ON LINEAR TRANSFORMATIONS WHICH PRESERVE THE DETERMINANT

BY
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Let S be the linear space of $n \times n$ real symmetric matrices and $\mathcal{P}$ be the cone of real positive definite matrices in S. Consider a linear transformation T on S to S such that

$$T(\mathcal{P}) \subseteq \mathcal{P} \tag{1}$$

and

$$\det(T(A)) = (\det A)c, \qquad A \in S \tag{2}$$

where c is a non-zero real constant. If M is an $n \times n$ non-singular matrix, let T_M denote the linear transformation on S defined by

$$T_M(A) \equiv MAM', \qquad A \in S; \tag{3}$$

and let G denote the set of such transformations T_M. It is obvious that if $T_M \in G$, then T_M satisfies (1) and (2). Our first theorem establishes the converse.

THEOREM 1. *If T is a linear transformation on S to S satisfying* (1) *and* (2), *then $T \in G$.*

Proof. Since $T(I) \in \mathcal{P}$, there exists a $B \in \mathcal{P}$ such that $T(I) = B^{-1}B^{-1}$.

Setting $U = T_B T$, we have that U satisfies (1) and (2) with $c = 1$ and $U(I) = I$. Since U is linear, we have

$$\det(\lambda I - A) = \det(U(\lambda I - A)) = \det(\lambda I - U(A)) \tag{4}$$

for $A \in S$ and real λ. Hence, the eigenvalues of A are the same as the eigenvalues of $U(A)$ for all $A \in S$. Now, an inner product on S is $\langle A_1, A_2\rangle \equiv \operatorname{tr} A_1 A_2$ (tr denotes trace). If A, $B \in S$ have the same eigenvalues, it is well known that $\operatorname{tr} A^2 = \operatorname{tr} B^2$. Thus, we see that

$$\langle A, A\rangle = \langle UA, UA\rangle = \langle U'UA, A\rangle \tag{5}$$

for all $A \in S$. Thus $U'U$ is the identity on S (see [1, p. 138]).

If $x \in R^n$ is a column vector, then $xx' \in S$. We also note that any positive semi-definite matrix of rank one is of the form xx' for some $x \in R^n$ and the only non-zero eigenvalue is $x'x$. Further, $\langle xx', yy'\rangle = (x'y)^2$. Now, let $\varepsilon_1, \cdots, \varepsilon_n$ be the standard orthonormal basis in R^n. Since $\varepsilon_i \varepsilon_i'$ is positive semi-definite

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of rank 1 with non-zero eigenvalue equal to 1, it follows that

$$U(\varepsilon_i \varepsilon_i') = x_i x_i', \qquad i = 1, \cdots, n \tag{6}$$

where $x_i' x_i = 1$. Furthermore,

$$\begin{aligned}(\varepsilon_i' \varepsilon_j)^2 &= \langle \varepsilon_i \varepsilon_i', \varepsilon_j \varepsilon_j' \rangle = \langle U'U\varepsilon_i \varepsilon_i', \varepsilon_j \varepsilon_j' \rangle \\ &= \langle U\varepsilon_i \varepsilon_i', U\varepsilon_j \varepsilon_j' \rangle = \langle x_i x_i', x_j x_j' \rangle \\ &= (x_i' x_j)^2\end{aligned}$$

so that x_i, $i = 1, \cdots, n$ is an orthonormal basis for R^n. Let Γ be the $n \times n$ orthogonal matrix with i^{th} row x_i' and define V on S by $V \equiv T_\Gamma U$. Then V satisfies (1) and (2) with $c = 1$ and $V\varepsilon_i \varepsilon_i' = \varepsilon_i \varepsilon_i'$, $i = 1, \cdots, n$, and

$$\det(\lambda I - A) = \det(\lambda I - VA). \tag{7}$$

Hence the eigenvalues of A and VA are the same. Now, fix $i < j$. Since $(\varepsilon_i + \varepsilon_j)(\varepsilon_i + \varepsilon_j)'$ is a rank one positive semidefinite matrix with non-zero eigenvalue equal to 2, there exists $x \in R^n$ such that

$$V(\varepsilon_i + \varepsilon_j)(\varepsilon_i + \varepsilon_j)' = xx' \tag{8}$$

where $x'x = 2$. Since $V\varepsilon_i \varepsilon_i' = \varepsilon_i \varepsilon_i'$, (8) can be written

$$xx' = \varepsilon_i \varepsilon_i' + \varepsilon_j \varepsilon_j' + V(\varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i').$$

However,

$$\begin{aligned}0 &= \langle \varepsilon_k \varepsilon_k', \varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i' \rangle \\ &= \langle V'V(\varepsilon_k \varepsilon_k'), \varepsilon_i \varepsilon_j' + \varepsilon_i \varepsilon_i' \rangle \\ &= \langle \varepsilon_k \varepsilon_k', V(\varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i') \rangle.\end{aligned}$$

Thus the i, i and j, j diagonal elements of $V(\varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i')$ are 0. Using (8) this implies that $(x^{(i)})^2 = (x^{(j)})^2 = 1$, where $x^{(k)}$ is the k^{th} element of the vector x. Since $x'x = 2$, we see that $x^{(k)} = 0$ for $k \neq i$, $k \neq j$, $x^{(i)} = \pm 1$ and $x^{(j)} = \pm 1$. Thus we have

$$V(\varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i') = xx' - \varepsilon_i \varepsilon_i' - \varepsilon_j \varepsilon_j' = \pm(\varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i'). \tag{9}$$

Noting that $\{\varepsilon_i \varepsilon_i', i = 1, \cdots, n\} \cup \{\varepsilon_i \varepsilon_j' + \varepsilon_j \varepsilon_i', i < j\}$ forms a basis for S we conclude that if $VA = C = \{C_{ij}\}$, then $C_{ij} = \xi_{ij} a_{ij}$ where $A = \{a_{ij}\}$, $\xi_{ij} = \pm 1$, and $\xi_{ii} = 1$.

Now, let $\eta_j = \xi_{1j}$ for $j = 1, \cdots, n$. We claim that $\xi_{ij} = \eta_i \eta_j$. To establish this claim, we first show that $\xi_{23} = \eta_2 \eta_3$. By assumption, $\det(VA) = \det(A)$ for all $A \in S$. For A, choose the matrix

$$A = \begin{pmatrix} B_1 & 0 \\ 0 & I \end{pmatrix} \quad \text{where } B_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and I is the $(n - 3) \times (n - 3)$ identity.

Then

$$V(A) = \begin{pmatrix} B_2 & 0 \\ 0 & I \end{pmatrix} \quad \text{where } B_2 = \begin{pmatrix} 0 & \eta_2 & \eta_3 \\ \eta_2 & 0 & \xi_{23} \\ \eta_3 & \xi_{23} & 1 \end{pmatrix},$$

and we then have $\det B_1 = \det B_2$. This yields the equation $\eta_2 \eta_3 \xi_{23} = 1$. Since $\xi_{23} = \pm 1$, we see that $\eta_2 \eta_3 = \xi_{23}$.

Now, by simply permuting rows and columns, it follows easily that $\xi_{ij} = \eta_i \eta_j$ for all i, j. Thus if we let $D \epsilon S$ be a diagonal matrix with i^{th} diagonal element η_i, then

$$VA = DAD \quad \text{for} \quad A \epsilon S. \tag{10}$$

Setting $M = B^{-1}\Gamma' D$, we have

$$\mathrm{T}(A) = MAM' \quad \text{for} \quad A \epsilon S. \tag{11}$$

Q.E.D.

Let $\mathcal{L}$ be the linear space of $n \times n$ real matrices. We want to extend the result of the above theorem to linear transformations on $\mathcal{L}$. First, we prove the following.

THEOREM 2. *Let M_1 and M_2 be two real $n \times n$ matrices such that*

$$\det(A + M_1) = \det(A + M_2) \quad \textit{for all} \quad A \epsilon S. \tag{12}$$

Then

$$M_1 = M_2 \quad \textit{or} \quad M_1 = M_2'.$$

Proof. We first write $M_i = A_i + N_i$, $i = 1, 2$ where A_i is symmetric and N_i is skew symmetric. Then (12) implies

(13)) $$\det(A + N_1) = \det(A + A_3 + N_2) \quad \text{for all} \quad A \epsilon S$$

where $A_3 = A_2 - A_1$ is symmetric. Now, write $A_3 = \Gamma D_0 \Gamma'$ where Γ is orthogonal and D_0 is diagonal. Then (13) implies that

$$\det(A + F) = \det(A + D_0 + G) \quad \text{for all} \quad A \epsilon S \tag{14}$$

where $F = \Gamma' N_1 \Gamma$ and $G = \Gamma' N_2 \Gamma$ are both skew symmetric. To establish the lemma, it is sufficient to show (14) implies that $D_0 = 0$ and that $F = G$ or $F = G'$.

Let $H = D_0 + G$ and note that (14) implies

$$\det(\lambda I + AF) = \det(\lambda I + AH) \tag{15}$$

for all non-singular $A \epsilon S$. However, (15) shows that for each non-singular $A \epsilon S$, the eigenvalues of AF are the same as the eigenvalues of AH. Thus,

$$\operatorname{tr}(AF)^2 = \operatorname{tr}(AH)^2 \tag{16}$$

for all non-singular $A \epsilon S$ and then (16) holds for all $A \epsilon S$ by continuity.[2] Writing out the left hand side of (16) explicitly, we have

$$\text{(17)} \qquad \operatorname{tr}(AF)^2 = \sum_i \sum_j \sum_k \sum_l a_{ik} f_{kj} a_{jl} f_{li},$$

where $A = \{a_{ij}\}$ and $F = \{f_{ij}\}$. Now, we desire the coefficient of $a_{\alpha\beta} a_{\gamma\delta}$ in (17). Due to the symmetry of $A = \{a_{ij}\}$, there are eight subscript combinations of (i, j, k, l) which yield a contributing term to the coefficient of $a_{\alpha\beta} a_{\gamma\delta}$ in (17). These are listed below:

Subscript Combination				Coefficient
$i = \alpha$,	$k = \beta$,	$j = \gamma$,	$l = \delta$	$f_{\beta\gamma} f_{\delta\alpha}$
$i = \alpha$,	$k = \beta$,	$j = \delta$	$l = \gamma$	$f_{\beta\delta} f_{\gamma\alpha}$
$i = \beta$,	$k = \alpha$,	$j = \gamma$,	$l = \delta$	$f_{\alpha\gamma} f_{\delta\beta}$
$i = \beta$,	$k = \alpha$,	$j = \delta$,	$l = \gamma$	$f_{\alpha\delta} f_{\gamma\beta}$
$i = \gamma$,	$k = \delta$	$j = \alpha$,	$l = \beta$	$f_{\delta\alpha} f_{\beta\gamma}$
$i = \gamma$,	$k = \delta$	$j = \beta$,	$l = \alpha$	$f_{\delta\beta} f_{\alpha\gamma}$
$i = \delta$,	$k = \gamma$,	$j = \alpha$,	$l = \beta$	$f_{\gamma\alpha} f_{\beta\delta}$
$i = \delta$,	$k = \gamma$,	$j = \beta$,	$l = \alpha$	$f_{\gamma\beta} f_{\alpha\delta}$

However, F is skew symmetric so that $f_{ij} = -f_{ji}$ for all i and j. Using this fact that adding the coefficients in the above table, we conclude that the coefficient of $a_{\alpha\beta} a_{\gamma\delta}$ in (17) is

$$\text{(18)} \qquad 4\{f_{\beta\gamma} f_{\delta\alpha} + f_{\beta\delta} f_{\gamma\alpha}\}.$$

Since (16) holds for all $A \epsilon S$, we conclude that

$$\text{(19)} \qquad f_{\beta\gamma} f_{\delta\alpha} + f_{\beta\delta} f_{\gamma\alpha} = h_{\beta\gamma} h_{\delta\alpha} + h_{\beta\delta} h_{\gamma\alpha}$$

for all $\alpha, \beta, \gamma, \delta$. Noting that $f_{\alpha\alpha} = 0$ and setting $\alpha = \beta = \gamma = \delta$ in (19) shows that $h_{\alpha\alpha} = 0$ for all α. Since $H = \{h_{ij}\} = D_0 + G, D_0$ is diagonal, and G is skew symmetric, it is clear that $D_0 = 0$. Thus we can write (19) as

$$\text{(20)} \qquad f_{\beta\gamma} f_{\delta\alpha} + f_{\beta\delta} f_{\gamma\alpha} = g_{\beta\gamma} g_{\delta\alpha} + g_{\beta\delta} g_{\gamma\alpha}$$

for all $\alpha, \beta, \gamma, \delta$. Setting $\alpha = \beta$ and $\gamma = \delta$ in (29), we have

$$\text{(21)} \qquad f_{\beta\gamma}^2 = g_{\beta\gamma}^2 \quad \text{for all} \quad \beta \quad \text{and} \quad \gamma.$$

Noting that $f_{\alpha\alpha} = g_{\alpha\alpha} = 0$ for all α, and using (20), first with $\delta = \beta$ and then with $\alpha = \gamma$, we have the two equations

$$\text{(22)} \qquad f_{\beta\gamma} f_{\beta\alpha} = g_{\beta\gamma} g_{\beta\alpha}$$

$$\text{(23)} \qquad f_{\beta\gamma} f_{\delta\gamma} = g_{\beta\gamma} g_{\delta\gamma}$$

for all $\alpha, \beta, \gamma, \delta$.

If $f_{\beta\gamma} = 0$ for all β and γ, then (21) shows that $F = G = 0$ and the lemma is established. In the case where $F \neq 0$, fix i and j such that $f_{ij} \neq 0$. From

[2] Those readers unfamiliar with continuity arguments in an algebraic setting might consult Chapter 1 of Bellman [2].

(22) and (23) we then have

$$f_{ij}^2 f_{i\alpha} f_{\delta j} = g_{ij}^2 g_{i\alpha} g_{\delta j} \tag{24}$$

for all α, δ, and then (21) shows that

$$f_{i\alpha} f_{\delta j} = g_{i\alpha} g_{\delta j} \quad \text{for all} \quad \alpha, \delta. \tag{25}$$

Now, setting $\beta = i$ and $\gamma = j$ in (20) and using (25), we conclude that

$$f_{ij} f_{\delta\alpha} = g_{ij} g_{\delta\alpha} \quad \text{for all} \quad \alpha, \delta. \tag{26}$$

Since $f_{ij}^2 = g_{ij}^2 \neq 0$, it follows that

$$f_{\delta\alpha} = g_{\delta\alpha} \quad \text{for all} \quad \alpha, \delta \tag{27}$$

or

$$f_{\delta\alpha} = -g_{\delta\alpha} \quad \text{for all} \quad \alpha, \delta.$$

However, F and G are skew symmetric so that either $F = G$ or $F = G'$. This establishes the theorem.

If $T_M \in G$, let $\tilde{T}_M$ denote the extension of T_M to $\mathfrak{L}$ given by $\tilde{T}_M N = MNM'$ for all $N \in \mathfrak{L}$. Also, let $\tilde{G}$ denote the set of $\tilde{T}_M$.

THEOREM 3. *Let T be a linear transformation on $\mathfrak{L}$ to $\mathfrak{L}$ such that*

$$T(\mathcal{P}) \subset \mathcal{P} \tag{28}$$

and

$$\det(TN) = c \det N \quad \textit{for} \quad N \in \mathfrak{L}, \tag{29}$$

where c is a non-zero real number. Then $T \in \tilde{G}$ or $TW \in \tilde{G}$ where W is the linear operation of transpose.

Proof. From (28) we have that $T(S) \subseteq S$. Applying Theorem 1 to the restriction of T to S, there exists a $\tilde{T}_M \in \tilde{G}$ such that

$$V = T\tilde{T}_M^{-1} \tag{30}$$

satisfies (29) with $c = 1$ and $V(A) = A$ for all $A \in S$. To establish the theorem, it is sufficient to show that V is the identity or $V = W$ on $\mathfrak{L}$.

Now, for each skew symmetric matrix F, (29) implies that

$$\det(A + V(F)) = \det(A + F) \quad \text{for all} \quad A \in S \tag{31}$$

and Theorem 2 shows that either $V(F) = F$ or $V(F) = F'$. Let $\mathfrak{F}$ denote the linear space of all $n \times n$ skew symmetric matrices and note that $\mathfrak{L} = S + \mathfrak{F}$. Also let

$$\mathfrak{F}_1 = \{F \mid F \in \mathfrak{F},\ V(F) = F\}, \qquad \mathfrak{F}_2 = \{F \mid F \in \mathfrak{F},\ V(F) = F'\}. \tag{32}$$

It is obvious that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are linear manifolds with only 0 in common and

$\mathfrak{F}_1 + \mathfrak{F}_2 = \mathfrak{F}$. However, the fact that every $F \in \mathfrak{F}$ is either in $\mathfrak{F}_1$ or $\mathfrak{F}_2$ shows that either $\mathfrak{F}_1 = \{0\}$ or $\mathfrak{F}_2 = \{0\}$. This completes the proof.

References

1. P. R. Halmos, *Finite-dimensional vector spaces*, second edition, Van Nostrand, Princeton, 1958.
2. R. Bellman, *Matrix analysis*, McGraw-Hill, New York, 1960.

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