geq 3 \quad \text{and} \quad \langle v + q - k - 1 \rangle \geq 2(v + q - k - 1).$$

Hence,

$$\langle K - k - 1 \rangle - \langle v + q - k - 1 \rangle \leq \langle K - k - 1 \rangle + 2(k + 1) - 2v - 2q. \tag{5}$$

Now,

$$2(k + 1) \leq \langle k + 1 \rangle \quad \text{for } k \geq 2. \tag{6}$$

Combining (4), (5), and (6),

$$\begin{aligned} &\langle K - k - u \rangle + \textstyle\sum_{j=1}^{r} \langle t_j \rangle \\ &\quad \leq \langle K - k - 1 \rangle + \langle k + 1 \rangle - 2v - 2q \\ &\quad \leq \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2v - q \quad \text{(Lemma 7)}. \end{aligned} \tag{7}$$

Here $k = 1$ can also be handled by an individual check; so far we have not had to enforce the condition that $k \geq 2$.

Subcase (b). $v + q = k + 3$. From (4),

$$\begin{aligned} &\langle K - k - u \rangle + \textstyle\sum_{j=1}^{r} \langle t_j \rangle \\ &\quad \leq \langle K - k - 1 \rangle - 3 \\ &\quad \leq \langle K - k \rangle + \langle k - \Phi(k) \rangle - \langle k + 1 \rangle - 3 \quad \text{(Lemma 7)}. \end{aligned} \tag{8}$$

Now for $k \geq 3$, $2(k + 3) < \langle k + 1 \rangle + 3$. Hence,

$$\begin{aligned} \langle K - k - u \rangle + \textstyle\sum_{j=1}^{r} \langle t_j \rangle &< \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2(k + 3) \\ &< \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2v - q. \end{aligned} \tag{9}$$

Here $k = 2$ can also be handled as a special case. The result, however, is not valid for $k = 1$ in this subcase.

Subcase (c). $v + q = k + 2$. From (4),

$$\begin{aligned} &\langle K - k - u \rangle + \textstyle\sum_{j=1}^{r} \langle t_j \rangle \\ &\quad \leq \langle K - k - 1 \rangle - 1 \\ &\quad \leq \langle K - k \rangle + \langle k - \Phi(k) \rangle - \langle k + 1 \rangle - 1 \quad \text{(Lemma 7)}. \end{aligned} \tag{10}$$

For $k \geq 3$,

$$2(k + 2) < \langle k + 1 \rangle + 1. \tag{11}$$

Hence,

$$\begin{aligned} \langle K - k - u \rangle + \textstyle\sum_{j=1}^{r} \langle t_j \rangle &< \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2(k + 2) \\ &< \langle K - k \rangle + \langle k - \Phi(k) \rangle - 2v - q. \end{aligned} \tag{12}$$

Again $k = 2$ can be handled by a special check, while the result is not valid for $k = 1$.

3. Statement of main result

G will denote a compact Lie group acting on a connected m-manifold M. The action of G on M is said to be *almost effective* if the normal subgroup K of G formed from all elements of G which act trivially on M is finite; an almost effective action is said to be *almost free* if G/K acts freely on M. Although Theorem A was stated in [2] in terms of almost effective actions, the proof given in [2] actually provides the statement as given here [2, p. 545 top].

A compact connected Lie group G can be expressed in the following form

$$G = (T^q \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_a)/N = \bar{G}/N$$

where T^q is a q-torus, $q \geq 0$ (T^0 is assumed to be trivial), each S_j is a compact connected, simply-connected simple Lie group and N is a finite normal subgroup of $\bar{G}$. If $q = 0$, G is called *semi-simple.*

We use the standard notation: A_r ($r \geq 2$, $r \neq 3$), B_r ($r \geq 1$), C_r ($r \geq 3$), D_r ($r \geq 3$), G_2, F_4, E_6, E_7, and E_8 for the classification of the compact simple Lie groups. The simply-connected representatives of the classes A, B, C and D are $SU(r + 1)$, Spin $(2r + 1)$, $Sp(r)$ and Spin $(2r)$ respectively. The simple observation that for G of type B, C or D, the dimension of G is of the form

$$\dim G = \langle l \rangle \quad \text{for some integer } l,$$

will be of particular future interest. We are now able to state our main result.

THEOREM B. *Let G be a compact Lie group acting effectively on a connected m-manifold M. Let k_i ($i = 0, 1, \cdots, s + 1$) be any sequence of positive integers satisfying the conditions:*

(a) $k_0 = m$,
(b) $k_{i+1} \leq \Phi(k_i), 0 \leq i \leq s$.

Then if the dimension of G falls into the range:

$$\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle < \dim G < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle$$

we have only three classes of possibilities.

In each case the action of G on M is transitive and G is semi-simple and locally isomorphic to

$$S_1 \oplus S_2 \oplus \cdots \oplus S_{s+1}$$

where the S_i are simple simply-connected Lie groups with, for $1 \leq i \leq s$, S_i of *type* B *or* D *and* $\dim S_i = \langle k_{i-1} - k_i \rangle$. *The three classes of possibilities are:*

(i) $k_s = 4$, $k_{s+1} = 1$, *and* S_{s+1} *isomorphic to* $SU(3)$.
(ii) $k_s = 6$, $k_{s+1} = 2$, *and* S_{s+1} *isomorphic to the exceptional Lie group* G_2.
(iii) $k_s = 10$, $k_{s+1} = 3$, *and* S_{s+1} *isomorphic to* $SU(6)$.

Condition (b) of Theorem B assures that $k_i \gg k_{i+1}$ for $0 \leq i \leq s$ Theorem B is the appropriate generalization of Theorem A as evidenced by the following proposition.

PROPOSITION 1. *Let* k_i $(i = 0, 1, \cdots, s + 1)$ *be a sequence of positive integers with*

$$k_{i+1} \leq \Phi(k_i), \quad 0 \leq i \leq s.$$

Then for $0 \leq r < s$,

$$\sum_{i=0}^{r} \langle k_i - k_{i+1} \rangle < \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle$$
$$< \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle \leq \sum_{i=0}^{r} \langle k_i - k_{i+1} \rangle + \langle k_{r+1} \rangle.$$

Proof. The first and second inequalities are clear. We prove the third inequality.
Now

$$\left[\sum_{i=r+1}^{s} (k_i - k_{i+1})\right] + 1 = k_{r+1} - k_{s+1} + 1 \leq k_{r+1}.$$

Applying Lemma 3(a),

$$\sum_{i=r+1}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle \leq \langle k_{r+1} \rangle$$

from which the result follows.

4. Proof of Theorem B

The first part of the following lemma appeared as Lemma 4 in [2]. The remaining parts are proved in an entirely analogous fashion and consequently depend upon knowing the maximal dimensions of proper closed subgroups of the compact simple Lie groups. This last information may be found in the table on p. 539 of [2].

LEMMA 9. *Let* G *be a compact connected simple Lie group acting almost effectively on a connected* m*-manifold* M. *Then*

(a) *If* G *is of type* A *or exceptional type,*

$$\dim G < \langle m - \Phi(m) \rangle \quad \textit{for } m \geq 17$$

$$\dim G < \langle m - \Phi(m) - 1 \rangle \quad \textit{for } m \geq 24.$$

(b) *If* G *is of type* C,

$$\dim G \leq \langle m - \Phi(m) \rangle \quad \textit{for } m \geq 8$$

$$\dim G \leq \langle m - \Phi(m) - 1 \rangle \quad \textit{for } m \geq 12.$$

Proof of Theorem B. We may suppose that G is connected for otherwise we would consider the action of its identity component on M. As mentioned previously G can be expressed in the form

$$G = (T^q \oplus S)/N \tag{1}$$

where S is a direct sum of compact simply-connected simple Lie groups. Let

$$\bar{G} = T^q \oplus S. \tag{2}$$

Now $\bar{G}$ acts almost effectively on M. Moreover it is known that $\bar{G}$ acts almost effectively and of course transitively on a *principal orbit* P (see [1, Chapter IX] for terminology) with

$$p = \dim P \leq m. \tag{3}$$

Consider the action of T^q on P. By [2, Lemma 3], S acts almost effectively and transitively on the compact manifold $M_0 = P/T^q$ where

$$m_0 = \dim M_0 = p - q. \tag{4}$$

We now restrict our attention to the action of S on M_0. Following the proof of [2, Theorem 1] we may decompose S as

$$S = V \oplus Q \oplus R \tag{5}$$

where

(α) V, Q and R are each direct sums of simple factor groups of S,
(β) V and R each act almost freely on M_0 with

$$\dim R \leq \dim V = v,$$

(γ) Q acts transitively and almost effectively on $M_1 = M_0/V$ where

$$m_1 = \dim M_1 = m_0 - v.$$

Moreover, we may express Q as

$$Q = S_1 \oplus S_2 \oplus \cdots \oplus S_r$$

where

(δ) S_j, $j = 1, 2, \cdots, r$, are simple factor groups of S with

$$\dim S_j \geq \dim S_{j+1}.$$

(ε) S_j acts almost effectively on the compact manifold

$$M_j = M_{j-1}/S_{j-1} \quad (S_0 = V).$$

Let l_j be the least integer such that

$$\dim S_j \leq \langle l_j \rangle. \tag{6}$$

We consider first the sequence $S_1, S_2, \cdots, S_d$ where

$$d = \min(s - 1, r). \tag{7}$$

The case $s = 1$ will be handled by later considerations. Since $k_{s+1} \geq 1$, it follows that

$$k_s \geq 3, \quad k_{s-1} \geq 10 \quad \text{and} \quad k_{s-2} \geq 66.$$

We show

$$\dim S_1 = \langle m - k_1 \rangle = \langle k_0 - k_1 \rangle$$

and that S_1 is of type B or D.

Now S_1 acts almost effectively on the compact connected m-dimensional manifold

$$N_1^m = M_1^{m_0 - v} \times S^{m - m_0 + v}.$$

(Here we agree that S^0 denotes a point rather than the actual 0-sphere.) Since $m = k_0 \geq 66$, it follows from Lemma 9 that if S_1 is of type A, C or exceptional type that

$$\dim S_1 \leq \langle m - \Phi(m) - 1 \rangle \leq \langle m - k_1 - 1 \rangle.$$

If S_1 is of type B or D, it follows from the form of the dimension of S_1 that $\dim S_1 = \langle l \rangle$ for some l. Moreover if $\dim S_1 \geq \langle m - k_1 + 1 \rangle$ we have by Proposition 1 that

$$\dim G \geq \dim S_1 \geq \langle m - k_1 + 1 \rangle > \langle m - k_1 \rangle + \langle k_1 \rangle$$
$$\geq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} + 1 \rangle$$

which of course is a contradiction to our assumption concerning the range of $\dim G$. Hence, if S_1 is of type B or D,

$$\dim S_1 = \langle m - k_1 \rangle \quad \text{or} \quad \dim S_1 \leq \langle m - k_1 - 1 \rangle.$$

It is sufficient therefore to eliminate the case where

$$\dim S_1 \leq \langle m - k_1 - 1 \rangle.$$

Now

$$\begin{aligned} \dim G = \dim \bar{G} &= \dim T^q + \dim S \\ &= q + \dim V + \dim R + \dim Q \leq q + 2v + \textstyle\sum_{j=1}^{r} \langle l_j \rangle. \end{aligned} \tag{8}$$

Since Q acts almost effectively on M_1, it follows from [2, Theorem 1] that

$$\textstyle\sum_{j=1}^{r} l_j \leq m_1 = m_0 - v = p - q - v \leq m - q - v. \tag{9}$$

Consequently,

$$\dim Q = \dim S_1 + \textstyle\sum_{j=2}^{r} \dim S_j \leq \langle m - k_1 - u \rangle + \textstyle\sum_{j=2}^{r} \langle l_j \rangle \tag{10}$$

where

(i) $v = 0$ or $v \geq 3$, $u \geq 1$, $k_1 \geq 10$,
(ii) $k_1 \leq \Phi(m)$,
(iii) $m - k_1 - u \geq l_j$, all $j \geq 2$,
(iv) $k_1 - v - q + u \geq 0$,
(v) $\sum_{j=2}^{r} l_j \leq k_1 - v - q + u$.

Hence we are precisely in the setting of Lemma 8. We conclude

$$\begin{aligned}(11)\qquad \dim Q &\leq \langle m - k_1\rangle + \langle k_1 - \Phi(k_1)\rangle - 2v - q \\ &\leq \langle m - k_1\rangle + \langle k_1 - k_2\rangle - 2v - q.\end{aligned}$$

Combining (8) and (11) we obtain

$$(12)\qquad \dim G \leq \langle m - k_1\rangle + \langle k_1 - k_2\rangle$$

which is a contradiction to our assumption concerning the range of $\dim G$. Hence $\dim S_1 = \langle m - k_1\rangle$ and S_1 is of type B or D. If $d \geq 2$, we continue with S_2.

Let α_1 = maximal dimension of the orbits of the action of S_1 on M_1. Then

$$(13)\qquad m - k_1 = l_1 \leq \alpha_1 .$$

Consider the almost effective action of S_2 on $M_2 = M_1/S_1$. By [2, Lemma 1],

$$(14)\qquad m_2 = \dim M_2 = m_1 - \alpha_1 .$$

We wish to show $\dim S_2 = \langle k_1 - k_2\rangle$ and that S_2 is of type B or D. Now S_2 acts almost effectively on the compact connected k_1-dimensional manifold

$$N_2^{k_1} = M_2^{m_1-\alpha_1} \times S^{m-m_1} \times S^{\alpha_1-(m-k_1)}.$$

Since $k_1 \geq 66$ it follows from Lemma 9 that if S_2 is of type A, C or exceptional type that $\dim S_2 \leq \langle k_1 - k_2 - 1\rangle$. As in the previous step for S_1 it is again sufficient to eliminate the case $\dim S_2 \leq \langle k_1 - k_2 - 1\rangle$. Now,

$$(15)\qquad \dim Q \leq \langle k_0 - k_1\rangle + \langle k_1 - k_2 - u\rangle + \textstyle\sum_{j=3}^{r} \langle l_j\rangle$$

where

(i) $v = 0$ or $v \geq 3$, $u \geq 1$, $k_2 \geq 10$,
(ii) $k_2 \leq \Phi(k_1)$,
(iii) $k_1 - k_2 - u \geq l_j$, all $j \geq 3$,
(iv) $k_2 - v - q + u \geq 0$,
(v) $\sum_{j=3}^{r} l_j \leq k_2 - v - q + u$.

It follows from Lemma 8 that

$$(16)\qquad \langle k_1 - k_2 - u\rangle + \textstyle\sum_{j=3}^{r} \langle l_j\rangle \leq \langle k_1 - k_2\rangle + \langle k_2 - k_3\rangle - 2v - q$$

and therefore

$$(17)\qquad \dim G \leq \langle k_0 - k_1\rangle + \langle k_1 - k_2\rangle + \langle k_2 - k_3\rangle$$

which is a contradiction.Hence $\dim S_2 = \langle k_1 - k_2\rangle$ and S_2 is of type B or D.

We continue this process until we have exhausted $S_1, S_2, \cdots, S_d$. In general

$$(18)\qquad \dim S_j = \langle k_{j-1} - k_j\rangle$$

and S_j is of type B or D ($j = 1, 2, \cdots, d$). In the $(j+1)^{\text{th}}$ step of the process ($1 \leq j \leq d-1$) we are concerned with

$$\alpha_j = \text{maximal dimension of the orbits of } S_j \text{ on } M_j .$$

$$m_{j+1} = \dim M_{j+1} = m_j - \alpha_j .$$

$$N_{j+1}^{k_j} = M_{j+1}^{m_j - \alpha_j} \times S^{k_{j-1} - m_j} \times S^{\alpha_j - (k_{j-1} - k_j)}.$$

Since at the j^{th} stage, $\dim S_j = \langle k_{j-1} - k_j \rangle$ it follows that

$$k_{j-1} - k_j \leq \alpha_j . \tag{19}$$

Using induction and (19) it is easily established that

$$m_{j+1} \leq k_j - v - q. \tag{20}$$

In later considerations we will be concerned with α_j and N_{j+1} for $j \geq d$ and, in these instances, (19) and (20) will still hold true.

Suppose first that $r \leq s - 1$. Now $d = r$ and

$$\dim Q = \textstyle\sum_{i=0}^{r-1} \langle k_i - k_{i+1} \rangle. \tag{21}$$

Moreover

$$\textstyle\sum_{i=0}^{r-1} (k_i - k_{i+1}) = k_0 - k_r = m - k_r \tag{22}$$

and by [2, Theorem 1],

$$m - k_r \leq \dim M_1 = m_1 \leq m - q - v. \tag{23}$$

Hence

$$q + v \leq k_r . \tag{24}$$

Now

$$\dim G \leq \dim Q + 2v + q \leq \textstyle\sum_{i=0}^{r-1} \langle k_i - k_{i+1} \rangle + 2v + q. \tag{25}$$

But since $r \leq s - 1$, $k_r \geq 10$ and

$$2v + q \leq 2k_r < \langle k_r - \Phi(k_r) \rangle \leq \langle k_r - k_{r+1} \rangle. \tag{26}$$

Hence from (25) and (26),

$$\dim G < \textstyle\sum_{i=0}^{r-1} \langle k_i - k_{i+1} \rangle + \langle k_r - k_{r+1} \rangle \leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle$$

which is a contradiction.

We suppose therefore from now on that $r \geq s$ and we consider two cases.

Case I. $\dim S_s > \langle k_{s-1} - k_s \rangle$. Due to our assumption concerning the range of $\dim G$, S_s must be of type A or exceptional type in this case. Now S_s acts almost effectively on $N_s^{k_{s-1}}$ of dimension k_{s-1} (N_s is defined in a completely analogous fashion to N_j for $j \leq d$). Hence by Lemma 9, $k_{s-1} \leq 16$. Therefore

$$10 \leq k_{s-1} \leq 16. \tag{27}$$

However it is now easily checked (for example, by using the table on p. 539 of [2]) that S_s must act transitively on N_s and, hence, on M_s. Therefore $r = s$.

For the remainder of Case I we assume $r = s$. Now by (20),

$$\dim M_s = m_s \leq k_{s-1} - v - q. \tag{28}$$

Since S_s acts almost effectively on M_s with $\dim S_s > \langle k_{s-1} - \Phi(k_{s-1})\rangle$ it is easily checked that

$$m_s = k_{s-1} \tag{29}$$

and hence

$$v = 0 = q. \tag{30}$$

For example if $k_{s-1} = 10$ and $m_s \leq 9$,

$$\dim S_s \leq \dim SU(5) = 24 < \langle 10 - 3\rangle \leq \langle k_{s-1} - \Phi(k_{s-1})\rangle.$$

Consequently $\bar{G} = Q$ and

$$\dim G = \textstyle\sum_{i=0}^{s-2} \langle k_i - k_{i+1}\rangle + \dim S_s\,. \tag{31}$$

If we consider the cases $11 \leq k_{s-1} \leq 16$ individually it is easily verified that

$$\dim S_s \leq \langle k_{s-1} - k_s\rangle + \langle k_s - k_{s+1}\rangle + \langle k_{s+1}\rangle$$

which combined with (31) is a contradiction to our assumption concerning the range of $\dim G$. For example, if $k_{s-1} = 12$, S_s must be isomorphic to $SU(7)$ and

$$\dim SU(7) = 48 < \langle 12 - 3\rangle + \langle 3 - 1\rangle + \langle 1\rangle.$$

We are left with the case $k_{s-1} = 10$. But here,

$$\begin{aligned}\dim S_s = \dim SU(6) &= 35 > \langle 10 - 3\rangle + \langle 3 - 1 + 1\rangle \\ &= \langle k_{s-1} - k_s\rangle + \langle k_s - k_{s+1} + 1\rangle.\end{aligned}$$

Combining this with (31) we again reach a contradiction. (Note that we must have $k_s = 3$ above for otherwise $\dim S_s < \langle k_{s-1} - k_s\rangle$.)

Case II. $\dim S_s \leq \langle k_{s-1} - k_s\rangle$. Recall that l_s denotes the least integer such that $\dim S_s \leq \langle l_s\rangle$.

By assumption, $l_s \leq k_{s-1} - k_s$. Since $k_s \geq 3$ we may use Lemma 8 in the usual fashion to conclude

$$l_s = k_{s-1} - k_s\,. \tag{32}$$

We consider two subcases of Case II.

Subcase (a). $r = s$. Now

$$\dim Q \leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1}\rangle. \tag{33}$$

Moreover since $l_{i+1} = k_i - k_{i+1}$, $i = 0, \cdots, s - 1$ we apply [2, Theorem 1] to conclude

(34) $m - k_s = \sum_{i=0}^{s-1} (k_i - k_{i+1}) \leq \dim M_1 = m_1 \leq m - q - v.$

Hence,

(35) $$q + v \leq k_s .$$

Now

(36) $$\dim G \leq \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + 2v + q.$$

If $k_s \geq 7$, $2v + q \leq 2k_s < \langle k_s - \Phi(k_s)\rangle \leq \langle k_s - k_{s+1}\rangle$ and from (36),

$$\dim G \leq \sum_{i=0}^{s} \langle k_i - k_{i+1} \rangle$$

which is a contradiction.

We assume for the remainder of Subcase (a) that

(37) $$3 \leq k_s \leq 6$$

and consider the individual cases. The cases $k_s = 4$, 5 and 6 give little difficulty. For example if $k_s = 5$, it follows from (35) that

$$2v + q \leq 8 < \langle k_s - \Phi(k_s)\rangle \leq \langle k_s - k_{s+1}\rangle$$

and hence from (36), $\dim G < \sum_{i=0}^{s} \langle k_i - k_{i+1} \rangle$.

The case $k_s = 3$ and $v = 3$, $q = 0$ appears to require a more subtle argument. Suppose first that $\dim S_s = \langle k_{s-1} - k_s \rangle$. Now

(38) $$\dim G = \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \dim V + \dim R + q.$$

Due to the range of $\dim G$,

(39) $4 = \langle k_s - k_{s+1}\rangle + \langle k_{s+1}\rangle < \dim V + \dim R + q < \langle k_s - k_{s+1} + 1\rangle = 6.$

Hence,

(40) $$\dim V + \dim R + q = 5.$$

But $\dim V = v = 3$, $q = 0$. Hence,

(41) $$\dim R = 2$$

which is impossible since R is a direct sum of simple groups. We assume therefore that $\dim S_s < \langle k_{s-1} - k_s \rangle$. By Lemma 8,

(42) $$\langle k_{s-1} - k_s - 1\rangle < \dim S_s < \langle k_{s-1} - k_s \rangle.$$

Consequently S_s is of type A or exceptional type. If we consider the almost effective action of S_s on $N_s^{k_{s-1}}$, we conclude from Lemma 9 that for $k_{s-1} \geq 17$,

$$\dim S_s < \langle k_{s-1} - \Phi(k_{s-1})\rangle \leq \langle k_{s-1} - k_s - 1\rangle$$

since $k_s = 3 \leq \Phi(k_{s-1}) - 1$. This contradicts, however, (42). We assume therefore

$$10 \leq k_{s-1} \leq 16. \tag{43}$$

However S_s acts almost effectively on $M_s^{m_s}$ with

$$m_s \leq k_{s-1} - v = k_{s-1} - 3. \tag{44}$$

A case by case analysis for $10 \leq k_{s-1} \leq 16$ verifies the non-existence of such an S_s satisfying (42). For example, if $k_{s-1} = 10$ and, consequently, $m_s \leq 7$,

$$\dim S_s \leq \dim SU(4) = 15 < \langle 6 \rangle \leq \langle k_{s-1} - k_s - 1 \rangle.$$

This concludes the case $k_s = 3$ and Subcase (a) of Case II.

Subcase (b). $r \geq s + 1$. From (32) we know that $l_s = k_{s-1} - k_s$. We wish first to eliminate the case $l_{s+1} \leq k_s - k_{s+1}$. If $l_{s+1} = k_s - k_{s+1}$, we may apply [2, Theorem 1] to conclude

$$\dim G \leq \textstyle\sum_{i=1}^{s} \langle k_i - k_{i+1} \rangle + \langle k_{s+1} \rangle.$$

Hence let us suppose $l_{s+1} \leq k_s - k_{s+1} - 1$. If $k_{s+1} \geq 2$, we may apply Lemma 8 directly to arrive at a contradiction. If $k_{s+1} = 1$ and *Lemma* 8 *is not applicable* then we must be in Case II, Subcase (b) or (c) of the proof of Lemma 8. Hence

$$v + q = k_{s+1} + 3 = 4 \quad \text{or} \quad v + q = k_{s+1} + 2 = 3. \tag{45}$$

By (4) of the proof of Lemma 8,

$$\begin{aligned} \textstyle\sum_{j \geq s+1} \langle l_j \rangle &\leq \langle k_s - k_{s+1} - 1 \rangle - \langle v + q - k_{s+1} - 1 \rangle \\ &\leq \langle k_s - k_{s+1} \rangle - (k_s - k_{s+1}) - \langle v + q - 2 \rangle \quad \text{(Lemma 4)} \\ &\leq \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle - k_s - \langle v + q - 2 \rangle. \end{aligned} \tag{46}$$

Now

$$\dim G \leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \sum_{j \geq s+1} \langle l_j \rangle + 2v + q. \tag{47}$$

Since S_{s+1} is a simple Lie group, $k_s - k_{s+1} - 1 \geq l_{s+1} \geq 2$. Hence,

$$k_s \geq 4. \tag{48}$$

Suppose first from (45) that $v + q = 4$. Then $2v + q \leq 7$ and it follows from (46) and (48) that

$$\begin{aligned} \textstyle\sum_{j \geq s+1} \langle l_j \rangle &\leq \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle - k_s - 3 \\ &\leq \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle - 2v - q. \end{aligned} \tag{49}$$

In light of (47) we have a contradiction. Hence we suppose $v + q = 3$. If $k_s \geq 5$, we obtain a contradiction as above by using (46). Assume then $k_s = 4$ and let

$$l_{s+1} = k_s - k_{s+1} - u, \quad u \geq 1.$$

Since $k_s = 4$, $k_{s+1} = 1$ and $l_{s+1} \geq 2$, it follows that $u = 1$. By [2, Theorem 1],

$$\begin{aligned}\dim G &\leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} - u \rangle + \langle k_{s+1} + u \rangle \\ &\leq \textstyle\sum_{i=0}^{s} \langle k_i - k_{i+1} \rangle .\end{aligned}$$

Hence we reach a contradiction and therefore from now on we suppose that

$$(50) \qquad l_{s+1} > k_s - k_{s+1} .$$

At this point we have the following data:

(α) S_i is of type B or D and $\dim S_i = \langle k_{i-1} - k_i \rangle$, $i = 1, 2, \cdots, s - 1$.
(β) $l_s = k_{s-1} - k_s$.
(γ) $\dim S_{s+1} > \langle k_s - k_{s+1} \rangle$.

Hence S_{s+1} is of type A or exceptional type and by Lemma 9

$$(51) \qquad 3 \leq k_s \leq 16.$$

Moreover S_{s+1} acts almost effectively on the compact manifold $N_{s+1}^{k_s}$ of dimension k_s .

We examine the individual cases for k_s . For $k_s \geq 6$, it follows that $k_i \geq 28$ ($i = 1, 2, \cdots, s - 1$) and by Lemma 9 and (β) above we conclude that

$$S_s \text{ is also of type B or D} \quad \text{and} \quad \dim S_s = \langle k_{s-1} - k_s \rangle .$$

(A) $k_s = 16$. Now $\dim S_{s+1} \leq \dim SU(9) = 80$. We may assume $k_{s+1} = \Phi(16) = 4$ for otherwise $\dim S_{s+1} \leq 80 < \langle k_s - k_{s+1} \rangle$. Now $l_{s+1} = 13 = k_s - k_{s+1} + 1$ and by [2, Theorem 1],

$$\begin{aligned}\dim G &\leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \dim S_{s+1} + \langle k_{s+1} - 1 \rangle \\ &\leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + 80 + \langle 3 \rangle \\ &< \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle .\end{aligned}$$

Hence we have eliminated the case $k_s = 16$.

(B) $k_s = 15, 14, 13, 11, 9, 7, 5$. In all these cases we lack the existence of an S_{s+1} satisfying (γ). For example, if $k_s = 11$,

$$\dim S_{s+1} \leq \dim SU(6) = 35 < \langle 8 \rangle = \langle k_s - \Phi(k_s) \rangle .$$

(C) $k_s = 12$. Now $\dim S_{s+1} \leq \dim SU(7) = 48$ and we may assume $k_{s+1} = \Phi(k_s) = 3$. Clearly $l_{s+1} = 10 = k_s - k_{s+1} + 1$ and by [2, Theorem 1],

$$\begin{aligned}\dim G &\leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \dim S_{s+1} + \langle k_{s+1} - 1 \rangle \\ &\leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + 48 + \langle 2 \rangle \\ &\leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle .\end{aligned}$$

(D) $k_s = 10$. Here, $\dim S_{s+1} \leq \dim SU(6) = 35$ and $k_{s+1} = 3$. Now

$$\langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle = 34 < 35 < 36 = \langle k_s - k_{s+1} + 1 \rangle .$$

Hence, we have an exceptional case for dim G with S_{s+1} isomorphic to $SU(6)$. Clearly $v = 0 = q$ and

$$\bar{G} = Q = S_1 \oplus S_2 \oplus \cdots \oplus S_{s+1}$$

where each S_i, $i \leq s$, is of type B or D. Finally to show the action of G on M is transitive we must show $p = m$.
We claim

$$m_j \leq k_{j-1} - (m - p), \qquad j = 1, 2, \cdots, s + 1$$

and we prove this fact by induction on j. Now $m_1 \leq p = k_0 - (m - p)$. Suppose then $m_t \leq k_{t-1} - (m - p)$, $1 \leq t \leq s$. We know $m_{t+1} = m_t - \alpha_t$. Hence from (19),

$$m_{t+1} \leq m_t - (k_{t-1} - k_t) \leq k_{t-1} - (m - p) - (k_{t-1} - k_t) \leq k_t - (m - p).$$

Now S_{s+1} acts almost effectively on M_{s+1} with

$$\dim M_{s+1} = m_{s+1} \leq k_s - (m - p) = 10 - (m - p).$$

Since S_{s+1} is isomorphic to $SU(6)$ we must have $p = m$.

(E) $k_s = 8$. $\dim S_{s+1} \leq \dim SU(5) = 24$ and $k_{s+1} = 2$. Now $l_{s+1} = 7$ and by [2, Theorem 1],

$$\dim G \leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + 24 + 1.$$

Since

$$\langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle = 24 < 24 + 1 < 28 = \langle k_s - k_{s+1} + 1 \rangle$$

we must have $q = 1$ for dim G to be in the correct range. But S_{s+1} acts almost effectively on M_{s+1} with

$$\dim M_{s+1} = m_{s+1} \leq k_s - v - q = 7$$

by (20). However this directly contradicts the fact that S_{s+1} is isomorphic to $SU(5)$. Hence the case $k_s = 8$ is eliminated.

(F) $k_s = 6$. Now $\langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle = 13$ and $\langle k_s - k_{s+1} + 1 \rangle = 15$. Hence S_{s+1} must be isomorphic to the exceptional Lie group G_2. As in (D) we have an exceptional case for dim G with $v = 0 = q$ and

$$\bar{G} = Q = S_1 \oplus S_2 \oplus \cdots \oplus S_{s+1}$$

where each S_i, $i \leq s$, is of type B or D. We show the transitivity of the action by the same method which was employed in (D).

(G) $k_s = 4$. Here $\dim S_{s+1} \leq \dim SU(3) = 8$ and since $l_{s+1} = 4 = k_s$,

$$\dim G \leq \textstyle\sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + \dim S_{s+1}$$

by [2, Theorem 1]. Since

$$\langle k_s - k_{s+1} \rangle + \langle k_{s+1} \rangle = 7 < 8 < 10 = \langle k_s - k_{s+1} + 1 \rangle$$

we once again have an exceptional case for dim G. Since now $k_s < 6$ we know that S_i only for $i \leq s - 1$ is of type B or D. It follows, however, that since $\dim G = \sum_{i=0}^{s-1} \langle k_i - k_{i+1} \rangle + 8$, we must have that $\dim S_s = \langle k_{s-1} - k_s \rangle$. Since $k_{s-1} \geq 15$, S_s is not of type C by Lemma 9. Moreover for $k_{s-1} \geq 17$, S_s is not of type A or exceptional type by Lemma 9. Finally, a simple check for $k_{s-1} = 15, 16$ verifies that S_s must be of type B or D. Again as in (D) and (F)

$$\bar{G} = Q = S_1 \oplus S_2 \oplus \cdots \oplus S_{s+1}$$

and G acts transitively on M.

(H) $k_s = 3$. Now $\dim S_{s+1} \leq \langle 3 \rangle$ and since S_{s+1} is simple,

$$\dim S_{s+1} = 3 \leq \langle k_s - \Phi(k_s) \rangle$$

which eliminates this case.

The proof is now complete with cases (G), (F), and (D) corresponding to the three classes of possibilities, (i), (ii), and (iii) respectively of Theorem B.

5. Final remarks

There are obvious examples of the three possibilities of Theorem B. For example, the product action of

$$G = SO(m - k_1 + 1) \oplus SO(k_1 - k_2 + 1) \oplus \cdots \oplus SO(k_{s-1} - 4 + 1) \oplus SU(3)$$

on

$$M^m = S^{m-k_1} \times S^{k_1-k_2} \times \cdots \times S^{k_{s-1}-4} \times P^2(C)$$

provides an example of (i).

In the statement of Theorem 1 of [2] a decomposition of G somewhat different from that assumed in the proof of Theorem B is used. In [2, Theorem 1] pairs of simple factor groups S_j isomorphic to Spin (3) in $\bar{G}$ are combined as copies of the non-simple Lie group Spin (4). If one checks through the proof of Theorem 1 in [2], it can be seen that this technicality does not affect the application of Theorem 1 in the proof of Theorem B. In particular, the above mentioned technicality does not actually arise in the consideration of the subgroup Q of $\bar{G}$.

Theorem B does not exhaust the total range of gaps. In particular, there are certainly additional gaps α where $\alpha < \langle m - \Phi(m) \rangle$. For example it can be verified that there is no effective pair (G, M^{20}) with $\dim M = 20$ and $\dim G = \langle 15 \rangle + 14$ (note $\langle 15 \rangle + 14 < \langle 20 - \Phi(20) \rangle$). If we restrict our attention to $\alpha > \langle m - \Phi(m) \rangle$ it can be verified that if α is a gap not covered by Theorem B, α must be in the range

$$\sum_{i=0}^{t-1} \langle k_i - k_{i+1} \rangle < \alpha < \sum_{i=0}^{t-1} \langle k_i - k_{i+1} \rangle + \langle k_t - \Phi(k_t) \rangle$$

where

(a) $k_0 = m$

(b) $k_i \leq \Phi(k_{i-1}),\ i \leq i \leq t$.

(Note that $k_1, k_2, \cdots, k_t$ are uniquely determined by α.) When we search for gaps α in the above range we run into a situation comparable to that where $\alpha < \langle m - \Phi(k_m) \rangle$. In the latter case Lemma 9 is not directly applicable and simple factor groups of type A and exceptional type enter significantly into the picture. In principle, the techniques of the proof of Theorem B could be used to track down all possible gaps. However, the program would appear hopelessly tedious, and the final listing of all possible gaps α particularly cumbersome.

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